COMPUTATIONAL PROPERTIES OF ROTATION PARAMETRIZATIONS

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Computational Properties of Rotation Parametrizations

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Abstract

There are various ways of representing a 3-D rotation in Euclidean space and some of them can be more sensitive to errors than others. We compare five well known parametrizations under circunstances encountered in image understanding systems by introducing perturbations on the parameters of each representation and measuring their influence on the output rotation. The exponential representation is found to be the least sensitive to these perturbations.

1 Introduction

In model based Computer Vision the 3-D position and orientation of an object needs to be represented ideally in the most exact and economic way. A vector gives the position of the object and its 3-D orientation is given by a rotation from a fixed reference orientation. Another use of the rotation is in specifying, along with the translation, the motion between two frames. There are various ways of representing a rotation. Rotation matrices (3 orthogonal vectors), Euler angles, unit quaternions, axis-angle pairs, and exponential vectors are most commonly used in Computer Vision. Other representations used are the Pauli spin matrices, Gibbs vectors and Cayley-Klein parameters.

The choice of which of these rotation parametrizations to use can be made more simple and more reliable if their computational properties, especially under the presence of errors, are better understood. The property of error linearity, and error sensitivity have been neglected by the literature covering the subject. This question is relevant if quantization effects are significant in the rotation representation (*i.e.* if a small number of bits are used to represent the rotation), or if the representation is affected by a fixed quanta of error (*e.g.* an arbitrary, but small error is considered). We offer in this paper an analysis of the property of error sensitivity of these five main parametrizations of rotation, and also discuss other properties such as the presence of duals and wrap-around, that can incur additional computational complexity. We show the results of experiments over a large range of perturbations on each of these representations to point out the differences between them.

2 Representing a 3-D Rotation

Most books in robotics and computer vision give a description of the main parametrizations used for rotations. A good treatment of these can be seen in Altmann [1], there is also a book by Kanatani [6], and the papers by Rooney [8] and Brown [2]. We give below a brief description of the equations for the five main parametrizations. For more details see, for example, [1].

2.1 3x3 Orthogonal Matrix

A 3x3 orthogonal matrix is formed by three orthogonal vectors related in the following way

$$\|\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\| = (\mathbf{r}_1 \times \mathbf{r}_2, \mathbf{r}_3) = (\mathbf{r}_2 \times \mathbf{r}_3, \mathbf{r}_1) = (\mathbf{r}_3 \times \mathbf{r}_1, \mathbf{r}_2) = 1,$$

and

$$\mathbf{R} = [\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3].$$

$\mathbf{R}$ represents a rotation if, and only if,

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I},$$

$$\|\mathbf{R}\| = 1.$$

A rotation $\mathbf{R}$ can be applied to a vector $\mathbf{v}_1$ resulting in $\mathbf{v}_2$ as

$$\mathbf{v}_2 = \mathbf{R}\mathbf{v}_1.$$

2.2 Euler Angles

There are sixteen ways of choosing three successive rotations about the axes [1]. Each one constitutes a particular set of Euler angles. One such set is the three angles R, S, and T defined as three successive rotations around the fixed axes during the transformation:

R is the first rotation around z
S is the second rotation around y
T is the third rotation around z

The rotation matrix related to this particular set of Euler angles is

$$\mathbf{R} = \begin{bmatrix} cosR * cosS * cosT - sinR * sinT & -cosR * cosS * sinT - sinR * cosT & cosR * sinS \\ sinR * cosS * cosT + cosR * sinT & -sinR * cosS * sinT + cosR * cosT & sinR * sinS \\ -sinS * cosT & sinS * sinT & cosS \end{bmatrix}$$

2.3 Unit axis and angle

A rotation can also be represented using a unit vector in the direction of the axis of rotation and its angle of rotation. If the angle is ϕ and the unit vector $\mathbf{n} = (n_x, n_y, n_z)$, the rotation matrix is:

$$\mathbf{R} = \begin{bmatrix} \mu + n_x^2\nu & n_x n_y \nu - n_z \tau & n_x n_z \nu + n_y \tau \\ n_y n_x \nu + n_z \tau & \mu + n_y^2 \nu & n_y n_z \nu - n_x \tau \\ n_z n_x \nu - n_y \tau & n_z n_y \nu + n_x \tau & \mu + n_z^2 \nu \end{bmatrix}$$

where,

$$\mu = cos\phi$$

$$\nu = 1 - cos\phi$$

$$\tau = sin\phi$$

2.4 Unit quaternion

A quaternion $\underline{q}$ can be used to represent a rotation around a unit axis $\mathbf{n} = (n_x, n_y, n_z)$ by an angle ϕ as follows,

$$\underline{q} = (q_0, \mathbf{q}),$$

where,

$$q_0 = cos(\phi/2),$$

$$\mathbf{q} = sin(\phi/2)\mathbf{n},$$

and,

$$\underline{q}^{-1} = (q_0, -\mathbf{q}).$$

the rule of multiplication in quaternion algebra, here shown as "*", for a quaternion $\underline{q}$ and a vector $\mathbf{v}$ is,

$$\underline{q} * \mathbf{v} = (cos(\phi/2), sin(\phi/2)(\mathbf{n} \times \mathbf{v})).$$

In order to rotate a vector $\mathbf{v}_1$ by ϕ around $\mathbf{n}$ we have,

$$\mathbf{v}_2 = \underline{q} * \mathbf{v}_1 * \underline{q}^{-1}.$$

2.5 Exponential representation

In the case of the exponential representation if $\mathbf{n}$ is the axis and ϕ is the angle of rotation we have,

$$\mathbf{r} = \phi\mathbf{n},$$

and since, $\phi = ||\mathbf{r}||$ and $||\mathbf{n}|| = 1$ Rodrigues' equation [7] gives

$$\mathbf{R} = \mathbf{I} + g_1\mathbf{H} + g_2\mathbf{H}^2,$$

where $\mathbf{I}$ is the identity matrix and,

$$g_1 = \frac{\sin||\mathbf{r}||}{||\mathbf{r}||},$$

$$g_2 = \frac{(1 - \cos||\mathbf{r}||)}{||\mathbf{r}||^2},$$

$$\mathbf{H} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}.$$

2.6 Discussion

The most used representation is the rotation matrix which has nine parameters derived from three orthogonal vectors. The Euler angles are also widely used and offer a very low redundancy since it needs only three parameters, the three angles. The quaternions and the axis-angle representations both need four parameters. However, as the axis $\mathbf{n}$ or quaternion $\underline{q}$ have unit length, there are still just three degrees of freedom. The quaternions provide a special algebra as observed by Hamilton [5] and their use as rotation parameters was first made by Rodrigues [7] (see [1] for some historical facts). The last one, the exponential representation, is relatively recent in computer vision [3] and uses only three parameters in the form of a vector in the direction of the axis of rotation with length equal to the angle of rotation.

Before looking at their behaviour under perturbations we would like to point out a set of other properties of these parametrizations. Since we are testing here the representations' sensitivity to perturbations it seems reasonable to

Representation	par.	n. oper. to rotate v	duals	wrap-around
3x3 ort. matrix	9	9*, 6+	n	n
Euler angles	3	25*, 10+, 3sin, 3cos	y	y
Unit axis and angle	4	33*, 16+, 1sin, 1cos	y	y
Unit quaternion	4	18*, 12+	y	y
Exponential	3	39*,20+,1$\sqrt{}$,1sin,1cos	y [1]	y

Table 1: Some Computational Properties of the Parametrizations

also look at their operation counts, in rotating a vector for example, and the presence of duals and wrap-around. The operation count is a useful measure of the time needed to compute a rotation with the representation. Table.1 lists the number of operations that are needed to rotate a given vector in each representation. The presence of duals means that there are two ways of setting the parameters for the same rotation, and it becomes a problem in averaging estimates of rotations. A wrap-around is caused when for small changes in a rotation a big change in the parameter values happens. In this case one has to work with a range of values and the representation becomes uncertain. Both count for the presence of error in estimating its final rotated position. These properties are listed in Table.1 and are indicative of a compromise that has to be made between the redundancy of the representation, its computational cost, and also whether it has duals or wrap-around.

One important computational property of the parametrizations is how they behave under error conditions that can be introduced anywhere in the parameters of the rotation? To answer that we have tested their performance with random noise added to the parametrizations for relatively large variations (*e.g.* for the range of values inside the unit sphere). It is possible that after the perturbation the set of parameters do not represent a rotation anymore. When this happens a normalization procedure has to be carried out (*i.e.* keeping the norm of the quaternion equal to one, the orthogonal matrix still orthogonal and with determinant equal to one, and also the axis of the axis and angle representation with length one). The normalization *per se* can not stop the perturbation in leading to a wrong result, but it keeps the influence of the perturbation within a limit.

We describe below experiments made with each of the representations for a large range of perturbations added to the rotations.

3 Experiments

We want to test the performance of those representations in the presence of random noise, in order to measure how sensitive they are to these perturbations. The experiment we did was to generate 5000 sample vectors, uniformly distributed on a complete unit sphere, to be applied a random rotation based on

[1] yes, if $\phi \in [0,2\pi]$, no if $\phi \in [0,\pi]$

a random angle of rotation, uniformly distributed in $[0,2\pi]$ and a random axis of rotation also uniformly distributed in all possible directions. Random noise was then added to each parameter of the representation using a normal distribution with zero mean, and standard deviation α. The steps were the following:

- For noise magnitude α from 0.0 to 1.0,
- Use 5000 samples for each α,
- Pick a random rotation and express it in the five parametrizations,
- Pick a random vector $\mathbf{v}_1$,
- Compute the rotated directions $\mathbf{v}_2 = \mathbf{R}\mathbf{v}_1$ for each of the representations,
- Perturb all the parameters of the representations with noise of size α giving rotation $\hat{\mathbf{R}}$, for each representation,
- Compute the perturbed directions $\hat{\mathbf{v}}_2 = \hat{\mathbf{R}}\mathbf{v}_1$, for each representation,
- Compute the difference (angle and distance) between $\hat{\mathbf{v}}_2$ and $\mathbf{v}_2$.

After perturbing the rotation a normalizing procedure was carried out to ensure that the parametrizations would still represent a rotation. This was done for the rotation matrix, the quaternion, and the axis-angle representation, the results are shown in Figures 1 and 2. The normalization for the rotation matrix is the same as in [4], for the quaternion and the axis-angle the procedure was to keep the norm and axis length equal to one. We used a hundred (100) samples of different magnitudes α in the range of [0.0,1.0], and for each α the mean distance and angle between the perturbed and the unperturbed vectors were computed.

The output measures show a function of the error sensitivity for perturbations going from zero to a complete corruption of the parameters.

4 Conclusions

By adding noise to the parameters of the representations we were looking for an overall sensitivity of the whole representation, and with that we could compare all of the five representations. It turns out that one has to be careful with this because the redundancy of the parametrization, *i.e.* more than three parameters poses different variations between each of the parameters for each representation.

However, as can be seen from the curves in Figures 1 and 2 all of the representations keep a steady ascendent curve not crossing each other unless of course for zero perturbation. The curves show that the quaternions for example exhibit a poor behaviour under perturbations in comparison with the

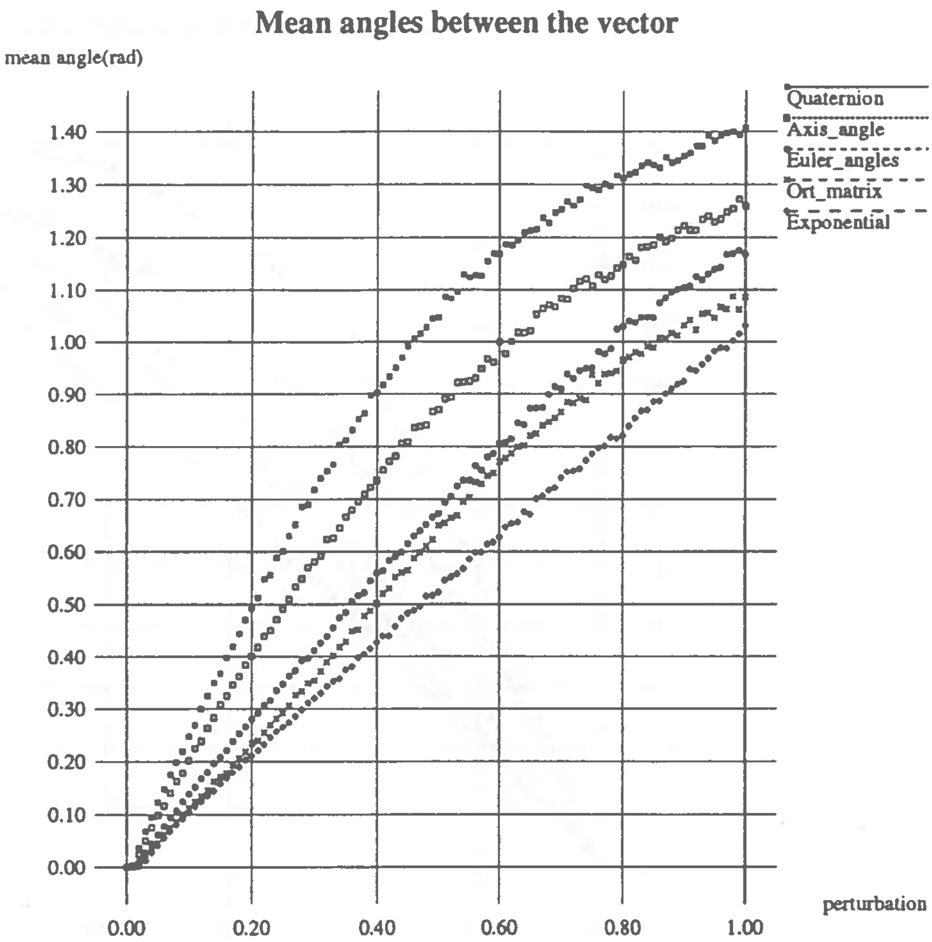

Figure 1: Mean angles between $\hat{\mathbf{v}}_2$ and $\mathbf{v}_2$

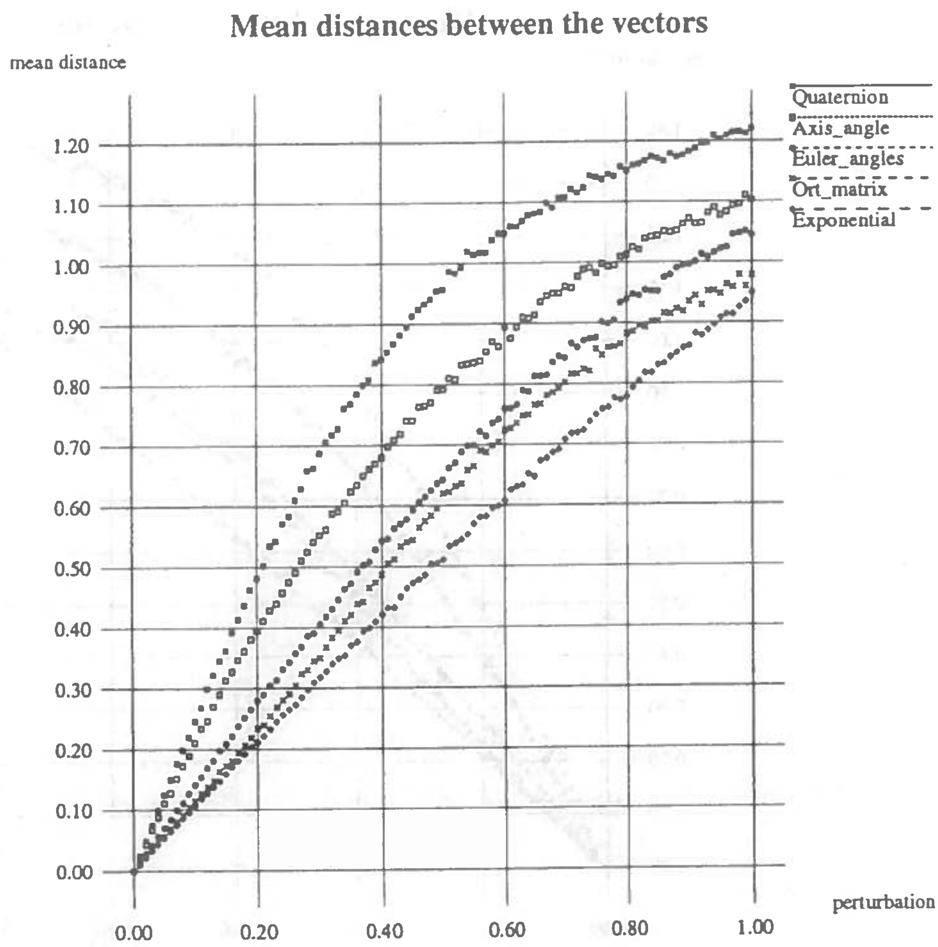

Figure 2: Mean distance between $\hat{v}_2$ and v_2

exponential representation, which it is shown to be the least sensitive to noise added in all the parameters at the same time. The 3x3 orthogonal rotation matrix also shows a good overall behaviour that can compensate for its large number of parameters. The Euler angles and the axis angle representation are not as good as the exponential but they still perform better than the quaternion.

Since normalization of the quaternion and axis-angle form has reduced their effective error relative to the other representations, their real performance for comparable errors must be worse. On the other hand, the perturbed rotation matrix is no longer strictly a rotation matrix, so any normalization to a true rotation matrix is likely to reduce its relative error. Because of the different structure of each representation, it is possible that the individual parameters of the representation may not be equally sensitive. We tested this hypothesis and concluded that the parameters of the rotation matrix and exponential representation are equally important. However, for the quaternion and axis-angle representations, the parameters that relate to the axis have greater influence, and for the Euler angle representation, the S angle has greater influence. However, in any case, after normalization, the difference between the representations shown in Figures 1 and 2 are more significant than the differing sensitivities in the individual parameters.

The final choice of which representation to use depends on other considerations also, such as: a) if one needs a more intuitive visualization of the rotation, or b) if some parameters are more easily extracted from the data or c) the algorithms one is working with favours particular properties. For the purpose of estimating rotations, the least number of parameters is best because only with the minimal parametrizations will the variance matrices (the uncertainty estimates) be non-singular[2]. This property of lowest redundancy favours also the exponential representation. However, as far as we can see at present, there is not yet an authoritative set of criteria to determine a best rotation representation for computer vision applications. Meanwhile the more we know the computational properties of those representations, the better the choice we can make of which one to use. This paper has presented two additional criteria useful for discriminating between the representations: (1) relative stability of the different representations and (2) relative importance of individual parameters in the representations.

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