Continuous Dcpos in Quantum Computing

Robert Furber
Aalborg University
Aalborg, Denmark
Email: furber@cs.aau.dk

***Abstract*—We show that the unit interval of a directed-complete C*-algebra is a continuous dcpo only if the C*-algebra is a product of finite-dimensional matrix algebras. Since Selinger showed that the positive cone of any product of finite-dimensional matrix algebras is continuous as a bounded directed-complete poset (in the process of showing that the categories CPM and $\mathbf{Q}$ are enriched in continuous dcpos) this is actually if and only if, and therefore shows that Selinger's result is the best possible. This means that attempts to define a quantum domain theory using infinite-dimensional W*-algebras or von Neumann algebras necessarily involve dcpos that are not continuous.**

We also show that even when the positive cone is continuous, the Scott and Lawson topologies are not suitable for computational realizability in the noncommutative case because the positive cone of a non-commutative matrix algebra does not have a countable base.

I. Introduction

In denotational semantics, we usually wish to interpret terms as functions, in the mathematical sense. Recursively-defined functions and while loops are then interpreted as limits of approximations by partial functions. That is to say, the partial functions involved are given an ordering, and the function defined by a recursive definition is given as the least upper bound of a sequence of approximants. This is one of the important points of using domain theory for denotational semantics.

Quantum mechanics does not work with arbitrary functions, but only with certain linear functions defined between vector spaces (subject to many technical conditions). So when giving a denotational semantics to quantum programming languages, it is natural to do it in terms of ordered vector spaces. In [1] Selinger defines a category $\mathbf{Q}$, having completely positive trace-reducing maps between tuples of matrices as hom sets. It is proved in [1, Lemma 6.4] that these hom sets are cpos and in [1, §6.5] this structure is used to form the fixed points needed for recursion. In [2, Example 2.7] Selinger shows that the poset of positive-semidefinite $n \times n$ matrices with trace ≤ 1 is continuous, and in [2, §4.3 and §5.1, Remark] deduces from this that $\mathbf{Q}$ is enriched in continuous dcpos. This does not involve showing that these dcpos belong to a cartesian-closed category of domains, but instead makes use of the Choi-Jamiołkowski isomorphism [3], [4] between the cone of completely positive maps $B(\mathcal{H}) \to B(\mathcal{K})$ and the positive cone of $B(\mathcal{H} \otimes \mathcal{K})$, when $\mathcal{H}, \mathcal{K}$ are finite-dimensional Hilbert spaces.

There is a duality for state and predicate transformers for quantum programs, first described by d'Hondt and Panangaden in [5]. We can illustrate it for the case of maps $B(\mathcal{H}) \to B(\mathcal{K})$, where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces[1]. We use the trace, but for an infinite-dimensional $\mathcal{H}$ the sum defining the trace might diverge or the trace might take different values for different bases, so we need to use *trace-class operators*, which are operators in $B(\mathcal{H})$ such that the trace of the positive part of their polar decomposition converges. We write $\mathrm{TC}(\mathcal{H})$ for the Banach space of trace-class operators. There is a bilinear pairing $\langle \text{-}, \text{-} \rangle : \mathrm{TC}(\mathcal{H}) \times B(\mathcal{H}) \to \mathbb{C}$ defined by $\langle \rho, a \rangle = \mathrm{tr}(\rho a)$. If we curry this pairing on the left, we get an isometry of Banach spaces $B(\mathcal{H}) \to \mathrm{TC}(\mathcal{H})^*$ (where $\mathrm{TC}(\mathcal{H})^*$ is the space of bounded linear maps $\mathrm{TC}(\mathcal{H}) \to \mathbb{C}$). If we curry it on the right, we get a bounded linear map $\mathrm{TC}(\mathcal{H}) \to B(\mathcal{H})^*$ that maps positive operators in $\mathrm{TC}(\mathcal{H})$ to Scott-continuous[2] maps $B(\mathcal{H}) \to \mathbb{C}$.

A bounded linear map $f : TC(\mathcal{H}) \to \mathrm{TC}(\mathcal{K})$ determines a dual bounded linear map $f^* : \mathrm{TC}(\mathcal{K})^* \to \mathrm{TC}(\mathcal{H})^*$, by taking $f^*(\phi) = \phi \circ f$. Using the isomorphisms defined by the pairings between TC and B, each bounded linear map $f : \mathrm{TC}(\mathcal{H}) \to \mathrm{TC}(\mathcal{K})$ determines a bounded linear map $f^* : B(\mathcal{K}) \to B(\mathcal{H})$. If f is completely positive trace-decreasing map, then f^* is a completely positive *subunital* map, *i.e.* $f^*(1_{\mathcal{K}}) \leq 1_{\mathcal{H}}$, where $1_{\mathcal{K}}$ and $1_{\mathcal{H}}$ are the identity maps on $\mathcal{K}$ and $\mathcal{H}$, respectively. So, in the terminology of [5], f is a state transformer, and f^* is the corresponding predicate transformer. We also have that if $g : B(\mathcal{K}) \to B(\mathcal{H})$ is a Scott-continuous[3] completely positive map, g^* maps Scott-continuous elements of $B(\mathcal{H})^*$ to Scott-continuous elements of $B(\mathcal{K})^*$, and thereby defines a bounded linear map $g^* : \mathrm{TC}(\mathcal{H}) \to \mathrm{TC}(\mathcal{K})$.

When all the details are carried out, this defines a contravariant equivalence between $\mathbf{Q}$ and the category of finite-dimensional C*-algebras and completely positive subunital maps, because objects of $\mathbf{Q}$ can be considered to be of the form $\mathrm{TC}(\mathbb{C}^{n_1}) \times \cdots \times \mathrm{TC}(\mathbb{C}^{n_i})$, and finite-dimensional C*-algebras are isomorphic to C*-algebras that are of the form $B(\mathbb{C}^{n_1}) \times \cdots \times B(\mathbb{C}^{n_i})$. Because the algebraic structure of $B(\mathcal{H})$ is easier to generalize than the definition of trace-class operators and density matrices, the usual starting point for generalizing to the infinite-dimensional case is C*-algebras, even though the maps point in the opposite direction from

[1] We allow infinite-dimensional Hilbert spaces in this example, even though [5] restricts to the finite-dimensional case, because it makes certain details in the extension to W*-algebras clearer.

[2] Known as *normal* in the operator algebra community.

[3] This is automatic for completely positive maps when $\mathcal{K}$ is finite-dimensional.

expected because everything is done in terms of predicate transformers rather than state transfomers.

We cannot use arbitrary C*-algebras for domain-theoretic purposes because it is not true that every infinite-dimensional C*-algebra has a bounded directed-complete positive cone. The C*-algebras that do have a bounded directed-complete positive cone and are separated by their Scott-continuous states[4] are exactly the W*-algebras [6, Definition 1] [7, Theorem A.4], which are C*-algebras equipped with a space that plays the rôle that $\mathrm{TC}(\mathcal{H})$ plays for $B(\mathcal{H})$. A research programme has developed, using W*-algebras as a starting point for quantum domain theory [7] [8, Chapter 3] [9]. For instance, in [10, §4.1] it is shown that the category of W*-algebras and Scott-continuous completely positive subunital maps is enriched in pointed dcpos, the infinite-dimensional analogue of [1, Lemma 6.4]. There are many non-trivial W*-algebras that may not be apparent from experience with the finite-dimensional case, for instance algebras of bounded measurable functions modulo null sets on σ-finite measure spaces (usually notated $L^\infty(X, \Sigma, \mu)$), and there are infinite-dimensional W*-algebras that come from infinite tensor products of finite-dimensional matrix algebras.

In view of the above, it is natural to ask which W*-algebras are have *continuous* positive cones, or in which the unit interval $[0,1]_A$ of a W*-algebra A is continuous, because the unit interval is isomorphic to the Scott-continuous completely positive maps $\mathbb{C} \to A$, so if the category of W*-algebras and Scott-continuous completely positive subunital maps is to be enriched in continuous domains, $[0,1]_A$ must be continuous. In this article, we show that Selinger's result is the best possible, *i.e.* that if a W*-algebra has a continuous positive cone, it is a product[5] of finite-dimensional matrix algebras. In fact, we do not need the "normal states" part of the definition of a W*-algebra in the proof, as it shows that if a *directed-complete C*-algebra* has continuous positive cone then it is a product of finite-dimensional matrix algebras. This can be seen as a "quantum analogue" of the theorem that a complete Boolean algebra is a continuous lattice iff it is atomic [11, Theorem I-4.20]. The proof proceeds by reducing the problem to a statement about projection operators. In fact, the lattice structure of the projection operators turns out to be essential, which is what Birkhoff and von Neumann called the "Logic of Quantum Mechanics" [12], by analogy to the rôle of lattice operations in Boolean algebra.

We conclude the introduction with a discussion of some related work. W*-algebras have a good notion of conditional expectation [13] that extends the usual one for $L^\infty(X, \Sigma, \mu)$, the bounded measurable functions on a probability space (X, Σ, μ). This has seen some use in the commutative case in [14], and it seems it would be difficult to formulate this using continuous dcpos.

There is some domain theoretic work [15] related to the *spectral order*, an ordering on the set of density matrices, where the totally mixed state ($\frac{1}{d}$ times the identity matrix) is the bottom element [15, §10.3.2 Definition 32 and Theorem 7], based on ordering states by the amount of information they contain. This is different from the Löwner order used here, where the 0 matrix is the least element. Additionally, the $\ll$ relation used there is different from the usual way-below relation [15, §10.2.4 Definition 11].

[4] This is necessary for every predicate transformer to correspond to a state transformer and vice-versa.

[5] We allow infinite products.

II. Definitions and Background

If P is a poset, a subset $S \subseteq P$ is *directed* if for each $a, b \in S$, there exists $c \in S$ such that $c \geq a$ and $c \geq b$. A poset P is *directed* if $P \subseteq P$ is directed. We will often refer to directed sets indexed by a poset, so we will say, for instance, let $(a_i)_{i \in I}$ bea directed set in P to mean that I is a directed poset, and the mapping $i \to a_i$ is a monotone map (and therefore the image $\{a_i \mid i \in I\}$ is a directed subset of P). Every directed set in P is of this form, by "self-indexing". We say a poset P is *directed complete* if every directed set $(a_i)_{i\in I}$ has a least upper bound, which is written $\bigvee_{i \in I} a_i$. If $S \subseteq P$, we just write $\bigvee S$. A poset P is *bounded directed complete* if for each directed set $(a_i)_{i \in I}$ that is bounded, *i.e.* such that there exists $b \in P$ such that for all $i \in I$, $a_i \leq b$, has a least upper bound $\bigvee_{i \in I} a_i$. For instance, $\mathbb{R}$ with its usual ordering is bounded directed complete but not directed complete.

First we recall the basic definitions. If D is a poset, $d, e \in D$, then we say e is *way below* d, or $e \ll d$, if for all directed sets $(d_i)_{i \in I}$ such that $\bigvee_{i \in I} d_i \geq d$, there exists $j \in I$ such that $e \leq d_j$. A poset is *continuous* if for all $d \in D$, the set $\twoheaddownarrow d = \{e \in D \mid e \ll d\}$ is directed, and $\bigvee \twoheaddownarrow d = d$. These notions are mostly used when D is not only a poset but a dcpo, but we allow the extension of the definition to posets.

For E a complex vector space, we define $\overline{E}$ to have the same underlying set and abelian group structure as E, but with scalar multiplication defined to be conjugated, *i.e.* if $z \in \mathbb{C}$ and $x \in \overline{E}$, we define $z \cdot_{\overline{E}} x = \overline{z} \cdot_E x$. This allows us to express "antilinear" maps as $\mathbb{C}$-linear maps $E \to \overline{E}$.

A *pre-Hilbert space* $(\mathcal{H}, \langle -, - \rangle)$ is a complex vector space $\mathcal{H}$ equipped with an inner product $\langle -, - \rangle : \overline{\mathcal{H}} \times \mathcal{H} \to \mathbb{C}$, which is to say

$$\begin{aligned} \langle \psi, \lambda_1 \phi_1 + \lambda_2 \phi_2 \rangle &= \lambda_1 \langle \psi, \phi_1 \rangle + \lambda_2 \langle \psi, \phi_2 \rangle \\ \overline{\langle \phi, \psi \rangle} &= \langle \psi, \phi \rangle \\ \langle \psi, \psi \rangle &\geq 0 \\ \langle \psi, \psi \rangle = 0 &\Rightarrow \psi = 0. \end{aligned}$$

Defining $\|\psi\| = \langle \psi, \psi \rangle^{\frac{1}{2}}$, $\|\text{-}\|$ is a norm. We say $(\mathcal{H}, \langle -, - \rangle)$ is a Hilbert space if it is a Banach space with respect to this norm, *i.e.* if it is a complete metric space under the metric $d(\psi, \phi) = \|\psi - \phi\|$.

We write $\mathrm{Ball}(E)$ for the closed unit ball of E, *i.e.* the set $\{x \in E \mid \|x\| \leq 1\}$. A linear map between normed spaces $f : E \to F$ is said to be *bounded* if $\{\|f(x)\| \mid x \in \mathrm{Ball}(E)\}$ is bounded in $\mathbb{R}_{\geq 0}$, *i.e.* if the real-valued function $x \mapsto \|f(x)\|$ is bounded in the usual sense on $\mathrm{Ball}(E)$. A convenient fact about linear maps between normed spaces is that they are bounded iff they are continuous [16, III.2.1], and the set of bounded linear maps $L(E, F)$ admits a norm, the *operator norm*, defined for $f : E \to F$

$$\|f\| = \sup\{\|f(x)\| \mid x \in \mathrm{Ball}(E)\}$$

If $\mathcal{H}$ is a Hilbert space, we write $B(\mathcal{H})$ for $L(\mathcal{H}, \mathcal{H})$, considered as a Banach space under the operator norm. The identity map is bounded, and bounded maps are closed under composition, making $B(\mathcal{H})$ a unital algebra, and each bounded map $f \in B(\mathcal{H})$ has an adjoint $f^* \in B(\mathcal{H})$, which is the unique map such that $\langle f^*(\psi), \phi\rangle = \langle \psi, f(\phi)\rangle$ for all $\psi, \phi \in \mathcal{H}$. It is easy to derive from this that $(g \circ f)^* = f^* \circ g^*$.

In the case that $\mathcal{H}$ is finite dimensional of dimension d, it is isomorphic to $\mathbb{C}^d$ with its usual inner product[6]. Then $B(\mathcal{H})$, as an algebra, is isomorphic to M_d, the algebra of $d \times d$ matrices of complex numbers. However, $B(\mathcal{H})$ has an extra piece of structure, the norm. This makes it a C*-algebra, which we define now.

A unital[7] C*-algebra is a $\mathbb{C}$-algebra A equipped with an antilinear operation $\text{-}^* : A \to \overline{A}$, and a norm $\|\text{-}\| : A \to \mathbb{R}_{\geq 0}$ such that A is a Banach *-algebra satisfying the C*-identity $\|a^*a\| = \|a\|^2$. In terms of axioms, this means that in addition to the $\mathbb{C}$-vector space axioms, we have

$$\begin{aligned} (\lambda a + \mu b)c &= \lambda ac + \mu bc & (ab)c &= a(bc) \\ 1a &= a & (ab)^* &= b^*a^* \\ (\lambda a + \mu b)^* &= \overline{\lambda}a^* + \overline{\mu}b^* & 1^* &= 1 \\ a^{**} &= a & \|a\| = 0 &\Leftrightarrow a = 0 \\ \|a + b\| &\leq \|a\| + \|b\| & \|\lambda a\| &= |\lambda|\|a\| \\ \|ab\| &\leq \|a\|\|b\| & \|a^*a\| &= \|a\|^2 \end{aligned}$$

and the condition that A must be complete in the metric $d(a, b) = \|a - b\|$.

These axioms imply certain others the reader might expect, such as distributivity of multiplication over linear combinations on the right side, $\|a^*\| = \|a\|$, and $\|1\| = 1$ in the case that A has a non-zero element. A linear map between C*-algebras $f : A \to B$ that preserves multiplication and -^* is called a *-homomorphism. A *-homomorphism is called *unital* if it preserves the unit element. Since *-homomorphisms only use the the equational part of the axioms of C*-algebras, the inverse of a bijective *-homomorphism is also a *-homomorphism, so we use *-isomorphism to refer to them. Additionally, *-homomorphisms are continuous with operator norm ≤ 1 [18, 1.3.7].

If A is a C*-algebra, and $B \subseteq A$ is a linear subspace that is also closed under -^* and multiplication, then B is called a *-subalgebra, and if it is also topologically closed with respect to the norm, it is a C*-algebra and we call it a C*-subalgebra of A.

For any Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ is a C*-algebra, and in fact the purpose of the C*-algebra axioms is to characterize the C*-subalgebras of $B(\mathcal{H})$. That is to say, every norm-closed *-subalgebra of $B(\mathcal{H})$ is a C*-algebra, and for any C*-algebra A there exists a Hilbert space $B(\mathcal{H})$, and a *-homomorphism $f : A \to B(\mathcal{H})$ that is an isomorphism onto its image [18, 2.6.1].

[6]Such isomorphisms correspond to orthonormal bases of $\mathcal{H}$.

[7]We will not be considering non-unital C*-algebras here because they are never directed complete. This follows from Proposition III.3 and the fact that AW*-algebras are unital [17, §3 Proposition 2].

Another important source of C*-algebras is that if X is a compact Hausdorff space, then the algebra of continuous $\mathbb{C}$-valued functions $C(X)$ is a C*-algebra, where the operations are defined pointwise from those on $\mathbb{C}$, and the norm of $a \in C(X)$ is defined to be $\|a\| = \sup\{|a(x)| \mid x \in X\}$. This C*-algebra is commutative, and for every commutative unital C*-algebra A, there exists a compact Hausdorff space X, unique up to homeomorphism, and a C*-isomorphism $A \cong C(X)$ [18, 1.4.1]. This is called *Gelfand duality*. It allows us to transfer algebraic facts about continuous functions to all commutative C*-algebras, or even to commuting elements of noncommutative C*-algebras.

As well as the operator topology, there are several other topologies that one can define on $B(\mathcal{H})$, each of which has a use under certain circumstances. For example, the weak topology is the unique topology such that a net of operators $(a_i)_{i \in I}$ converges to an operator a iff for all $\phi, \psi \in \mathcal{H}$, $\langle \phi, a_i(\psi)\rangle \to \langle \phi, a(\psi)\rangle$ in the usual topology of $\mathbb{C}$. A *von Neumann algebra* is a *-subalgebra of $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, that is closed in the weak operator topology. This is equivalent to being closed in all the other commonly used topologies on $B(\mathcal{H})$ that are not the norm topology [19, I.3.4 Theorem 2]. A C*-algebra that is C*-isomorphic to a von Neumann algebra is called a W*-algebra, and the most commonly used characterization of these is as C*-algebras A such that there exists a Banach space A_*, such that A is *isometric* to $(A_*)^*$, *i.e.* the dual norm of A_* agrees with the norm of A under the isomorphism. Bounded directed sets of self-adjoint elements in W*-algebras have suprema [20, 1.7.4].

We describe here how certain notions from finite-dimensional matrix theory are specializations of concepts in C*-algebra theory. An element a of a C*-algebra A is *invertible* if there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$. The element a^{-1} is unique, and is called the *inverse* of A. In the case of $B(\mathcal{H})$ for $\mathcal{H}$ finite-dimensional, the invertible elements are the nonsingular matrices. An element of a C*-algebra $u \in A$ is called *unitary* if u^* is the inverse of u.

The *spectrum* of an element of a C*-algebra $a \in A$, which we write $\mathrm{sp}(a)$, is defined by

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible}\}.$$

The spectrum $\mathrm{sp}(a)$ is a compact subset of $\mathbb{C}$ [21, Proposition I.2.3]. Recall that λ is an eigenvalue of a matrix a iff $a - \lambda 1$ is not invertible[8]. The elements of $\mathrm{sp}(a)$ are called *spectral values* of a, and eigenvalues have their usual definition. Eigenvalues are always spectral values, but it is not necessarily the case that all spectral values are eigenvalues. As $\mathrm{sp}(a^*) = \{\overline{\lambda} \in \mathbb{C} \mid \lambda \in \mathrm{sp}(a)\}$, self-adjoint elements,

[8]This characterization is used to show that the eigenvalues are the roots of the characteristic polynomial.

i.e. those such that $a^* = a$, have $\mathrm{sp}(a) \subseteq \mathbb{R}$. The opposite implication does not hold, even for 2×2 matrices (*e.g.* $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$).

Although we do not consider non-unital C*-algebras, we do consider non-unital *-homomorphisms, and it is convenient to have a way of turning them into unital ones. If A is a C*-algebra, we define its *unitization* $\tilde{A}$ to have $A \times \mathbb{C}$ as its underlying vector space, the multiplication and -* defined pointwise, $(1,1)$ as the unit, and the norm defined as $\|(a,\alpha)\| = \max\{\|a\|, |\alpha|\}$. We embed A in $\tilde{A}$ by the map $a \mapsto (a,0)$, which is a non-unital *-homomorphism [18, 1.3.8]. The *quasi-spectrum* $\mathrm{sp}'(a)$ of an element $a \in A$ is $\mathrm{sp}((a,0))$, as calculated in $\tilde{A}$.

Lemma II.1. *Let $a \in A$, for A a unital C*-algebra. Then* $\mathrm{sp}'(a) = \mathrm{sp}(a) \cup \{0\}$.

Proofs that are either standard, known, or purely technical, such as the proof of the previous lemma, are contained in the appendix.

A nontrivial consequence of the axiomatics of C*-algebras is that C*-algebras admit a translation-invariant order. An element a of a C*-algebra A is called *positive* if it is of the form b^*b for $b \in A$. The following is the C*-algebraic version of a well-known characterization of positive semi-definite matrices.

Lemma II.2. *The following are equivalent for an element $a \in A$ of a unital C*-algebra.*

(i) a is self-adjoint and $\mathrm{sp}(a) \subseteq \mathbb{R}_{\geq 0}$.
(ii) a is self-adjoint and $\mathrm{sp}'(a) \subseteq \mathbb{R}_{\geq 0}$.
(iii) There exists $b \in A$ such that $a = b^*b$.
(iv) There exists a self-adjoint $b \in A$ such that $a = b^2$.

We write A_+ for the set of positive elements. The positive elements form a cone (*i.e.* are closed under addition and multiplication by nonnegative reals, and $A_+ \cap -A_+ = \{0\}$), which implies that the order defined by $a \leq b \Leftrightarrow b - a \in A_+$ is a partial order. If $a \in A$ is positive, as it is self-adjoint, the C*-subalgebra that it generates is commutative, so is canonically isomorphic to $C(X)$ for some compact Hausdorff space X, in which a takes values in $[0,\infty)$. Therefore we can take its positive square root $a^{\frac{1}{2}}$. We say a linear map $f : A \to B$ between C*-algebras is *positive* if it maps positive elements of A to positive elements of B. For linear maps, positivity is equivalent to monotonicity. It is easy to show that any *-homomorphism is positive.

Lemma II.3. *Let $\mathcal{H}$ be a Hilbert space and $a \in B(\mathcal{H})$. The following are equivalent:*

(i) a is positive.
(ii) For all $\psi \in \mathcal{H}$, $\langle \psi, a(\psi) \rangle \geq 0$.

For the proof, see [18, 1.6.7].

Therefore positive operators on finite-dimensional Hilbert spaces are positive semidefinite matrices by another name. The characterization above implies that $a \leq b$ iff for all $\psi \in \mathcal{H}$, $\langle \psi, a(\psi) \rangle \leq \langle \psi, b(\psi) \rangle$, and this is how the Löwner order was originally defined [22], rather than by using a cone.

A function between posets $f : P \to Q$ is called an *order-embedding* if it is monotone and order-reflecting, *i.e.* for all $x, y \in P$, $x \leq y \Leftrightarrow f(x) \leq f(y)$. The antisymmetry axiom implies that order-embeddings are injective, but it is easy to find injective monotone maps that are not order embeddings (there is one involving 2-element posets). This problem does not occur with *-homomorphisms between C*-algebras.

Lemma II.4. *Let A, B be C* algebras and $f : A \to B$ an injective *-homomorphism (not necessarily preserving the unit). Then f is an order-embedding.*

Proof. As f is a *-homomorphism, it is positive and therefore monotone. We show that f reflects positive elements, and deduce that it is an order-embedding from this. Let $a \in A$ and suppose that $f(a)$ is positive. Then $f(a^*) = f(a)^* = f(a)$, so by injectivity, a is self-adjoint. By [18, 1.3.10 (i)], $\mathrm{sp}'(f(a))$ in A is the same as $\mathrm{sp}'(f(a))$ in $f(A)$. As f is injective, it is an isomorphism onto its image [18, 1.8.3], and so $\mathrm{sp}'(f(a)) = \mathrm{sp}'(a)$. Therefore a is positive by Lemma II.2 (i).

We can therefore show that f is order reflecting as follows. If $f(a) \leq f(b)$, then $f(b-a) = f(b) - f(a)$ is positive, so $b - a$ is positive, *i.e.* $a \leq b$. □

We will call a C*-algebra A *directed-complete*[9] if the set of self-adjoint elements $\mathrm{SA}(A)$ is bounded directed-complete, *i.e.* for each directed set $(a_i)_{i \in I}$ in $\mathrm{SA}(A)$ that has an upper bound (there exists a $b \in \mathrm{SA}(A)$ such that for all $i \in I$, $b \geq a_i$), there is a *least* upper bound. Every W*-algebra is directed-complete, as we saw earlier, but there are other examples.

In many arguments, we need to use certain special elements of C*-algebras, called projections. A *projection*[10] in a C*-algebra A is a self-adjoint element p such that $p^2 = p$. We write $\mathrm{Proj}(A)$ for the set of projections. For each projection $p \in B(\mathcal{H})$, the range of p is a closed subspace of $\mathcal{K} \subseteq \mathcal{H}$. The mapping that takes a projection in $B(\mathcal{H})$ to its range is a poset isomorphism between $\mathrm{Proj}(B(\mathcal{H}))$ and the closed subspaces of $\mathcal{H}$, ordered by inclusion [24, §26 Theorem 4, §29 Theorem 2], and this is the reason for the name *projection*. The projections in a C*-algebra need not form a lattice, under the order coming from A [25, Lemma 2.1]. However, the projections do form a lattice in the finite-dimensional case, and we shall see that if a C*-algebra is directed-complete then its projections do form a lattice, so we will be concentrating on this case. We take this opportunity to summarize certain facts about projections. For any $p \in \mathrm{Proj}(A)$, we write $p^\perp = 1 - p$, because it projects onto the orthogonal complement in the Hilbert space case [24, §27 Theorem 3].

Lemma II.5. *Let A be a C*-algebra.*

(i) If $p, q \in \mathrm{Proj}(A)$, then $p \leq q$ iff $q - p$ is a projection.
(ii) If $p, q \in \mathrm{Proj}(A)$, then $p \leq q$ iff $pq = p$ iff $qp = p$.
(iii) If $p, q \in \mathrm{Proj}(A)$ are commuting projections, $pq = p \wedge q$.
(iv) If $p, q \in \mathrm{Proj}(A)$ and $p \leq q$, then $q - p = q \wedge p^\perp$.

[9] Also known as *monotone-complete*.

[10] Also called a *projector*, such as in [23, §2.1.6].

(v) The mapping $a \mapsto 1 - a$ is an isomorphism $[0,1]_A \to [0,1]_A^{\mathrm{op}}$.
(vi) For all $q \in \mathrm{Proj}(A)$, the mapping $p \mapsto q - p$ is an isomorphism $\downarrow q \to (\downarrow q)^{\mathrm{op}}$.

Given a compact Hausdorff space X, the projections in $C(X)$ are continuous functions taking values in $\{0,1\}$, and therefore are indicator functions of clopen subsets of X, so form a Boolean algebra. By Gelfand duality, this carries over to all commutative unital C*-algebras.

Given an operator $a \in B(\mathcal{H})$, we define its *null space* or *kernel* to be $\ker(a) = a^{-1}(0)$, which by linearity and continuity of a is a closed subspace of $\mathcal{H}$. We define its *support* to be the orthogonal complement of this, and its *support projection* $\mathrm{supp}(a)$ to be the projection onto the support. An operator is injective iff its kernel is $\{0\}$, and therefore iff $\mathrm{supp}(a) = 1$. In the case that $\mathcal{H}$ is finite-dimensional, an operator $\mathcal{H} \to \mathcal{H}$ is injective iff it is invertible, so $\mathrm{supp}(a) = 1$ characterizes invertible operators. This does not hold if $\mathcal{H}$ is infinite-dimensional.

The following lemma is important in the next section.

Lemma II.6. *Let $a, b \in B(\mathcal{H})$ be positive, and $p = \mathrm{supp}(a)$, with $\mathcal{K}$ the corresponding subspace. If $b \leq a$, then $\ker(a) \subseteq \ker(b)$ and so $\mathrm{supp}(b) \leq \mathrm{supp}(a)$, and $b = bp = pb = pbp$, and $b \in B(\mathcal{K})$.*

Proposition II.7. *Let A be a C*-algebra*

(a) The following are equivalent:
  - *(i) A is bounded directed complete.*
  - *(ii) A_+ is bounded directed complete.*
  - *(iii) $[0,1]_A$ is a dcpo.*

(b) The following are equivalent when A is a bounded directed complete C-algebra:*
  - *(i) A_+ is continuous.*
  - *(ii) $[0,1]_A$ is continuous.*

In view of the above, we will simply say a C*-algebra A is *directed complete* if we mean that A or A_+ is bounded directed complete or $[0,1]_A$ is directed complete, and we will say that A is continuous if we mean that A_+ or $[0,1]_A$ is continuous. For technical reasons, we are unable in the infinite-dimensional case to prove that if A is continuous as a poset under its natural order, then A_+ and $[0,1]_A$ are continuous, and we do not know of any counterexample either. However, the posets that we really use for (mixed state) quantum computing come from A_+, not A itself, so this is not really a difficulty.

We will also need the notion of a product of C*-algebras. If $(A_i)_{i \in I}$ is an I-indexed family of C*-algebras, we define the product $\prod_{i \in I} A_i$ to have underlying set

$$\prod_{i \in I} A_i = \{(a_i)_{i \in I} \mid \forall i \in I. a_i \in A_i \text{ and } \exists \alpha \in \mathbb{R}_{\geq 0}. \forall i \in I. \|a_i\| < \alpha\},$$

i.e. it is the elements of the set-theoretic product for which the sequence of norms $\|a_i\|$ forms a sequence bounded uniformly in i. The unit is the constant 1 sequence, the vector space operations, multiplication and -* operation are defined pointwise, and the norm is defined by

$$\|(a_i)\| = \sup_{i \in I} \|a_i\|.$$

This is sometimes called the *direct sum* of C*-algebras, because if the C*-algebras are all C*-subalgebras of $B(\mathcal{H}_i)$ one gets a C*-subalgebra of $B(\oplus_{i \in I} \mathcal{H}_i)$, but we find this name misleading because the reader might blithely expect it to be a biproduct of C*-algebras, which it is not, even in the case that I is finite. Dixmier [18, 1.3.3] calls it the product, and we do too, because it is the categorical product in $\mathbf{C^*Alg}$, the category of unital C*-algebras and unital *-homomorphisms.

Proposition II.8. *The C*-algebra $\prod_{i \in I} A_i$, defined above, equipped with the projections $(\pi_i)_{i \in I}$ defined such that $\pi_j((a_i)) = a_j$, is the categorical product of $(A_i)_{i \in I}$.*

The forgetful functor $U : \mathbf{C^*Alg} \to \mathbf{Set}$ that takes a C*-algebra to its underlying set does *not* preserve products. However, the forgetful functor $\mathrm{Ball} : \mathbf{C^*Alg} \to \mathbf{Set}$, taking a C*-algebra to its closed unit ball, not only preserves products, as seen above, but in fact has a left adjoint making $\mathbf{C^*Alg}$ monadic over $\mathbf{Set}$ by this functor [26] [27, Lemma 3.1].

Selinger proved that products of finite-dimensional matrix algebras are continuous [2, Example 2.7].

Theorem II.9. (Selinger). *If $\mathcal{H}$ is a finite-dimensional Hilbert space, $B(\mathcal{H})$ is a continuous directed-complete C*-algebra.*

The following consequence is also given in [2, Example 2.7].

Theorem II.10. (Selinger). *Let X be a set, and $(\mathcal{H}_x)_{x \in X}$ a family of finite-dimensional Hilbert spaces. Then $\prod_{x \in X} B(\mathcal{H}_x)$ is a continuous directed-complete C*-algebra. The way-below relation is characterized by $(a_x)_{x \in X} \ll (b_x)_{x \in X}$ iff there exists a finite subset $S \subseteq X$ such that $a_x = 0$ for all $x \in X \backslash S$, and for all $x \in S$ $a_x \ll b_x$.*

III. Characterization of Continuous Directed-Complete C*-algebras

In this section, we show that if the unit interval in a directed-complete C*-algebra A is continuous, then A is a product of finite-dimensional matrix algebras. This includes the case that A is a W*-algebra, but there are directed-complete C*-algebras that are not W*-algebras[11]. Nik Weaver has already shown that the projection lattice of a W*-algebra A with trivial centre is continuous iff A is finite-dimensional [28], [29]. However, an induced subdcpo of a continuous dcpo is not necessarily continuous [11, Exercise I-2.19]. So we cannot use this result directly.

To circumvent this problem, we use the following lemma, which is an alteration of the statement of [11, Theorem I-2.7]

[11] An example of one is the bounded Borel-measurable functions on $[0,1]$ modulo meagre sets (if it were modulo sets of Lebesgue measure 0, this would be a W*-algebra).

with essentially the same proof, but that, as we shall see, is better adapted to directed-complete C*-algebras. For clarity, for each join or meet we take, we write the poset in which it is intended to be interpreted, so $\bigvee^{E}_{i\in I} x_i$ is the least upper bound of $(x_i)_{i\in I}$ in the poset E.

Lemma III.1. *Let D be a continuous dcpo, and $E \subseteq D$ a complete lattice in the induced ordering, such that the inclusion mapping preserves all non-empty meets, and directed joins. Then E is a continuous lattice.*

Proof. We use condition (DD) of [11, Theorem I-2.7], which is to say, let J be a set, $\{K_j\}_{j\in J}$ a J-indexed family of posets, $\{x_{j,k}\}_{j\in J,k\in K_j}$ be a family of elements in E such that for all $j \in J$, $\{x_{j,k}\}_{k\in K_j}$ is directed, then we want to show

$$\bigwedge^{E}_{j\in J}\bigvee^{E}_{k\in K_j} x_{j,k} = \bigvee^{E}_{f\in M}\bigwedge^{E}_{j\in J} x_{j,f(j)}, \tag{1}$$

where M is the set of functions f mapping $j \in J$ to some $f(j) \in K_j$, and we have written E above the lattice operations to emphasize that they should be calculated in E, rather than D. This is a kind of distributivity property that holds iff E is a continuous lattice by [11, Theorem I-2.7].

The proof is by showing the inequality in each direction.

- $\geq$:
  This holds in any complete lattice, so the proof does not depend on D, so we do not need to use the notation above that emphasizes which poset the joins and meets are calculated in, as they will all be calculated in E. We have

$$\forall f \in M, j \in J. \bigwedge_{j'\in J} x_{j',f(j')} \leq x_{j,f(j)} \leq \bigvee_{k\in K_j} x_{j,k} \quad \text{so}$$
$$\forall j \in J. \bigvee_{f\in M}\bigwedge_{j'\in J} x_{j',f(j')} \leq \bigvee_{k\in K_j} x_{j,k} \quad \text{so}$$
$$\bigvee_{f\in M}\bigwedge_{j\in J} x_{j,f(j)} \leq \bigwedge_{j\in J}\bigvee_{k\in K_j} x_{j,k}.$$

- $\leq$:
  We use the continuity of D in the following way. If we want to show that $x \leq y$ in a continuous dcpo, we can show that for all $z \ll x$, we have $z \leq y$. Then $x = \bigvee \mathord{\downarrow\!\!\downarrow} x \leq y$. Therefore what we want to show is that if $y \in D$ and $y \ll \bigwedge^{E}_{j\in J}\bigvee^{E}_{k\in K_j} x_{j,k}$, then $y \leq \bigvee^{E}_{f\in M}\bigwedge^{E}_{j\in J} x_{j,f(j)}$. We start with

$$y \ll \bigwedge^{E}_{j\in J}\bigvee^{E}_{k\in K_j} x_{j,k} \leq \bigvee^{E}_{k\in K_j} x_{j,k} = \bigvee^{D}_{k\in K_j}$$

  for all $j \in J$, by the assumption that the inclusion of E in D preserves directed joins. So by the definition of way below, for all $j \in J$ there exists $g(k) \in K_j$ such that $y \leq x_{j,g(j)}$, which defines a function $g \in M$. As this holds for all $j \in J$,

$$y \leq \bigwedge^{E}_{j\in J} x_{j,g(j)} \leq \bigvee^{D}_{f\in M}\bigwedge^{E}_{j\in J} x_{j,f(j)} = \bigvee^{E}_{f\in M}\bigwedge^{E}_{j\in J} x_{j,f(j)},$$

  where the join over all f is directed because given two elements $f, g \in M$, we can find (using the axiom of choice if necessary) $h \in M$ such that $h(j) \geq f(j), g(j)$ for all $j \in J$ by directedness of K_j. □

In order to continue the proof, will need to use the fact that directed-complete C*-algebras are AW*-algebras[12]. This fact is known to experts, but does not seem to have made its way into textbooks, so we give a proof here. First we must define AW*-algebras. To do this, we need some definitions. If A is a *-algebra, and $S \subseteq A$ the *right annihilator* of S, $R(S)$ is defined to be

$$R(S) = \{a \in A \mid \forall s \in S. sa = 0\}$$

In order to imagine what $R(S)$ is, it may help to consider the case of $C(X)$. If $a \in C(X)$ is a complex-valued function, $R(\{a\})$ is the set of functions that vanish wherever a is nonzero.

We can also define the *commutant* of S, written S':

$$S' = \{a \in A \mid \forall b \in S. ab = ba\},$$

i.e. S' is the set of elements that commute with everything in S.

Definition III.2. *Let A be a C*-algebra. The following four conditions are equivalent and define what it is for A to be an AW*-algebra.*

(i) *A is a* Baer **-ring,* i.e. *for all $S \subseteq A$, there exists a projection $p \in A$ such that $R(S) = pA$.*
(ii) *The projections of A form a complete lattice and A is a* Rickart **-ring,* i.e. *for all $a \in A$, there exists a projection $p \in A$ such that $R(\{a\}) = pA$.*
(iii) *Every set of orthogonal projections in A has a supremum and A is a Rickart *-ring.*
(iv) *Every set of orthogonal projections in A has a supremum and every maximal commutative *-subalgebra of A is generated by its projections.*

Proposition III.3. *Every directed-complete C*-algebra is an AW*-algebra.*

Proof. Let A be a directed-complete C*-algebra. To show that A is an AW*-algebra, it suffices to show that every maximal commutative *-subalgebra B is directed-complete [30, Proposition 1.4]. Let $(a_i)_{i\in I}$ be a bounded directed set of self-adjoint elements of B, and $b = \bigvee_{i\in I} a_i$, as calculated in A. By [30, Lemma 1.6], $b \in B$. As B is order-embedded in A (Lemma II.4), b is also the supremum of $(a_i)_{i\in I}$ in B. Therefore B is directed-complete. □

[12] Whether the converse is true is an open problem.

We need the following purely technical lemma about positive operators and projections on a Hilbert space.

Lemma III.4. *Let A be a C*-algebra, $a \in [0,1]_A$, and $p \in \mathrm{Proj}(A)$. Then the following are equivalent:*

(i) $a \le p$
(ii) $a = ap$
(iii) $a = pa$
(iv) $a = pap$

Care is required in interpreting the following proposition, because $\mathrm{Proj}(A)$ is not a sublattice of the set of self-adjoint operators on A, even in the case of $A = M_2$ [31, Lemma 7]. As before, for extra clarity we write down the poset in which each join or meet is intended to be interpreted, so $\bigvee^{E}_{i\in I} x_i$ is the least upper bound of $(x_i)_{i\in I}$ in the poset E.

Proposition III.5. *Let A be an AW*-algebra. Then the inclusion map $\mathrm{Proj}(A) \to [0,1]_A$ preserves all lattice operations.*

Proof. Let $(p_i)_{i\in I}$ be a family of projections, and let $p = \bigwedge^{\mathrm{Proj}(A)}_{i\in I} p_i$. We want to show that $\bigwedge^{[0,1]_A}_{i\in I} p_i = p$. As p is a lower bound for $(p_i)_{i\in I}$ in $[0,1]_A$, it suffices to show that p is greater than any lower bound $a \in [0,1]_A$ for $(p_i)_{i\in I}$.

So let $a \in [0,1]_A$ such that for all $i \in I$, $a \le p_i$. By Lemma III.4, $a = ap_i$ for all $i \in I$. If we define $q_i = 1 - p_i$ for all $i \in I$ and $q = 1 - p$, we have $q = \bigvee^{\mathrm{Proj}(A)}_{i\in I} q_i$ by the fact that the map $p \mapsto 1 - p$ is an isomorphism of $\mathrm{Proj}(A)$ with its opposite (Lemma II.5 (vi)).

As $ap_i = a$, we have $aq_i = 0$ for all $i \in I$. By [17, §3 Proposition 6], this implies $aq = 0$, and therefore $ap = a$, which by Lemma III.4 implies $a \le p$. Therefore $p = \bigwedge^{[0,1]_A}_{i\in I} p_i$.

It then follows from the fact that $a \mapsto 1-a$ is a isomorphism of $[0,1]_A$ with its opposite (Lemma II.5 (v)), and restricts to a such a map on $\mathrm{Proj}(A)$ as well, that the inclusion morphism $\mathrm{Proj}(A) \to [0,1]_A$ preserves joins as well, and so $\mathrm{Proj}(A)$ is a complete sublattice of $[0,1]_A$. □

We can now make full and effective use of Lemma III.1.

Proposition III.6. *If A is a continuous directed complete C*-algebra, A is an AW*-algebra with $\mathrm{Proj}(A)$ a continuous lattice.*

Proof. By Proposition II.7, $[0,1]_A$ is a continuous dcpo, and by Proposition III.3, A is an AW*-algebra, so by Proposition III.5, the inclusion $\mathrm{Proj}(A) \hookrightarrow [0,1]_A$ satisfies the conditions of Lemma III.1, and therefore $\mathrm{Proj}(A)$ is a continuous lattice. □

We can now prove that certain projection lattices are not continuous. An *atom* in a poset P with a bottom element 0 is an element $a \in P$ such that there is no element strictly between a and 0. We say a poset is *atomic* if for each $b \in P$ there is an atom $a \le b$. A poset is *atomless* if it has no atoms.

The following observation is due to Nik Weaver in the case of a W*-algebra. [28], [29]

Lemma III.7. (Weaver). *Let A be an AW*-algebra. If $\mathrm{Proj}(A)$ is continuous, then it is atomic.*

In the commutative case, we have the following.

Lemma III.8. *Let A be a commutative AW*-algebra. If $\mathrm{Proj}(A)$ is continuous then $\mathrm{Proj}(A) \cong \mathcal{P}(X)$ for some set X.*

Proof. The projection lattice of a commutative C*-algebra AW*-algebra is a complete Boolean algebra, because the commutativity implies that it is a Boolean algebra, and Definition III.2 (ii) implies that it is a complete lattice. The fact that $\mathrm{Proj}(A) \cong \mathcal{P}(X)$ then follows from [11, Theorem I-4.20]. However, we can prove it directly from Lemma III.7. That lemma implies that $\mathrm{Proj}(A)$ is atomic.

In an atomic complete Boolean algebra, we have that every element $p \in \mathrm{Proj}(A)$ is the supremum of the set of atoms below it. For if there were some element p where this were not the case, $p - \bigvee\{a \in \mathrm{Proj}(A) \mid a \text{ atom and } a \le p\}$ is a non-zero element with no atom below it.

Let X be the set of atoms of $\mathrm{Proj}(A)$. Define $f : \mathcal{P}(X) \to \mathrm{Proj}(A)$ by $f(S) = \bigvee S$. It is clear that f preserves joins. As $\bigvee_{x\in X\setminus S} x \vee \bigvee_{x\in S} x = X$ and $\bigvee_{x\in X\setminus S} x \wedge \bigvee_{x\in S} x = 0$, the uniqueness of complements in Boolean algebras implies f preserves complements, and is therefore a complete Boolean homomorphism. If $f(S) = f(S')$, then suppose for a contradiction that there is an element $x \in S$ and $x \notin S'$. We have $x \le f(S) = f(S')$, but $x \wedge f(S') = x \wedge \bigvee_{x'\in S'} x' = \bigvee_{x'\in S'} x \wedge x' = 0$, a contradiction. As this is symmetrical in S and S', we have $S = S'$, so f is injective. As every element of $\mathrm{Proj}(A)$ is the supremum of the atoms below it, f is surjective, and therefore a (complete) Boolean isomorphism. □

We will require the notion of an AW*-subalgebra. For the benefit of the reader, we condense [17, §4 Definitions 3 and 4] and [17, §3 Definition 4]. Given an AW*-algebra and an element $a \in A$, and taking p to be the unique projection such that $pA = R(\{a\})$, we define the *right projection* $RP(a)$ to be $1 - p$ [17, §3 Proposition 3, Definition 4].

Definition III.9. *Let A be an AW*-algebra and $B \subseteq A$ a *-subalgebra. We say that it is an AW*-subalgebra if*

(i) *B is norm-closed,* i.e. *B is a C*-subalgebra.*
(ii) *If $x \in B$ then $RP(x) \in B$ (as calculated in A).*
(iii) *If $(p_i)_{i\in I}$ is a nonempty family of projections in B, $\bigvee_{i\in I} p_i \in B$ (the join being calculated in $\mathrm{Proj}(A)$).*

By [17, §4 Proposition 8 (i)], if $B \subseteq A$ is an AW*-subalgebra of an AW*-algebra A, then B is an AW*-algebra.

Lemma III.10. *Let A be an AW*-algebra and $B \subseteq A$ an AW*-subalgebra. Then $\mathrm{Proj}(B) \subseteq \mathrm{Proj}(A)$ has the induced ordering and the inclusion map $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ preserves arbitrary joins and nonempty meets. It preserves all joins iff the unit element of A is contained in B.*

The following is the combination of the previous lemma with Lemma III.1 that we will use twice.

Corollary III.11. *Let A be an AW*-algebra such that $\mathrm{Proj}(A)$ is continuous, and $B \subseteq A$ an AW*-subalgebra. Then $\mathrm{Proj}(B)$ is continuous.*

Proof. The inclusion $\mathrm{Proj}(B) \subseteq \mathrm{Proj}(A)$ satisfies the hypotheses of Lemma III.1[13] by Lemma III.10. □

For a C*-algebra A, the *centre* $Z(A)$ is defined to be the set of elements that commute with every element of A. For an AW*-algebra A, if $Z(A)$ is as small as possible, consisting only of multiplies of the identity element, we say that A is a factor. Contrariwise, $Z(A) = A$ iff A is commutative. The projections in the centre $\mathrm{Proj}(Z(A))$ are called the *central projections* of A.

Lemma III.12. *Let A be an AW*-algebra and p a central projection,* i.e. $p \in \mathrm{Proj}(Z(A))$.

- *(i) Let $a \in A$. Then $pa = ap = pap$.*
- *(ii) The element $a \in A$ is in $pAp = pA = Ap$ iff $a = pa$ (and therefore iff $a = ap$ or $a = pap$).*
- *(iii) pAp is an AW*-subalgebra of A, with unit element p.*
- *(iv) The map $\pi_p : A \to pAp$ defined by $\pi_p(a) = pap$ (equivalently pa or ap) is a unital *-homomorphism.*
- *(v)* $Z(pAp) = pZ(A)p$, i.e. *the centre of pAp is the image of the centre of A.*

Proposition III.13. *Let A be an AW*-algebra such that $\mathrm{Proj}(A)$ is continuous. Then $\mathrm{Proj}(Z(A)) \cong \mathcal{P}(X)$ for some set X, and $A \cong \prod_{x\in X} xAx$ where each xAx is an AW*-factor such that $\mathrm{Proj}(xAx)$ is continuous.*

Proof. By [17, §4 Proposition 8 (v)] $Z(A)$ is an AW*-subalgebra of A. By the continuity of $\mathrm{Proj}(A)$, $\mathrm{Proj}(Z(A))$ is continuous (Corollary III.11). Therefore $\mathrm{Proj}(Z(A))$ is isomorphic to $\mathcal{P}(X)$, where X is the set of atoms of $\mathrm{Proj}(Z(A))$ (Lemma III.8).

The atoms of $\mathrm{Proj}(Z(A))$ form a disjoint family of central projections whose join is 1, so we can apply [17, §10 Proposition 2] to conclude that the mapping $\phi : A \to \prod_{x\in X} xAx$ defined by $\phi(a) = (xax)_{x\in X}$ is an isomorphism.

By Lemma III.12 (iii), xAx is an AW*-algebra for all $x \in X$. If p is a central projection in xAx, then $p \in Z(A)$ and $px = xp = x$ by Lemma III.12 (v) and (ii). By Lemma II.5 (ii), $p \le x$, so as x is an atom, either $p = x$ or $p = 0$. Since commutative AW*-algebras are the closed $\mathbb{C}$-linear span of their projections [17, Proposition 1 (3)], it follows that $Z(xAx)$ is the linear span of x, and therefore xAx is a factor.

Finally, as xAx is an AW*-subalgebra of A, $\mathrm{Proj}(xAx)$ continuous (Corollary III.11). □

As in Lemma III.7, the following observation is due to Nik Weaver [28], [29] in the W*-algebra case.

[13] In fact [11, Theorem I-2.7] would work unaltered here, but not in Proposition III.6.

Proposition III.14. (Weaver). *An AW*-factor A has $\mathrm{Proj}(A)$ continuous iff there exists a finite-dimensional Hilbert space $\mathcal{H}$ such that $A \cong B(\mathcal{H})$.*

We can now state and prove the precise characterization of directed-complete C*-algebras with continuous positive cone or effect algebra.

Theorem III.15.
- *(i) A directed-complete C*-algebra is continuous iff it is of the form $\prod_{x\in X} B(\mathcal{H}_x)$ where $\mathcal{H}_x$ is finite-dimensional*[14].
- *(ii) The projection lattice $\mathrm{Proj}(A)$ of an AW*-algebra A is continuous iff A is of the form $\prod_{x\in X} B(\mathcal{H}_x)$ where $\mathcal{H}_x$ is finite-dimensional.*

Proof. Let A be a directed complete C*-algebra that is continuous. Then by Proposition III.6, A is an AW*-algebra and $\mathrm{Proj}(A)$ a continuous lattice.

Therefore we are in the situation of (ii). If A is an AW*-algebra with $\mathrm{Proj}(A)$ a continuous lattice, then $A \cong \prod_{x\in X} xAx$ (Proposition III.13) and by Proposition III.14 $xAx \cong B(\mathcal{H}_x)$ for a finite-dimensional Hilbert space $\mathcal{H}_x$ for all $x \in X$. So we have proved the forward implication of both (i) and (ii).

The backward implication of (i) follows from Theorem II.10. The backward implication of (ii) then follows by Proposition III.6. □

We close this section with some remarks about countably based and effective dcpos. For this we need the notion of a *basis* for a continuous dcpo [11, Definition III-4.1]. To avoid confusion with the linear notion of basis for a vector space, we will refer to this as a *base* instead. A *base* of a dcpo D is a set $B \subseteq D$ such that for all $d \in D$, $\twoheaddownarrow d \cap B$ is directed, and $d = \bigvee \twoheaddownarrow d \cap B$. A dcpo D has a base iff it is continuous, and it is immediate from the definition that if D is continuous, D is a base. We can also define a base for a continuous poset, in a similar way, where we do not require all directed joins to exist. The *weight* of a continuous poset D, written $w(D)$, is the minimum cardinality of a base [11, Definition III-4.4]. For any continuous dcpo D, the weight of D is the same as the weight[15] of the Scott and Lawson topologies [11, Theorem III-4.5].

In order to define computable elements and computable functions in domain theory, we need to be able to recursively enumerate a base, much as one recursively enumerates the rationals when defining computable reals. Since recursively enumerable sets are countable, this requires the dcpo to have a countable base.

Lemma III.16. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space of dimension ≥ 2. Then any base of $[0,1]_{B(\mathcal{H})}$ has cardinality $2^{\aleph_0}$, so $w([0,1]_{B(\mathcal{H})}) = 2^{\aleph_0}$.*

Proof. Let $B \subseteq [0,1]_{B(\mathcal{H})}$ be a base. Consider the set $P \subseteq [0,1]_{B(\mathcal{H})}$ of projections onto 1-dimensional subspaces of $\mathcal{H}$.

[14] Where the product is in the category of C*-algebras.

[15] In the topological sense, meaning the minimal cardinality of a family of open sets generating the topology.

For each $p \in P$, as $p = \bigvee \downarrow\!\!\downarrow p \cap B$, there must exist $b_p \in B$ such that $b_p \neq 0$ and $b_p \ll p$, so in particular, $b_p \leq p$. Fix such a b_p for each $p \in P$. We show the mapping $p \mapsto b_p$ is injective as follows. Let $p, p' \in P$ be such that $b_p = b_{p'}$. Let $\mathcal{K}$ and $\mathcal{K}'$ be the 1-dimensional subspaces of $\mathcal{H}$ corresponding to p and p', respectively. By Lemma II.6, as $b_p \leq p$, $b_p \in B(\mathcal{K})$ and likewise $b_{p'} \in B(\mathcal{K}')$. Since $b_p = b_{p'} \neq 0$ and $B(\mathcal{K})$ and $B(\mathcal{K}')$ are 1-dimensional, $\mathcal{K} = \mathcal{K}'$, so $p = p'$.

As $2 \leq \dim(\mathcal{H}) < \infty$, the cardinality of P and $B(\mathcal{H})$ are both $2^{\aleph_0}$, so the injectivity of $p \mapsto b_p$ and the fact that $B \subseteq B(\mathcal{H})$ imply $2^{\aleph_0} \leq |B| \leq 2^{\aleph_0}$. □

Proposition III.17. *Let A be a directed-complete C^*-algebra such that $[0,1]_A$ has a countable base, as a dcpo. Then $A \cong \ell^\infty(X)$, where X is a countable set. In particular, A is commutative.*

Proof. Since $[0.1]_A$ has a countable base B, it is continuous, so $A \cong \prod_{i \in I} B(\mathcal{H}_i)$, with $\mathcal{H}_i$ finite-dimensional, by Theorem III.15 (i). If there were an $i \in I$ such that $\dim(\mathcal{H}_i) \geq 2$, then $\pi_i : [0,1]_A \to [0,1]_{B(\mathcal{H}_i)}$ is a Scott continuous surjective map. So $\pi_i(B)$ would be a countable base for $[0,1]_{B(\mathcal{H}_i)}$ by [11, Proposition III-4.12], which contradicts Lemma III.16. Therefore for all $i \in I$, $\mathcal{H}_i$ is 0 or 1-dimensional.

If $\dim(\mathcal{H}) = 0$, then $B(\mathcal{H}) \cong \{0\}$, the ring with $0 = 1$, and if $\dim(\mathcal{H}) = 1$, then $B(\mathcal{H}) \cong \mathbb{C}$, mapping the identity map to $1 \in \mathbb{C}$. So, defining $X = \{i \in I \mid \dim(\mathcal{H}_i) = 1\}$. Then $A \cong \ell^\infty(X)$, including the case when $X = \emptyset$. So all that remains is to prove that X is countable. We do this by constructing an injection $X \to B$.

Let $\delta_x : X \to \mathbb{C}$ be the function that takes the value 1 at x and 0 everywhere else. We reuse B for the image of the countable base in $\ell^\infty(X)$, which is a countable base for the dcpo $[0,1]^X$. For each $x \in X$, $\delta_x = \bigvee \downarrow\!\!\downarrow \delta_x \cap B$, so there exists $b_x \in B$ such that $b_x \neq 0$ and $b_x \leq \delta_x$. Fix such a b_x for each $x \in X$. We show $x \mapsto b_x$ is injective as follows. If $x, x' \in X$ such that $b_x = b_{x'}$, if $x \neq x'$ then $0 \leq b_x(x') \leq \delta_x(x') = 0$, so $0 = b_x(x') = b_{x'}(x')$. As $b_{x'} \leq \delta_{x'}$, this shows $b_{x'} = 0$, which is a contradiction. Therefore $x = x'$, and so $x \mapsto b_x$ is injective. Since B is countable, X is countable. □

This shows that we cannot use the Scott and Lawson topologies to define computable elements as is usually done in effective versions of domain theory, such as [32, Definition 3.1] [33, §7, Definition 1]. Notions of computability based on the norm topology (such as the definition of a computable metric space from [34, Definition 8.1.2]) do not have this problem, at least in the finite-dimensional case, as $B(\mathcal{H})$ is a separable Banach space if $\mathcal{H}$ is finite-dimensional.

Acknowledgements: Many thanks to Mathys Rennela for discussion about the ideas used here. Robert Furber has been financially supported by the Danish Council for Independent Research, Project 4181-00360.

REFERENCES

[1] P. Selinger, "Towards a Quantum Programming Language," *Mathematical Structures in Computer Science*, vol. 14, no. 4, pp. 527–586, 2004. 1, 2

[2] ——, "Towards a Semantics for Higher-Order Quantum Computation," in *Proceedings of the 2nd International Workshop on Quantum Programming Languages*, ser. TUCS General Publications, vol. 33. Turku Centre for Computer Science, 2004, pp. 127–143. 1, 5, 11

[3] M.-D. Choi, "Completely Positive Linear Maps on Complex Matrices," *Linear Algebra and its Applications*, vol. 10, no. 3, pp. 285 – 290, 1975. 1

[4] A. Jamiołkowski, "Linear Transformations which Preserve Trace and Positive Semidefiniteness of Operators," *Reports on Mathematical Physics*, vol. 3, no. 4, pp. 275 – 278, 1972. 1

[5] E. d'Hondt and P. Panangaden, "Quantum Weakest Preconditions," *Mathematical Structures in Computer Science*, vol. 16, no. 3, pp. 429–451, 2006. 1

[6] R. V. Kadison, "Operator Algebras with a Faithful Weakly-Closed Representation," *Annals of Mathematics*, vol. 64, no. 1, pp. 175–181, 1956. 2

[7] M. Rennela, "Operator Algebras in Quantum Computation," Master's thesis, Université Paris 7 Diderot, 2013, http://arxiv.org/abs/1510.06649. 2

[8] ——, "Enrichment in Categorical Quantum Foundations," Ph.D. dissertation, Radboud University, Nijmegen, 2018. 2

[9] K. Cho, "Semantics for a Quantum Programming Language by Operator Algebras," *New Generation Computing*, vol. 34, no. 1, pp. 25–68, 2016. 2

[10] M. Rennela, "Towards a Quantum Domain Theory: Order-enrichment and Fixpoints in W*-algebras," *Electronic Notes in Theoretical Computer Science*, vol. 308, pp. 289–307, 2014, proceedings of the 30th Conference on the Mathematical Foundations of Programming Semantics (MFPS XXX). 2

[11] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott, *Continuous Lattices and Domains*, ser. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2003, vol. 93. 2, 5, 6, 7, 8, 9, 13

[12] G. Birkhoff and J. von Neumann, "The Logic of Quantum Mechanics," *Annals of Mathematics*, vol. 37, no. 4, pp. 823–843, 1936. 2

[13] M. Takesaki, "Conditional expectations in von Neumann algebras," *Journal of Functional Analysis*, vol. 9, no. 3, pp. 306–321, 1972. 2

[14] P. Chaput, V. Danos, P. Panangaden, and G. D. Plotkin, "Approximating Markov Processes by Averaging," in *ICALP*, 2009. 2

[15] B. Coecke and K. Martin, "A Partial Order on Classical and Quantum States," in *New Structures for Physics*, B. Coecke, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 593–683. 2

[16] J. B. Conway, *A Course In Functional Analysis, Second Edition*, ser. Graduate Texts in Mathematics. Springer Verlag, 1990, vol. 96. 2

[17] S. K. Berberian, *Baer *-Rings*, ser. Grundlehren der mathematischen Wissenschaften. Springer, 1972, vol. 195. 3, 7, 8, 14, 15

[18] J. Dixmier, *C*-algebras*, ser. North-Holland Mathematical Library. North-Holland Publishing Company, 1977, vol. 15. 3, 4, 5, 10, 12

[19] ——, *Von Neumann Algebras*, ser. North-Holland Mathematical Library. North-Holland Publishing Company, 1981, vol. 27. 3

[20] S. Sakai, *C*-algebras and W*-algebras*, ser. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1971, vol. 60. 3, 13

[21] M. Takesaki, *Theory of Operator Algebra*. Springer Verlag, 1979, vol. 1. 3

[22] K. Löwner, "Über monotone Matrixfunktionen," *Mathematische Zeitschrift*, vol. 38, no. 1, pp. 177–216, Dec 1934. 4

[23] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. Cambridge University Press, 2011. 4

[24] P. R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*. Chelsea Publishing Company, 1951. 4, 10

[25] A. J. Lazar, "AF Algebras with a Lattice of Projections," *Mathematica Scandinavica*, vol. 50, pp. 135–144, 1982. 4

[26] D. van Osdol, "C*-algebras and Cohomology," in *Categorical Topology: Proceedings of the International Conference held at the University of Toledo, Ohio, U.S.A. August 1–5, 1983*, ser. Sigma Series in Pure Mathematics, H. L. Bentley, H. Herrlich, M. Rajagopalan, and H. Wolff, Eds. Heldermann Verlag, 1983, vol. 5, pp. 582–587. 5

[27] J. W. Pelletier and J. Rosický, "On the Equational Theory of C*-algebras," *Algebra Universalis*, vol. 30, pp. 275–284, 1993. 5

[28] N. Weaver (https://mathoverflow.net/users/23141/nik-weaver), "Projections in a W*-algebra as a continuous lattice?" MathOverflow. [Online]. Available: https://mathoverflow.net/q/133037 5, 7, 8

[29] user46855 (https://mathoverflow.net/users/46855/user46855), "Projections in a W*-algebra as a continuous lattice?" MathOverflow. [Online]. Available: https://mathoverflow.net/q/157384 5, 7, 8
[30] K. Saitô and J. D. M. Wright, "On Defining AW*-algebras and Rickart C*-algebras," *The Quarterly Journal of Mathematics*, 2015. 6
[31] R. V. Kadison, "Order Properties of Bounded Self-Adjoint Operators," *Proceedings of the American Mathematical Society*, vol. 2, no. 3, pp. 505–510, 1951. 7
[32] M. B. Smyth, "Effectively Given Domains," *Theoretical Computer Science*, vol. 5, no. 3, pp. 257 – 274, 1977. 9
[33] G. Plotkin, "Domains (Pisa Notes)," http://homepages.inf.ed.ac.uk/gdp/publications/Domains_a4.ps, 1983. 9
[34] A. Weihrauch, *Computable Analysis: An Introduction*, ser. Texts in Theoretical Computer Science. Springer, 2000. 9
[35] H. H. Schaefer, *Topological Vector Spaces*, ser. Graduate Texts in Mathematics. Springer Verlag, 1966, vol. 3. 13
[36] I. Kaplansky, "Projections in Banach Algebras," *Annals of Mathematics*, vol. 53, no. 2, pp. 235–249, 1951. [Online]. Available: http://www.jstor.org/stable/1969540 14
[37] ——, "Algebras of Type I," *Annals of Mathematics*, vol. 56, no. 3, pp. 460–472, 1952. 15

APPENDIX

Lemma II.1. *Let $a \in A$, for A a unital C*-algebra. Then* $\mathrm{sp}'(a) = \mathrm{sp}(a) \cup \{0\}$.

Proof. We first show that $0 \in \mathrm{sp}'(a)$. Suppose for a contradiction that $0 \notin \mathrm{sp}'(a)$, so there exists $(b, \beta) \in \tilde{A}$ such that $(a, 0)(b, \beta) = (1, 1)$. Then $0 \cdot \beta = 1$, which is impossible.

We show $\mathrm{sp}(a) \subseteq \mathrm{sp}'(a)$ by showing that $\mathbb{C} \setminus \mathrm{sp}'(a) \subseteq \mathbb{C} \setminus \mathrm{sp}(a)$. Suppose $\lambda \in \mathbb{C} \setminus \mathrm{sp}'(a)$. Then there exists $(b, \mu) \in \tilde{A}$ such that $(b, \mu)(a - \lambda, -\lambda) = (1, 1) = (a - \lambda, -\lambda)(b, \mu)$, so in particular, $b(a - \lambda) = 1 = (a - \lambda)b$. Therefore $\lambda \notin \mathrm{sp}(a)$. This completes the part of the proof that shows $\mathrm{sp}(a) \cup \{0\} \subseteq \mathrm{sp}'(a)$.

To show $\mathrm{sp}'(a) \subseteq \{0\} \cup \mathrm{sp}(a)$, we prove that $\mathbb{C} \setminus (\{0\} \cup \mathrm{sp}(a)) \subseteq \mathbb{C} \setminus \mathrm{sp}'(a)$. If $\lambda \in \mathbb{C} \setminus (\{0\} \cup \mathrm{sp}(a))$, then there exists $b \in A$ such that $b(a - \lambda) = 1 = (a - \lambda)b$. We therefore have $(b, -\frac{1}{\lambda})(a - \lambda, -\lambda) = (1, 1) = (a - \lambda, -\lambda)(b, -\frac{1}{\lambda})$, so $\lambda \notin \mathrm{sp}'(a)$. □

Lemma II.2. *The following are equivalent for an element $a \in A$ of a unital C*-algebra.*

(i) a *is self-adjoint and* $\mathrm{sp}(a) \subseteq \mathbb{R}_{\geq 0}$.
(ii) a *is self-adjoint and* $\mathrm{sp}'(a) \subseteq \mathbb{R}_{\geq 0}$.
(iii) There exists $b \in A$ *such that* $a = b^* b$.
(iv) There exists a self-adjoint $b \in A$ *such that* $a = b^2$.

Proof. The equivalence of (i) and (ii) follows from Lemma II.1, and the equivalence of (ii),(iii) and (iv) follows from [18, 1.6.1], after observing that the assumption there that a is self-adjoint is not needed for (ii) and (iii), because $b^* b$ is self-adjoint for all b and therefore b^2 is self-adjoint if b is. □

Using the characterization of positive elements in $B(\mathcal{H})$ (Lemma II.3), we can prove the following fact about self-adjoint elements. For ease of notation, we write $x \geq S$ for an element and a set S to mean x is greater than every element of S.

Lemma A.1. *Let $a \in A$ be an element of a C*-algebra.*

(i) $\mathrm{sp}(-a) = -\mathrm{sp}(a)$.
(ii) Let $\alpha \in \mathbb{C}$. *Then* $\mathrm{sp}(a + \alpha 1) = \mathrm{sp}(a) + \alpha$, i.e. *shifting the operator by* α *shifts its spectrum by* α.
(iii) If a *is self-adjoint and* $\alpha \in \mathbb{R}$, $\alpha 1 \leq a$ *iff* $\alpha \leq \mathrm{sp}(a)$, *and* $\alpha 1 \geq a$ *iff* $\alpha \geq \mathrm{sp}(a)$.
(iv) If a *is self-adjoint, there exist* $\alpha, \beta \in \mathbb{R}$ *such that* $\beta 1 \leq a \leq \alpha 1$.

Proof. (i) First, observe that if a is invertible, with inverse a^{-1}, then $-a^{-1}$ is an inverse to $-a$. Therefore λ is outside the spectrum of a iff $a - \lambda 1$ is invertible iff $-a - (-\lambda)1$ is invertible iff $-\lambda$ is outside the spectrum of $-a$.

(ii) Let $\lambda \in \mathbb{C}$. We have that $(a + \alpha 1) - \lambda 1 = a - (\lambda - \alpha)1$, so $\lambda \in \mathrm{sp}(a + \alpha 1)$ iff $\lambda - \alpha \in \mathrm{sp}(a)$ iff $\lambda \in \mathrm{sp}(a) + \alpha$.

(iii) By part (ii) above and part (i) of Lemma II.2, $\alpha 1 \leq a$ iff $a - \alpha 1$ is positive iff $\mathrm{sp}(a) - \alpha \geq 0$ iff $\alpha \leq \mathrm{sp}(a)$. So by part (i) above, $\alpha 1 \geq a$ iff $-\alpha 1 \leq -a$ iff $-\alpha \leq \mathrm{sp}(-a)$ iff $-\alpha \leq -\mathrm{sp}(a)$ iff $\alpha \geq \mathrm{sp}(a)$.

(iv) As a is self-adjoint, $\mathrm{sp}(a) \subseteq \mathbb{R}$, and as it is compact, it has an upper and a lower bound. So we pick $\alpha \geq \mathrm{sp}(a)$ and $\beta \leq \mathrm{sp}(a)$. By the previous part, $\beta 1 \leq \alpha \leq \alpha 1$. □

Lemma II.5. *Let A be a C*-algebra.*

(i) If $p, q \in \mathrm{Proj}(A)$, *then* $p \leq q$ *iff* $q - p$ *is a projection.*
(ii) If $p, q \in \mathrm{Proj}(A)$, *then* $p \leq q$ *iff* $pq = p$ *iff* $qp = p$.
(iii) If $p, q \in \mathrm{Proj}(A)$ *are commuting projections,* $pq = p \wedge q$.
(iv) If $p, q \in \mathrm{Proj}(A)$ *and* $p \leq q$, *then* $q - p = q \wedge p^{\perp}$.
(v) The mapping $a \mapsto 1 - a$ *is an isomorphism* $[0, 1]_A \to [0, 1]_A^{\mathrm{op}}$.
(vi) For all $q \in \mathrm{Proj}(A)$, *the mapping* $p \mapsto q - p$ *is an isomorphism* $\downarrow q \to (\downarrow q)^{\mathrm{op}}$.

Proof. Throughout, we use the fact that we can represent a C*-algebra in $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ to transfer facts about projections on Hilbert space.

(i) See [24, §29 Theorem 3].
(ii) See [24, §29 Theorem 2].
(iii) See [24, §30 Theorem 2].
(iv) As $p \leq q$, p commutes with q by (ii), so $1 - p$ commutes with q by linearity. Therefore $q \wedge p^{\perp} = q(1 - p) = q - qp = q - p$, by (iii) and (ii) in turn.
(v) It is a self-inverse bijection because $1 - (1 - a) = a$. It is an order-reversing isomorphism because

$$1 - b \leq 1 - a \Leftrightarrow 1 - a - 1 + b \in A_+ \Leftrightarrow b - a \in A_+ \Leftrightarrow a \leq b.$$

(vi) First we need to show that if $p \leq q$, then $q - p \in \downarrow q$. By (i) it is a projection, and as $q - (q - p) = p$, $q - p \leq q$. It is a self-inverse bijection because $q - (q - p) = p$. It is an order-reversing isomorphism because for all p, p' projections that are $\leq q$,

$$q - p \leq q - p' \Leftrightarrow q - p' - q + p \in A_+ \Leftrightarrow p - p' \in A_+ \Leftrightarrow p' \leq p.$$

□

Lemma A.2. *Let $\mathcal{H}$ be a Hilbert space.*

(i) If $a \in B(\mathcal{H})_+$, then $\psi \in \ker(a)$ iff $\langle \psi, a(\psi) \rangle = 0$.
(ii) Let $a, b \in B(\mathcal{H})_+$. Then

$$\text{supp}(a+b) = \text{supp}(a) \vee \text{supp}(b).$$

Proof.

(i) If $\psi \in \ker(a)$, then $\langle \psi, a(\psi) \rangle = \langle \psi, 0 \rangle = 0$. For the other direction, $0 = \langle \psi, a(\psi) \rangle = \langle a^{\frac{1}{2}}(\psi), a^{\frac{1}{2}}(\psi) \rangle = \|a^{\frac{1}{2}}(\psi)\|^2$, so $a^{\frac{1}{2}}(\psi) = 0$. Therefore $a(\psi) = a^{\frac{1}{2}}(a^{\frac{1}{2}}(\psi)) = a^{\frac{1}{2}}(0) = 0$, so $\psi \in \ker(a)$.

(ii) First we show that $\ker(a+b) = \ker(a) \wedge \ker(b)$, and then the statement follows from the fact that $-^{\perp}$ is an order-reversing bijection. If $\psi \in \ker(a) \wedge \ker(b)$, then $(a+b)(\psi) = a(\psi) + b(\psi) = 0+0 = 0$, so $\psi \in \ker(a+b)$. For the other direction, if $\psi \in \ker(a+b)$, then by part (i), $\langle \psi, (a+b)(\psi) \rangle = 0$, so $\langle \psi, a(\psi) \rangle + \langle \psi, b(\psi) \rangle = 0$. As a and b are positive, this implies $\langle \psi, a(\psi) \rangle = 0 = \langle \psi, b(\psi) \rangle$, which implies $\psi \in \ker(a) \wedge \ker(b)$ by part (i). □

Lemma II.6. *Let $a, b \in B(\mathcal{H})$ be positive, and $p = \text{supp}(a)$, with $\mathcal{K}$ the corresponding subspace. If $b \leq a$, then $\ker(a) \subseteq \ker(b)$ and so $\text{supp}(b) \leq \text{supp}(a)$, and $b = bp = pb = pbp$, and $b \in B(\mathcal{K})$.*

Proof. If $\psi \in \ker(a)$, then

$$0 \leq \langle \psi, b(\psi) \rangle \leq \langle \psi, a(\psi) \rangle = \langle \psi, 0 \rangle = 0,$$

so by Lemma A.2 (i), $\psi \in \ker(b)$. Therefore $\ker(a) \subseteq \ker(b)$, and it follows by the fact that $-^{\perp}$ is order reversing that $\text{supp}(b) \leq \text{supp}(a)$.

So for each $\phi \in \mathcal{H}$

$$b(\phi) = b((1-p)(\phi) + p(\phi)) = b((1-p)(\phi)) + b(p(\phi)) = b(p(\phi))$$

because $1-p$ is the projection onto $\ker(a) = \mathcal{K}^{\perp}$. Therefore $b = bp$. Taking adjoints, $b = b^* = p^* b^* = pb$, and combining these two facts, $b = bp = pbp$. So b vanishes on $\mathcal{K}^{\perp}$, and its range lies in $\mathcal{K}$, so $b \in B(\mathcal{K})$. □

We need some results about how directed suprema behave under multiplication and the relationship between different notions of directed completeness and continuity.

Lemma A.3. *Let A be a C*-algebra, $(a_i)_{i \in I}$ a directed set that has a supremum a.*
(i) Let $\beta \in \mathbb{R}_{\geq 0}$. Then $\beta a = \bigvee_{i \in I} \beta a_i$.
(ii) Let $b \in A$. Then $a + b = \bigvee_{i \in I} (a_i + b)$.

Proof.

(i) If $\beta = 0$, then this is true because $0 = 0$. If $\beta \neq 0$, we reason as follows. We have $a_i \leq a$, so $a - a_i \in A_+$, so $\beta a - \beta a_i \in A_+$, as it is a cone, so $\beta a \geq \beta a_i$. Therefore βa is an upper bound for $(\beta a_i)_{i \in I}$. Suppose $b \geq \beta a_i$ for all $i \in I$. Then $\beta^{-1} b \geq a_i$ for all $i \in I$, so $\beta^{-1} b \geq a$, and therefore $b \geq \beta a$.

(ii) As $a_i \leq a$, $a - a_i \in A_+$, so $a + b - (a_i + b) \in A_+$, so $a_i + b \leq a + b$ for all $i \in I$, and therefore $a + b$ is an upper bound for $(a + b_i)_{i \in I}$. Suppose $c \geq a_i + b$ for all $i \in I$. Then $c - b \geq a_i$ for all $i \in I$, so $c - b \geq a$, and $c \geq a + b$. □

The following is proved in the more general case of a directed-complete normed cone in [2, Lemma 2.16].

Lemma A.4 (Selinger)**.** *Let A be a C*-algebra, and $a, b \in A$, and $\beta \in \mathbb{R}_{>0}$. Then $a \ll b$ implies $\beta a \ll \beta b$.*

Proof. Let $a \ll b$, and let $(a_i)_{i \in I}$ be a directed set with supremum $\bigvee_{i \in I} a_i \geq \beta b$. By Lemma A.3 (i), $\bigvee_{i \in I} \beta^{-1} a_i = \beta^{-1} \bigvee_{i \in I} a_i \geq b$, so there exists $i \in I$ such that $\beta^{-1} a_i \geq a$. Therefore $a_i \geq \beta a$ for this i. As this holds for any directed set with supremum exceeding βb, we have proved $\beta a \ll \beta b$. □

Proposition II.7. *Let A be a C*-algebra*

(a) The following are equivalent:
(i) A is bounded directed complete.
(ii) A_+ is bounded directed complete.
(iii) $[0,1]_A$ is a dcpo.

(b) The following are equivalent when A is a bounded directed complete C-algebra:*
(i) A_+ is continuous.
(ii) $[0,1]_A$ is continuous.

Proof.

(a)
- (i) ⇒ (ii):
Let $(a_i)_{i \in I}$ be a bounded directed set in A_+. Then it is a bounded directed set in A, so there exists $a = \bigvee_{i \in I} a_i$. Pick $j \in I$, and hen $a \geq a_j \geq 0$, so $a \in A_+$. Therefore A_+ is bounded directed complete.
- (ii) ⇒ (iii):
Let $(a_i)_{i \in I}$ be a directed set in $[0,1]_A$. As $1 \geq a_i$ for all $i \in I$, it is a bounded set in A_+ and so has a supremum $a \in A_+$. As 1 is an upper bound for $(a_i)_{i \in I}$, $a \leq 1$ so is the supremum in $[0,1]_A$. Therefore $[0,1]_A$ is directed complete.
- (iii) ⇒ (i):
Let $(a_i)_{i \in I}$ be a directed set that is bounded above. Pick $i_0 \in I$, and define $J = \uparrow i_0$. Then $(a_j)_{j \in J}$ is cofinal in $(a_i)_{i \in I}$, because $(a_i)_{i \in I}$ is directed. Define $(b_j)_{j \in J}$ by $b_j = a_j - a_{i_0}$. Let $b \in A$ be an upper bound for $(a_i)_{i \in I}$ (equivalently for $(a_j)_{j \in J}$), and therefore $b - a_{i_0}$ is an upper bound for $(b_j)_{j \in J}$. By Lemma A.1 (iv) there exists $n \in \mathbb{N}$ such that $n \cdot 1 \geq b - a_{i_0}$, so $b_j \leq n \cdot 1$ for all $j \in J$. We can therefore define $(c_j)_{j \in J}$ by $c_j = \frac{1}{n} b_j$, which is a directed set in $[0,1]_A$. Let $c = \bigvee_{j \in J} c_j$. By Lemma A.3 (i), $nc = \bigvee_{j \in J} b_j$, and by Lemma A.3 (ii), $nc + a_{i_0} = \bigvee_{j \in J} a_j = \bigvee_{i \in I} a_i$.

(b)
- (i) ⇒ (ii):
Let $a \in [0,1]_A$. Then $a \in A_+$ and $\twoheaddownarrow a \cap [0,1]_A = \twoheaddownarrow a \cap A_+$ because $b \ll a$ implies $b \leq a \leq 1$. So $\twoheaddownarrow a \cap [0,1]_A$ is directed and $a = \bigvee \twoheaddownarrow a \cap [0,1]_A$, proving $[0,1]_A$ is continuous.
- (ii) ⇒ (i):
Let $a \in A_+$. By Lemma A.1 (iv), there exists $n \in \mathbb{N}$ such that $a \leq n \cdot 1$. Therefore $0 \leq \frac{1}{n} a \leq 1$. By the

assumption that $[0,1]_A$ is continuous, $\downarrow\frac{1}{n}a$ is directed and $\bigvee\downarrow\frac{1}{n}a = \frac{1}{n}a$. If $b \in n\downarrow\frac{1}{n}a$ then $\frac{1}{n}b \ll \frac{1}{n}a$ so $b \ll a$ (Lemma A.4). Similarly if $b \ll a$, $b \in n\downarrow\frac{1}{n}a$, so $n\downarrow\frac{1}{n}a = \downarrow a$. As $\downarrow\frac{1}{n}a$ is directed, $\downarrow a = n\downarrow\frac{1}{n}a$ is directed, and $\bigvee\downarrow a = \bigvee n\downarrow\frac{1}{n}a = n\bigvee\downarrow\frac{1}{n}a = n\frac{1}{n}a = a$, by Lemma A.3 (i). □

Proposition II.8. *The C*-algebra $\prod_{i\in I} A_i$, defined above, equipped with the projections $(\pi_i)_{i\in I}$ defined such that $\pi_j((a_i)) = a_j$, is the categorical product of $(A_i)_{i\in I}$.*

Proof. The purely algebraic axioms of C*-algebras are easily verified for $\prod_{i\in I} A_i$ pointwise, and the axioms for the norm are verified using the universal property of the supremum. We have that $\|a_i\| \leq \sup_{i\in I}\|a_i\|$ for all $i \in I$, so if we have a Cauchy sequence $(a_{ij})_{i\in I,j\in\mathbb{N}}$ in $\prod_{i\in I} A_i$, for each $i \in I$ $(a_{ij})_{j\in\mathbb{N}}$ is a Cauchy sequence in A_i, so converges to an element b_i, but we still need to show that $\|(b_i)_{i\in I}\|$ is bounded to prove that $(b_i)_{i\in I}$ is an element of $\prod_{i\in I} A_i$. Given $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $j, k \geq \mathbb{N}$, $\|(a_{ij}) - (a_{ik})\| < 1$, *i.e.* for all $i \in I$, $\|a_{ij} - a_{ik}\| < 1$. Since $a_{ij} \to b_i$, for all $\epsilon' > 0$ there exists a $k_i \in \mathbb{N}$ such that $\|a_{ik} - b_i\| < \epsilon'$. By the triangle inequality, for all $i \in I$, all $j \geq N$ and all $\epsilon' > 0$, $\|a_{ij} - b_i\| < 1 + \epsilon'$, so $\|a_{ij} - b_i\| \leq 1$. So for all $i \in I$ and $j \geq N$

$$\|b_i\| = \|b_i - a_{ij} + a_{ij}\| \leq \|b_i - a_{ij}\| + \|a_{ij}\| = 1 + \|a_{ij}\|.$$

If we pick some $j \geq N$, there is a bound, uniform in I, $\alpha \geq \|a_{ij}\|$, so $1 + \alpha \geq \|b_i\|$ for all $i \in I$. This proves $(b_i)_{i\in I} \in \prod_{i\in I} A_i$, so $\prod_{i\in I} A_i$ is complete in its norm, and a C*-algebra.

Because the C*-algebra operations are defined pointwise, $\pi_i : \prod_{i\in I} A_i \to A_i$ is easily seen to be a unital *-homomorphism for each $i \in I$. So we only have to prove the universal property of the product. Given a family of unital *-homomorphisms $(f_i)_{i\in I}$ where $f_i : B \to A_i$, B being a unital C*-algebra, we define $\langle f_i\rangle_{i\in I} : B \to \prod_{i\in I} A_i$ as follows, for each $b \in B$:

$$\langle f_i\rangle(b) = (f_i(b))_{i\in I}.$$

It is clear from the fact that the operations are defined pointwise that if this defines an element of $\prod_{i\in I} A_i$ for each $b \in B$, then $\langle f_i\rangle$ is a unital *-homomorphism, $\pi_i\circ\langle f_i\rangle = f_i$ for each $i \in I$ and $\langle f_i\rangle$ is the unique *-homomorphism with this property, so we only need to prove that $\langle f_i\rangle(b) \in \prod_{i\in I} A_i$.

As each f_i is a unital *-homomorphism, it has operator norm $\|f_i\| \leq 1$ [18, 1.3.7]. Therefore for all $i \in I$, $\|f_i(b)\| \leq \|b\|$, so we have proven that for each $b \in B$, $\langle f_i\rangle(b) = (f_i(b))_{i\in I}$ is uniformly bounded in I and therefore an element of $\prod_{i\in I} A_i$, as required. □

For $\mathcal{H}$ a Hilbert space, we write $\mathrm{SA}(\mathcal{H})$ for the $\mathbb{R}$-Banach space of self-adjoint operators. It is helpful to recall that $\mathrm{sp}(a)$ is simply the set of eigenvalues of a for a finite-dimensional $\mathcal{H}$.

Lemma A.5. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. The following sets are the same.*

(i) The norm unit ball: $U = \{a \in \mathrm{SA}(\mathcal{H}) \mid \|a\| \leq 1\}$.
(ii) The interval from -1 to 1 in the Löwner order: $[-1,1]_{\mathcal{H}} = \{a \in \mathrm{SA}(\mathcal{H}) \mid -1 \leq a \leq 1\}$.
(iii) The set of self-adjoint operators with eigenvalues in $[-1,1]$: $\{a \in \mathrm{SA}(\mathcal{H}) \mid \mathrm{sp}(a) \subseteq [-1,1]\}$.

Proof. The equivalence of (ii) and (iii) follows directly from Lemma A.1.

- $\|a\| \leq 1 \Rightarrow \mathrm{sp}(a) \subseteq [-1,1]$:
  Suppose for a contradiction that $\|a\| \leq 1$ and there exists an eigenvalue with $|\lambda| > 1$. Let ψ be an eigenvector of Hilbert norm 1 with eigenvalue λ. Then
  $$\|a(\psi)\|^2 = \langle\psi, a^2(\psi)\rangle = \lambda^2\langle\psi,\psi\rangle = \lambda^2 > 1.$$
  Taking square roots, $\|a(\psi)\| > 1$, which, as $\|\psi\| = 1$, contradicts $\|a\| \leq 1$.
- $\mathrm{sp}(a) \subseteq [-1,1] \Rightarrow \|a\| \leq 1$:
  Let $(\psi_i)_{i\in I}$ be an orthonormal basis of eigenvectors of a, $(\lambda_i)_{i\in I}$ the corresponding eigenvalues. Let $\phi \in \mathcal{H}$ with $\|\phi\| \leq 1$, which we write as $\sum_{i\in I}\alpha_i\psi_i$, so $\sum_{i\in I}\overline{\alpha}_i\alpha_i \leq 1$. By the defining property of eigenvectors
  $$\begin{aligned}\|a(\phi)\|^2 &= \left\langle \sum_{i\in I}\lambda_i\alpha_i\psi_i, \sum_{i\in I}\lambda_i\alpha_i\psi_i \right\rangle \\ &= \sum_{i\in I}\lambda_i^2\overline{\alpha}_i\alpha_i \leq 1\end{aligned}$$
  because $\sum_{i\in I}\overline{\alpha}_i\alpha_i \leq 1$ and $0 \leq \lambda_i^2 \leq 1$, by the initial assumption. Taking square roots, $\|a(\psi)\| \leq 1$, so $\|a\| \leq 1$ in the operator norm. □

Lemma A.6. *The following are equivalent, for an operator $a \in B(\mathcal{H})$, $\mathcal{H}$ finite-dimensional:*

(i) $a \in \mathrm{int}\,(B(\mathcal{H})_+)$, i.e. *$a$ is in the norm interior of the positive cone.*
(ii) $a \in B(\mathcal{H})_+$ and a is invertible.

Proof. We prove this by showing that for $a \in B(\mathcal{H})_+$, it is in the interior of $B(\mathcal{H})_+$ iff it is invertible. So $a \in \mathrm{int}\,(B(\mathcal{H})_+)$ iff there exists $\epsilon > 0$ such that $[a - \epsilon 1, a + \epsilon 1] \subseteq B(\mathcal{H})_+$ by Lemma A.5 (ii). Then $[a - \epsilon 1, a + \epsilon 1] \subseteq B(\mathcal{H})_+$ iff $a - \epsilon 1 \in B(\mathcal{H})_+$, and this in turn holds iff $\mathrm{sp}(a) > \epsilon$, by Lemma A.1 (iii). As $\mathrm{sp}(a)$ is closed, there exists an $\epsilon > 0$ such that $\mathrm{sp}(a) > \epsilon$ iff $0 \neq \mathrm{sp}(a)$, which holds iff a is invertible. □

Lemma A.7. *Let a be a non-zero positive operator on $\mathcal{H}$, $\mathcal{H}$ finite-dimensional, and $p = \mathrm{supp}(a)$. Then there exists an $N \in \mathbb{N}$ such that for all $i \geq N$, $a - 2^{-i}p$ is positive, and*

$$\bigvee_{i=N}^{\infty}(a - 2^{-i}p) = a.$$

in the positive cone of $B(\mathcal{H})$.

Proof. As a is positive, it is self-adjoint, and so as it is non-zero, it has a non-zero eigenvalue. Let $(\psi_j)_{j\in J}$ be an orthonormal basis of eigenvectors for a (J a finite set), $(\lambda_j)_{j\in J}$ their corresponding eigenvalues, and let $K \subseteq J$ be the indices such that $\lambda_j \neq 0$. Then $(\psi_k)_{k\in K}$ spans the support of a, because each ψ_k is orthogonal to the null space of a, and every vector in the support of a is expressible in terms of $(\psi_j)_{j\in J}$, but cannot use any of the ψ_j with $\lambda_j = 0$.

Let $\lambda > 0$ be the smallest nonzero eigenvalue of a. Let N be the smallest $N \in \mathbb{N}$ such that $2^{-N} \le \lambda$, so for all $i \ge N$ and $k \in K$, $2^{-i} \le \lambda_k$. Let $\phi \in \mathcal{H}$, and express it in terms of eigenvectors as $\sum_{j\in J} \alpha_j \psi_j$. Then

$$\begin{aligned}\langle \phi, a(\phi)\rangle &= \left\langle \phi, a\left(\sum_{j\in J} \alpha_j \psi_j\right)\right\rangle \\ &= \left\langle \phi, \sum_{j\in J} \alpha_j \lambda_j \psi_j\right\rangle \\ &= \left\langle \phi, \sum_{k\in K} \alpha_k \lambda_k \psi_k\right\rangle \\ &= \sum_{k\in K} \lambda_k \langle \phi, \alpha_k \psi_k\rangle \\ &\ge \sum_{k\in K} 2^{-i} \langle \phi, \alpha_k \psi_k\rangle \\ &= \left\langle \phi, 2^{-i} \sum_{k\in K} \alpha_k \psi_k\right\rangle \\ &= \langle \phi, 2^{-i} p(\phi)\rangle,\end{aligned}$$

so $a \ge 2^{-i}p$, *i.e.* $a - 2^{-i}p$ is positive.

Now we prove the least upper bound property. As projections are positive, $2^{-i}p \ge 0$ for all $i \ge N$, so $a + 2^{-i}p \ge a$, and therefore $a \ge a - 2^{-i}p$ for all $i \in \mathbb{N}$, making a an upper bound. Now suppose b is a positive operator and $a - 2^{-i} \le b$ for all $i \ge N$. Then $a - b \le 2^{-i}p$ and for all $\psi \in \mathcal{H}$

$$\langle \psi, (a-b)(\psi)\rangle \le \langle \psi, 2^{-i}p(\psi)\rangle \le 2^{-i}\langle \psi, p(\psi)\rangle \le 2^{-i}\|\psi\|^2.$$

As this holds for all $i \ge N$, we apply the archimedean property of the reals to deduce that $\langle \psi, (a-b)(\psi)\rangle \le 0$, and therefore $a - b \le 0$ and so $a \le b$. This proves a is the least upper bound. □

Using the above, we can characterize the way-below relation on positive operators. This differs from Selinger's characterization, but it is not difficult to prove that the two characterizations are equivalent (which they must be).

Lemma A.8. *Let a, b be positive operators on $\mathcal{H}$, where $\mathcal{H}$ is finite-dimensional, and $p = \mathrm{supp}(a)$. Then*

$$b \ll a \Leftrightarrow \exists \epsilon > 0. b \le a - \epsilon p$$

Proof.

- $\Rightarrow$:
  By Lemma A.7 and the fact that $b \ll a$, there exists $i \in \mathbb{N}$ such that $b \le a - 2^{-i}p$, so we can take $\epsilon = 2^{-i}$.
- $\Leftarrow$:
  Suppose that $b \le a - \epsilon p$ for some $\epsilon > 0$. Let $(c_i)_{i\in I}$ be a directed set of positive operators with supremum $c \ge a$. Let $\mathcal{K}$ be the support of c. If $\mathcal{K} = \{0\}$, then $c = 0$, so $a = b = 0$ and therefore $b \ll a$. So we now assume that $\mathcal{K} \ne \{0\}$ and therefore $c \ne 0$. As $c_i \le c$ for all $i \in I$ and $a, b \le c$, all these operators can be restricted to elements of $B(\mathcal{K})$ by Lemma II.6, and by Lemma II.4, $\bigvee_{i\in I} c_i = c$ in $B(\mathcal{K})$ and all other order relations that hold in $B(\mathcal{H})$ continue to hold in $B(\mathcal{K})$. In $B(\mathcal{K})$, we have $\mathrm{supp}(c) = 1$.
  As $a \le c$, $c - a$ is positive, and since $(c - a) + a = c$, we have, by Lemma A.2 (ii), $\mathrm{supp}(c - a) \vee \mathrm{supp}(a) = \mathrm{supp}(c) = 1$, these supports being calculated in $B(\mathcal{K})$. Then

$$\begin{aligned}\mathrm{supp}(c - (a - \epsilon p)) &= \mathrm{supp}((c - a) + \epsilon p) \\ &= \mathrm{supp}(c - a) \vee \mathrm{supp}(\epsilon p) \\ &= \mathrm{supp}(c - a) \vee \mathrm{supp}(a) = 1,\end{aligned}$$

  using Lemma A.2 (ii) again. Therefore $c - (a - \epsilon p)$ is invertible, by the finite-dimensionality of $\mathcal{K}$, and so $c - (a - \epsilon p)$ is in the norm interior of $B(\mathcal{K})_+$ (Lemma A.6). By Lemma A.3 (ii), $c - (a - \epsilon p) = \bigvee_{i\in I} c_i - (a - \epsilon p)$, and so $(c_i - (a - \epsilon p))_{i\in I}$ converges to $c - (a - \epsilon p)$ in the weak-* topology [20, 1.7.4]. As all vector space topologies agree on finite-dimensional spaces [35, I.3.2], the convergence is also holds in the norm topology, so there exists $i \in I$ such that $c_i - (a - \epsilon p) \in B(\mathcal{K})_+$. Therefore $c_i \ge a - \epsilon p \ge b$ in $B(\mathcal{K})$, so $b \le c_i$ in $B(\mathcal{H})$ by Lemma II.4. This proves $b \ll a$. □

Theorem II.9. (Selinger). *If $\mathcal{H}$ is a finite-dimensional Hilbert space, $B(\mathcal{H})$ is a continuous directed-complete C*-algebra.*

Proof. All we need to show is that for all positive operators $a \in B(\mathcal{H})$, the set $\twoheaddownarrow a$ is directed and $a = \bigvee \twoheaddownarrow a$. This holds automatically if $a = 0$, so we reduce to the case that $a \ne 0$. As before, let $p = \mathrm{supp}(a)$. If b_1, b_2 are positive operators such that $b_1, b_2 \ll a$, then by Lemma A.8, there exist $\epsilon_1, \epsilon_2 > 0$ such that $b_i \le a - \epsilon_i p$ for $i \in \{1, 2\}$. If we take $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, then $b_1, b_2 \le a - \epsilon p$, and by Lemma A.8, $a - \epsilon p \ll a$. This proves that $\twoheaddownarrow a$ is directed.

Since for each $\epsilon > 0$ there exists an $i \in \mathbb{N}$ such that $2^{-i} \le \epsilon$, Lemma A.8 shows that if $b \ll a$, then there exists $i \in \mathbb{N}$ such that $b \le a - 2^{-i}p \ll a$, *i.e.* positive elements of the form $a - 2^{-i}p$ are cofinal in $\twoheaddownarrow a$. As a is not zero, we have that $\bigvee \twoheaddownarrow a = \bigvee_{i=N}^{\infty}(a - 2^{-i}p) = a$, by Lemma A.7. □

Theorem II.10. (Selinger). *Let X be a set, and $(\mathcal{H}_x)_{x\in X}$ a family of finite-dimensional Hilbert spaces. Then $\prod_{x\in X} B(\mathcal{H}_x)$ is a continuous directed-complete C*-algebra. The way-below relation is characterized by $(a_x)_{x\in X} \ll (b_x)_{x\in X}$ iff there exists a finite subset $S \subseteq X$ such that $a_x = 0$ for all $x \in X \backslash S$, and for all $x \in S$ $a_x \ll b_x$.*

Proof. For convenience, we write $A_x = B(\mathcal{H}_x)$ and $A = \prod_{x\in X} A_x$. By Theorem II.9, each A_x is continuous. Therefore, by Proposition II.7 (b), the unit interval $[0, 1]_{A_x}$ is a continuous dpco for all $x \in X$. Because the operations are defined pointwise, it is clear that $(a_x)_{x\in X}$ is positive in A iff each $a_x \in A_x$ is positive. Therefore the order on $[0, 1]_A$ is the product ordering, and, as each element is uniformly bounded above by 1 and below by 0, $[0,1]_A$ is the poset product $\prod_{x\in X}[0,1]_{A_x}$.

By [11, Proposition I-2.1 (ii)], $\prod_{x\in X}[0,1]_{A_x}$ is a continuous dcpo, and $(a_x)_{x\in X} \ll (b_x)_{x\in X}$ iff there exists a finite set $S \subseteq X$ such that $a_x = 0$ except when $x \in S$, and $a_x \ll b_x$ for all $x \in X$, which, as $0 \ll b_x$, is satisfied iff $a_x \ll b_x$ for all $x \in S$.

Therefore, by Proposition II.7 (b), A is continuous. ☐

Definition III.2. *Let A be a C*-algebra. The following four conditions are equivalent and define what it is for A to be an AW*-algebra.*

(i) A is a Baer *-ring, i.e. *for all $S \subseteq A$, there exists a projection $p \in A$ such that $R(S) = pA$.*
(ii) The projections of A form a complete lattice and A is a Rickart *-ring, i.e. *for all $a \in A$, there exists a projection $p \in A$ such that $R(\{a\}) = pA$.*
*(iii) Every set of orthogonal projections in A has a supremum and A is a Rickart *-ring.*
*(iv) Every set of orthogonal projections in A has a supremum and every maximal commutative *-subalgebra of A is generated by its projections.*

Proof. The equivalence of the first three is shown in [17, §4 Proposition 1]. Part (iv) is actually the original definition of an AW*-algebra and is proved to imply (i) in [36, Theorem 2.3]. To show (iii) implies (iv), we only need to show the second part holds. So let A be a C*-algebra satisfying (iii), and let $B \subseteq A$ be a maximal commutative *-subalgebra. By [17, §3 Proposition 9 (5)], A'' is commutative and contains A, so as A is maximal, $A'' = A$. By [17, §4 Proposition 8 (iv)], this implies A is a commutative AW*-algebra (in the sense of (i)). Commutative AW*-algebras are isomorphic to $C(X)$ for a stonean space X [17, §7 Theorem 1], so are generated by their projections. ☐

The following standard lemma characterizes when products of positive elements are positive.

Lemma A.9. *Let A be a C*-algebra and $a, b \in A$ positive. Then ab is positive iff $ab = ba$.*

Proof. If ab is positive, then it is self-adjoint, so $ab = (ab)^* = b^*a^* = ba$. For the other direction, suppose that $ab = ba$. Then a and b generate a commutative C*-subalgebra of A, which, by Gelfand duality, is isomorphic to $C(X)$ for some X. Since positive elements of $C(X)$ correspond to functions taking values in $\mathbb{R}_{\geq 0}$, ab is positive. ☐

Lemma III.4. *Let A be a C*-algebra, $a \in [0,1]_A$, and $p \in \mathrm{Proj}(A)$. Then the following are equivalent:*

(i) $a \leq p$
(ii) $a = ap$
(iii) $a = pa$
(iv) $a = pap$

Proof. In the proof, we observe that all these relations are preserved and reflected by isomorphisms. As for every C*-algebra A there exists a Hilbert space $\mathcal{H}$ such that A is isomorphic to a C*-subalgebra of $B(\mathcal{H})$, we can reduce to proving the equivalence of (i)-(iv) for A a C*-subalgebra of $B(\mathcal{H})$.

- (i) ⇒ (ii), (ii) ⇔ (iii), (iii) ⇒ (iv):
  If we apply Lemma II.6 to a and p, using the fact that $p = \mathrm{supp}(p)$, we get (i) ⇒ (ii), and the proof used in that Lemma to show that (ii) ⇒ (iii) actually also shows (iii) ⇒ (ii), and it is clear that (ii) and (iii) together imply (iv).
- (iv) ⇒ (ii): We have $a = pap$ and therefore $ap = pap^2 = pap = a$.
- (ii),(iii) ⇒ (i): We want to show that $a \leq p$, *i.e.* $p - a \in A_+$. By (ii), we have
  $$p - a = p - ap = (1-a)p$$
  and similarly by (iii), $p - a = p(1-a)$, so $(1-a)$ commutes with p. Then p is positive, and $(1-a)$ is positive because $a \in [0,1]_A$, so by Lemma A.9, $p - a = (1-a)p$ is positive. ☐

Lemma III.7. (Weaver). *Let A be an AW*-algebra. If $\mathrm{Proj}(A)$ is continuous, then it is atomic.*

Proof. We prove the contrapositive, *i.e.* that if $\mathrm{Proj}(A)$ is not atomic, it is not continuous. To clarify, in the following we say an element p "has no atoms below it" to mean there is no atom $a \in \mathrm{Proj}(A)$ such that $a \leq p$. So let $p \in P$ be an element with no atoms below it, which must exist if $\mathrm{Proj}(A)$ is not atomic. We will show that $\downarrow\!\!\downarrow p = \{0\}$, so $p \neq \bigvee \downarrow\!\!\downarrow p$. Let $q \leq p$ and $q \neq 0$. If q had an atom below it, so would p, so q has no atoms below it. Therefore we can construct a decreasing sequence such that $q_1 = q$, $q_{i+1} \leq q_i$ and $q_{i+1} \neq q_i$ and $q_{i+1} \neq 0$ inductively.

Define $q' = \bigwedge_{i=1}^{\infty} q_i$ and $p_i = p - (q_i - q')$. Using Lemma II.5 (i), we see that $q' \leq q_i$ implies $q - q_i$ is a projection, and $q_i - q' \leq q_i \leq q \leq p$ implies $p - (q_i - q')$ is a projection, so p_i is a projection. Now, as for all $i \in \mathbb{N}$, $q^{\perp} \wedge q_i = 0$, we have $(p \wedge q^{\perp}) \wedge (q_i \wedge {q'}^{\perp}) = 0$, and therefore, as $q_i \neq 0$, $q_i - q' \not\leq p \wedge q^{\perp} = p - q$ (Lemma II.5 (iv)), so $q \not\leq p - q_i + q' = p_i$. But

$$\begin{aligned}
\bigvee_{i=1}^{\infty} p_i &= \bigvee_{i=1}^{\infty} (p - (q_i - q')) \\
&= p - \bigwedge_{i=1}^{\infty} (q_i - q') && \text{Lemma II.5 (vi)} \\
&= p - \bigwedge_{i=1}^{\infty} q_i \wedge (q')^{\perp} && \text{Lemma II.5 (iv)} \\
&= p - \left(\bigwedge_{i=1}^{\infty} q_i\right) \wedge (q')^{\perp} \\
&= p - q' \wedge (q')^{\perp} \\
&= p.
\end{aligned}$$

Therefore $(p_i)_{i\in\mathbb{N}}$ shows that $q \not\ll p$. So $\downarrow\!\!\downarrow p = \{0\}$, and as $p \neq 0$, $p \neq \bigvee \downarrow\!\!\downarrow p$, proving that $\mathrm{Proj}(A)$ is not continuous. ☐

Lemma III.10. *Let A be an AW*-algebra and $B \subseteq A$ an AW*-subalgebra. Then* $\text{Proj}(B) \subseteq \text{Proj}(A)$ *has the induced ordering and the inclusion map* $\text{Proj}(B) \to \text{Proj}(A)$ *preserves arbitrary joins and nonempty meets. It preserves all joins iff the unit element of A is contained in B.*

Proof. First, observe that $\text{Proj}(A)$ is order-embedded in A and $\text{Proj}(B)$ is order-embedded in B, and B is order-embedded in A by Lemma II.4, so $\text{Proj}(B)$ is order-embedded in $\text{Proj}(A)$.

By Definition III.9 (iii), nonempty suprema are preserved by the inclusion map, and as B is a *-subalgebra, 0 is preserved as well, showing all suprema are preserved.

To show that non-empty meets are preserved, it helps to factorize the inclusion map into two maps. Let $u \in B$ be the unit element of B (which exists because B is an AW*-algebra). Now $\text{Proj}(B) \subseteq {\downarrow} u \subseteq \text{Proj}(A)$. If $(p_i)_{i\in I}$ be a non-empty family of projections in ${\downarrow} u$. By the nonemptiness, if $q \in A$ and $q \leq p_i$ for all $i \in I$, then $q \in {\downarrow} u$. Therefore the inclusion map ${\downarrow} u \to \text{Proj}(A)$ preserves non-empty meets. Now, as the complement of an element $a \in \text{Proj}(B)$ is $u - a$, and this is also true for ${\downarrow} u$, the inclusion map $\text{Proj}(B) \to \text{Proj}(A)$ preserves complements. As it preserves joins, it preserves meets. Therefore the composite inclusion map $\text{Proj}(B) \to \text{Proj}(A)$ preserves non-empty meets.

As the unit element is the empty meet, the inclusion map preserves all joins iff $u \in A$. □

Lemma III.12. *Let A be an AW*-algebra and p a central projection,* i.e. $p \in \text{Proj}(Z(A))$.

(i) Let $a \in A$. Then $pa = ap = pap$.
(ii) The element $a \in A$ is in $pAp = pA = Ap$ iff $a = pa$ (and therefore iff $a = ap$ or $a = pap$).
(iii) pAp is an AW-subalgebra of A, with unit element p.*
*(iv) The map $\pi_p : A \to pAp$ defined by $\pi_p(a) = pap$ (equivalently pa or ap) is a unital *-homomorphism.*
(v) $Z(pAp) = pZ(A)p$, i.e. *the centre of pAp is the image of the centre of A.*

Proof.

(i) $pa = ap$ follows from $p \in Z(A)$, and therefore $pap = p^2a = pa$.

(ii) First, by (i), $pAp = pA = Ap$. By definition, $a \in pA$ iff there is some $b \in A$ such that $a = pb$. So $a \in pA$ implies $pa = p^2b = pb = a$. Conversely, if $pa = a$, then immediately $a \in pA$. By (i) these statements hold equally well for $ap = a$ and $pap = a$.

(iii) See [17, §4 Proposition 8 (iii)] for the proof that pAp is an AW*-subalgebra of A. It is then easy to see that p is the unit element, because $ppap = pap = papp$ for all $a \in A$. It follows that the inclusion morphism $pAp \to A$ is a *-homomorphism, but is not unital unless $p = 1$.

(iv) Since (i) shows that $\pi_p(a) = pa$, we will work with this definition, as it is slightly simpler. If $\alpha a + \beta b$ is a $\mathbb{C}$-linear combination in A, then $\pi_p(\alpha a + \beta b) + p(\alpha a + \beta b) = \alpha pa + \beta pb = \alpha\pi_p(a) + \beta\pi_p(b)$, proving linearity. For all $a \in A$, we have $\pi_p(a^*) = pa^* = a^*p = (pa)^* = \pi_p(a)^*$, so π_p preserves the -* operation. If $a, b \in A$, then $\pi_p(ab) = pab = p^2ab = papb = \pi_p(a)\pi_p(b)$. Finally, $\pi_b(1) = p1 = p$, which is the unit element of pAp by (iii), so $\pi_b : A \to pAp$ is a unital *-homomorphism.

(v) If $a \in Z(A)$, then as $p \in Z(A)$ and the centre is a *-subalgebra of A, $pap \in Z(A)$, so $pZ(A)p \subseteq Z(A)$. As every element of pAp is an element of A, $pZ(A)p \subseteq Z(pAp)$. For the opposite inclusion, suppose that $a \in Z(pAp)$, *i.e.* $pap = a$ and a commutes with all elements of pAp. We show that $a \in Z(A)$, and therefore $a \in pZ(A)p$ (because $a = pap$). It follows from $a \in pAp$ that $(1 - p)a = 0$. Let $b \in A$, and

$$\begin{aligned} ba &= (bp + b(1-p))a \\ &= bpa + b(1-p)a \\ &= bpa && \text{because } (1-p)a = 0 \\ &= (pbp)a && \text{part (i)} \\ &= a(pbp) && \text{because } a \in Z(pAp) \\ &= apb && \text{part (i)} \\ &= ab && \text{part (ii).} \end{aligned}$$

Therefore $Z(pAp) = pZ(A)p$. □

Proposition III.14. (Weaver). *An AW*-factor A has* $\text{Proj}(A)$ *continuous iff there exists a finite-dimensional Hilbert space $\mathcal{H}$ such that $A \cong B(\mathcal{H})$.*

Proof. The fact that $\text{Proj}(B(\mathcal{H}))$ is continuous if $\mathcal{H}$ is finite-dimensional follows from Theorem II.9 and Proposition III.6. So we only need to show that if $\text{Proj}(A)$ is continuous, $A \cong B(\mathcal{H})$ for $\mathcal{H}$ finite-dimensional. By Lemma III.7, $\text{Proj}(A)$ must be atomic. By [17, §15, Theorem 1, (4)], there is a central projection h_4 such that h_4A is a *discrete AW*-algebra* and $(1 - h_4)A$ is a *continuous*[16] *AW*-algebra* (see [17, §15, Definition 3] for the definitions of these). Since A is a factor, the only central projections are 0 and 1, so either A is a discrete AW*-algebra or a continuous AW*-algebra. If A were continuous, then the only abelian projection (see [17, §15, Definition 2] for the definition of this) is 0. So by [17, §19 Lemma 1] every projection other than 0 contains a strictly smaller non-zero projection (see [17, §14, Proposition 2 and Corollary 1] for why an AW*-algebra "has PC"). As $\text{Proj}(A)$ is atomic, this is false, so A must be discrete, or a *type I AW*-algebra* [17, §15, Definition 4].

Therefore, by [37, Lemma 1], there exists a Hilbert space $\mathcal{H}$ such that $A \cong B(\mathcal{H})$. So all we need to do is show that $\mathcal{H}$ cannot be infinite dimensional. By taking an orthonormal basis, identify $\mathcal{H}$ with $\ell^2(\kappa)$ for some cardinal κ. We use $(e_\alpha)_{\alpha\in\kappa}$ for the basis vectors, the functions taking the value 0 everywhere except for at α, where they take the value 1. Define $p = |e_0\rangle\langle e_0|$ and for $i \in \omega$, define

$$\psi_n = \frac{e_0 + \frac{1}{n+1}e_{n+1}}{\sqrt{\frac{n+2}{n+1}}}$$

[16] As will soon be apparent, it is important not to confuse this notion with the notion of continuity for dcpos.

so that $p_n = |\psi_n\rangle\langle\psi_n|$ is the projection onto the span of $e_0 + \frac{1}{n+1}e_{n+1}$. So $\bigvee_{i=0}^{n} p_i$ is the projection onto the span of $\{e_0 + e_1, e_0 + \frac{1}{2}e_2, \ldots, e_0 + \frac{1}{n+1}e_{n+1}\}$. It is clear that e_0 is not in this subspace for any $n \in \omega$, so $p \not\leq \bigvee_{i=0}^{n} p_i$ for any $n \in \omega$. However, as

$$\left\| e_0 + \frac{1}{n}e_n - e_0 \right\| = \left\| \frac{1}{n}e_n \right\| = \frac{1}{n} \to 0$$

we have that e_0 is in the closure of the span $\{e_0 + e_1, e_0 + \frac{1}{2}e_2, \ldots\}$, so $p \leq \bigvee_{i=0}^{\infty} p_i$. We can then define $q_0 = \bigvee_{\alpha\in\kappa\setminus\omega} |e_\alpha\rangle\langle e_\alpha|$, $q_n = q_{n-1} \vee p_{n-1}$, for $n > 1$ in ω, and we have a chain of projections such that $\bigvee_{i=0}^{\infty} q_i = 1$, and $p \not\leq q_i$ for any $i \in \omega$. So $p \not\ll 1$.

We can re-run this argument for any projection onto a 1-dimensional subspace, by extending a unit vector ψ contained in that subspace to an orthonormal basis and identifying ψ with e_0. Therefore no projection onto a 1-dimensional subspace is way below 1. As every non-zero projection contains a 1-dimensional subspace, this shows that the only projection that is way below 1 is 0, so 1 is not the supremum of elements way below it, and $\mathrm{Proj}(B(\mathcal{H}))$ is not continuous. $\square$