Some No-Go Results in Quantum Domain Theory (Preprint)

Robert Furber

September 19, 2022

Abstract

We show that the type of approximation used in the Solovay-Kitaev theorem cannot be carried out in quantum domain theory. Specifically, there is no countable set D of completely positive maps such that all completely positive maps can be expressed as least upper bounds of directed subsets of D, even for the case of 2×2 matrices. Via a well-known isomorphism, this negative result can be carried over to the forward light cone of 4D Minkowski spacetime, considered as a domain.

We also establish that the effects (equivalently, 2-valued POVMs) on a C*-algebra form a continuous dcpo iff it is a (possibly infinite) product of finite-dimensional matrix algebras, so there are no nontrivial infinite-dimensional C*-algebras that have a continuous dcpo for an effect algebra.

1 Introduction

The idea behind domain-theoretic semantics of programs can be summarized as follows. We admit that we write programs that fail, so we represent (denote) programs as partial functions. The denotations of programs form partially ordered sets, with less-defined programs being lower in the order than more-defined ones. We can then represent while loops and recursive functions as limits of an iteration $f^n(\perp)$ of a function f, starting with a completely undefined function $\perp$ (see, for example [1, §4.4 and p. 59]). This limit is defined in terms of the order as a least upper bound, and in the classical applications one can avoid the use of any metric or topology to define this approximation, using only the theory of directed-complete partial orders (dcpos).

In the usual formulation of mixed-state quantum computing, the set of completely positive maps forms a cone, and restricting to trace-reducing maps (Schrödinger picture) or subunital maps (Heisenberg picture) gives us a dcpo, and this approach has been used in [2, 3, 4] to define semantics for quantum programs in a domain-theoretic manner.

The full set of unitary matrices, the "gates" in quantum computing, forms a continuum. Of course, a continuum has never been a problem in computing, even since Turing's first paper [5, 6], as long as it is approximated using a countable set. So in quantum computing we pick a finite set of unitary gates (*e.g.* Clifford + T, *a.k.a.* Hadamard, phase and $\frac{\pi}{8}$, as described in [7, §4.5]) generating a countable dense subgroup of all unitaries, and if we include measurement operators we can even define a countable dense set of superoperators.

The question then arises: if one expresses this process of approximation in the form of a program, can it be given a quantum domain-theoretic semantics using the usual order on superoperators? It is not unreasonable to ask, as there are domain-theoretic approaches for classical programs that do things such as computing integrals, for example [8, 9]. We show that the answer is no – given any countable set X of completely positive subunital maps, there must be completely positive subunital maps that we cannot express by taking least upper bounds of directed sets in X (Theorem 3.11).

This shows that we cannot define approximation with just the order-theoretic structure of the space of completely positive maps, we need the topology. It also shows that the domain theoretic topologies, the Scott topology and the Lawson topology, are not the right ones to use. We can also get a domain-theoretic non-approximation result for spacetime as a corollary (Theorem 3.12), by using the isomorphism between $3+1$-dimensional Minkowski space and space of 2×2 self-adjoint matrices defined by the Pauli matrices.

The proof of Theorem 3.11 uses the C*-algebra structure of $n \times n$ matrices to define positivity. The approach to quantum domain theory using this notion of positivity has been generalized to the infinite dimensional case, using W*-algebras, for example in [10] [11, Chapter 3] [12]. Every finite dimensional C*-algebra is a W*-algebra, and in infinite dimensions they carry over many properties from the finite-dimensional case not possessed by infinite-dimensional C*-algebras. It is known that the superoperators between W*-algebras form a dcpo in this case [13, §4.1]. As the structure of a continuous dcpo was useful in the previous proof, we can ask if any infinite-dimensional W*-algebras form continuous dcpos. We show that for a C*-algebra A, the poset $[0,1]_A$ is a continuous dcpo iff A is a (possibly infinite) product of finite-dimensional matrix algebras (Theorem 4.17). This can be seen as a "quantum analogue" of the theorem that a complete Boolean algebra is a continuous dcpo iff it is of the form $\mathcal{P}(X)$ [14, Theorem I-4.20].

The proof proceeds by nontrivially reducing the problem of when the projection lattice in an AW*-algebra is continuous (even though the statement of the result makes no mention of projections or AW*-algebras). The fact that projection lattices of von Neumann algebras are only continuous if they are products of finite-dimensional matrix algebras was established by Weaver [15]. We extend this result to C*-algebras whose unit interval is directed-complete. The reduction of the continuity of $[0,1]_A$ to the continuity of $\mathrm{Proj}(A)$ is nontrivial because it is *not* the case that a sub-dcpo of a continuous dcpo is continuous. It requires a technical result (Proposition 4.7) relating projections and effects in an AW*-algebra: the inclusion map $\mathrm{Proj}(A) \hookrightarrow [0,1]_A$ preserves all joins and meets in $\mathrm{Proj}(A)$, so $\mathrm{Proj}(A)$ is a "sublattice" of $[0,1]_A$, even though $[0,1]_A$ is not a lattice if A is noncommutative.

2 Definitions and Background

The following section serves to collect basic definitions and background information used in the rest of the article.

If P is a poset, a subset $S \subseteq P$ is *directed* if for each $a, b \in S$, there exists $c \in S$ such that $c \geqslant a$ and $c \geqslant b$. A poset P is *directed* if $P \subseteq P$ is directed. We will often refer to directed sets indexed by a poset, so we will say, for instance, let $(a_i)_{i \in I}$ be a directed set in P to mean that I is a directed poset, and the mapping $i \to a_i$ is a monotone map (and therefore the image $\{a_i \mid i \in I\}$ is a directed subset of P). Every directed set in P is of this form, by "self-indexing". We say a poset P is *directed complete* if every directed set $(a_i)_{i \in I}$ has a least upper bound, which is written $\bigvee_{i \in I} a_i$. If $S \subseteq P$, we just write $\bigvee S$. A poset P is *bounded directed complete* if for each directed set $(a_i)_{i \in I}$ that is bounded, *i.e.* such that there exists $b \in P$ such that for all $i \in I$, $a_i \leqslant b$, has a least upper bound $\bigvee_{i \in I} a_i$. For instance, $\mathbb{R}$ with its usual ordering is bounded directed complete but not directed complete.

If D is a poset, $d, e \in D$, then we say e is *way below* d, or $e \ll d$, if for all directed sets $(d_i)_{i \in I}$ such that $\bigvee_{i \in I} d_i \geqslant d$, there exists $j \in I$ such that $e \leqslant d_j$. A poset is *continuous* if for all $d \in D$, the set $\twoheaddownarrow d = \{e \in D \mid e \ll d\}$ is directed, and $\bigvee \twoheaddownarrow d = d$. These notions are mostly used when D is not only a poset but a dcpo, but we allow the extension of the definition to posets.

For E a complex vector space, we define $\overline{E}$ to have the same underlying set and abelian group structure as E, but with scalar multiplication defined to be conjugated, *i.e.* if $z \in \mathbb{C}$ and $x \in \overline{E}$, we define $z \cdot_{\overline{E}} x = \overline{z} \cdot_E x$. This allows us to express "antilinear" maps as $\mathbb{C}$-linear maps $E \to \overline{E}$. We take our Hilbert space inner products to be antilinear on the left and linear on the right – they that prefer it the other way should swap left and right arguments when appropriate.

For a normed space E, we write $\mathrm{Ball}(E)$ for the closed unit ball of E, *i.e.* $\{x \in E \mid \|x\| \leqslant 1\}$. A linear map between normed spaces $f : E \to F$ is said to be *bounded* if the set $\{\|f(x)\| \mid x \in \mathrm{Ball}(E)\}$ is bounded in $\mathbb{R}_{\geqslant 0}$, *i.e.* if the real-valued function $x \mapsto \|f(x)\|$ is bounded in the usual sense on $\mathrm{Ball}(E)$. A convenient fact about linear maps between normed spaces is that they are bounded iff they are continuous [16, III.2.1], and the set of bounded linear maps $L(E, F)$ admits a norm, the *operator norm*, defined for $f : E \to F$

$$\|f\| = \sup\{\|f(x)\| \mid x \in \mathrm{Ball}(E)\}$$

If $\mathcal{H}$ is a Hilbert space, we write $B(\mathcal{H})$ for $L(\mathcal{H}, \mathcal{H})$, considered as a Banach space under the operator norm. The identity map is bounded, and bounded maps are closed under composition, making $B(\mathcal{H})$ a unital algebra, and each bounded map $f \in B(\mathcal{H})$ has an adjoint $f^* \in B(\mathcal{H})$, which is the unique map such that $\langle f^*(\psi), \phi \rangle = \langle \psi, f(\phi) \rangle$ for all $\psi, \phi \in \mathcal{H}$. It is easy to derive from this that $(g \circ f)^* = f^* \circ g^*$.

In the case that $\mathcal{H}$ is finite dimensional of dimension d, it is isomorphic to $\mathbb{C}^d$ with its usual inner product.[1] Then $B(\mathcal{H})$, as an algebra, is isomorphic to M_d, the algebra of $d \times d$ matrices of complex numbers. However, $B(\mathcal{H})$ has an extra piece of structure, the norm. This makes it a C*-algebra, which we define now.

A unital[2] C*-algebra is a $\mathbb{C}$-algebra A equipped with an antilinear operation $\text{-}^* : A \to \overline{A}$, and a norm $\|\text{-}\| : A \to \mathbb{R}_{\geqslant 0}$ such that A is a Banach *-algebra satisfying the C*-identity $\|a^* a\| = \|a\|^2$. In terms of axioms, this means that in

[1]Such isomorphisms correspond to orthonormal bases of $\mathcal{H}$.

[2]We will not be considering non-unital C*-algebras here because they are never directed complete. This follows from Proposition 4.4 and the fact that AW*-algebras are unital [17, §3 Proposition 2].

addition to the $\mathbb{C}$-vector space axioms, we have

$$\begin{aligned} (\lambda a + \mu b)c &= \lambda ac + \mu bc & (ab)c &= a(bc) \\ 1a &= a & (ab)^* &= b^* a^* \\ (\lambda a + \mu b)^* &= \overline{\lambda} a^* + \overline{\mu} b^* & 1^* &= 1 \\ a^{**} &= a & \|a\| = 0 &\Leftrightarrow a = 0 \\ \|a + b\| &\leqslant \|a\| + \|b\| & \|\lambda a\| &= |\lambda| \|a\| \\ \|ab\| &\leqslant \|a\| \|b\| & \|a^* a\| &= \|a\|^2 \end{aligned}$$

and the condition that A must be complete in the metric $d(a, b) = \|a - b\|$.

These axioms imply certain others the reader might expect, such as distributivity of multiplication over linear combinations on the right side, $\|a^*\| = \|a\|$, and $\|1\| = 1$ in the case that A has a non-zero element. A linear map between C*-algebras $f : A \to B$ that preserves multiplication and -* is called a *-homomorphism. A *-homomorphism is called *unital* if it preserves the unit element. Since *-homomorphisms only use the the equational part of the axioms of C*-algebras, the inverse of a bijective *-homomorphism is also a *-homomorphism, so we use *-isomorphism to refer to them. Additionally, *-homomorphisms are continuous with operator norm $\leqslant 1$ [18, 1.3.7].

If A is a C*-algebra, and $B \subseteq A$ is a linear subspace that is also closed under -* and multiplication, then B is called a *-subalgebra, and if it is also topologically closed with respect to the norm, it is a C*-algebra and we call it a C*-subalgebra of A.

For any Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ is a C*-algebra, and in fact the purpose of the C*-algebra axioms is to characterize the C*-subalgebras of $B(\mathcal{H})$. That is to say, every norm-closed *-subalgebra of $B(\mathcal{H})$ is a C*-algebra, and for any C*-algebra A there exists a Hilbert space $B(\mathcal{H})$, and a *-homomorphism $f : A \to B(\mathcal{H})$ that is an isomorphism onto its image [18, 2.6.1].

Another important source of C*-algebras is that if X is a compact Hausdorff space, then the algebra of continuous $\mathbb{C}$-valued functions $C(X)$ is a C*-algebra, where the operations are defined pointwise from those on $\mathbb{C}$, and the norm of $a \in C(X)$ is defined to be $\|a\| = \sup\{|a(x)| \mid x \in X\}$. This C*-algebra is commutative, and for every commutative unital C*-algebra A, there exists a compact Hausdorff space X, unique up to homeomorphism, and a C*-isomorphism $A \cong C(X)$ [18, 1.4.1]. This is called *Gel'fand duality.* It allows us to transfer algebraic facts about continuous functions to all commutative C*-algebras, or even to commuting elements of noncommutative C*-algebras.

We describe here how certain notions from finite-dimensional matrix theory are specializations of concepts in C*-algebra theory. An element a of a C*-algebra A is *invertible* if there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$. The element a^{-1} is unique, and is called the *inverse* of A. In the case of $B(\mathcal{H})$ for $\mathcal{H}$ finite-dimensional, the invertible elements are the nonsingular matrices. An element of a C*-algebra $u \in A$ is called *unitary* if u^* is the inverse of u.

The *spectrum* of an element of a C*-algebra $a \in A$, which we write $\mathrm{sp}(a)$, is defined by

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible}\}.$$

The spectrum $\mathrm{sp}(a)$ is a compact subset of $\mathbb{C}$ [19, Proposition I.2.3]. Recall that

λ is an eigenvalue of a matrix a iff $a - \lambda 1$ is not invertible[3]. The elements of $\mathrm{sp}(a)$ are called *spectral values* of a, and eigenvalues have their usual definition. Eigenvalues are always spectral values, but it is not necessarily the case that all spectral values are eigenvalues. As $\mathrm{sp}(a^*) = \{\overline{\lambda} \in \mathbb{C} \mid \lambda \in \mathrm{sp}(a)\}$, self-adjoint elements, *i.e.* those such that $a^* = a$, have $\mathrm{sp}(a) \subseteq \mathbb{R}$. The opposite implication does not hold, even for 2×2 matrices (*e.g.* $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$).

Although we do not consider non-unital C*-algebras, we do consider non-unital *-homomorphisms, and it is convenient to have a way of turning them into unital ones. If A is a C*-algebra, we define its *unitization* $\tilde{A}$ to have $A \times \mathbb{C}$ as its underlying vector space, the multiplication and -* defined pointwise, $(1,1)$ as the unit, and the norm defined as $\|(a, \alpha)\| = \max\{\|a\|, |\alpha|\}$. We embed A in $\tilde{A}$ by the map $a \mapsto (a, 0)$, which is a non-unital *-homomorphism [18, 1.3.8]. The *quasi-spectrum* $\mathrm{sp}'(a)$ of an element $a \in A$ is $\mathrm{sp}((a, 0))$, as calculated in $\tilde{A}$.

Lemma 2.1. *Let $a \in A$, for A a unital C*-algebra. Then* $\mathrm{sp}'(a) = \mathrm{sp}(a) \cup \{0\}$.

Proof. We first show that $0 \in \mathrm{sp}'(a)$. Suppose for a contradiction that $0 \notin \mathrm{sp}'(a)$, so there exists $(b, \beta) \in \tilde{A}$ such that $(a, 0)(b, \beta) = (1, 1)$. Then $0 \cdot \beta = 1$, which is impossible.

We show $\mathrm{sp}(a) \subseteq \mathrm{sp}'(a)$ by showing that $\mathbb{C} \backslash \mathrm{sp}'(a) \subseteq \mathbb{C} \backslash \mathrm{sp}(a)$. Suppose $\lambda \in \mathbb{C} \backslash \mathrm{sp}'(a)$. Then there exists $(b, \mu) \in \tilde{A}$ such that $(b, \mu)(a - \lambda, -\lambda) = (1, 1) = (a - \lambda, -\lambda)(b, \mu)$, so in particular, $b(a - \lambda) = 1 = (a - \lambda)b$. Therefore $\lambda \notin \mathrm{sp}(a)$. This completes the part of the proof that shows $\mathrm{sp}(a) \cup \{0\} \subseteq \mathrm{sp}'(a)$.

To show $\mathrm{sp}'(a) \subseteq \{0\} \cup \mathrm{sp}(a)$, we prove that $\mathbb{C} \backslash (\{0\} \cup \mathrm{sp}(a)) \subseteq \mathbb{C} \backslash \mathrm{sp}'(a)$. If $\lambda \in \mathbb{C} \backslash (\{0\} \cup \mathrm{sp}(a))$, then there exists $b \in A$ such that $b(a - \lambda) = 1 = (a - \lambda)b$. We therefore have $(b, -\frac{1}{\lambda})(a - \lambda, -\lambda) = (1, 1) = (a - \lambda, -\lambda)(b, -\frac{1}{\lambda})$, so $\lambda \notin \mathrm{sp}'(a)$. □

A nontrivial consequence of the axiomatics of C*-algebras is that C*-algebras admit a translation-invariant order. An element a of a C*-algebra A is called *positive* if it is of the form b^*b for $b \in A$. The following is the C*-algebraic version of a well-known characterization of positive semi-definite matrices.

Lemma 2.2. *The following are equivalent for an element $a \in A$ of a unital C*-algebra.*

(i) a is self-adjoint and $\mathrm{sp}(a) \subseteq \mathbb{R}_{\geqslant 0}$.

(ii) a is self-adjoint and $\mathrm{sp}'(a) \subseteq \mathbb{R}_{\geqslant 0}$.

(iii) There exists $b \in A$ such that $a = b^*b$.

(iv) There exists a self-adjoint $b \in A$ such that $a = b^2$.

Proof. The equivalence of (i) and (ii) follows from Lemma 2.1, and the equivalence of (ii),(iii) and (iv) follows from [18, 1.6.1], after observing that the assumption there that a is self-adjoint is not needed for (ii) and (iii), because b^*b is self-adjoint for all b and therefore b^2 is self-adjoint if b is. □

We write A_+ for the set of positive elements. The positive elements form a cone (*i.e.* are closed under addition and multiplication by nonnegative reals, and

[3]This characterization is used to show that the eigenvalues are the roots of the characteristic polynomial.

$A_+ \cap -A_+ = \{0\}$), which implies that the order defined by $a \leqslant b \Leftrightarrow b - a \in A_+$ is a partial order. We write $[0,1]_A$ for the set

$$[0,1]_A = \{a \in A \mid 0 \leqslant a \leqslant 1\} = A_+ \cap (1 - A_+).$$

This is known as either the *unit interval* or the *effect algebra* of the C*-algebra A.

If $a \in A$ is positive, as it is self-adjoint, the C*-subalgebra that it generates is commutative, so is canonically isomorphic to $C(X)$ for some compact Hausdorff space X, in which a takes values in $[0,\infty)$. Therefore we can take its positive square root $a^{\frac{1}{2}}$. We say a linear map $f : A \to B$ between C*-algebras is *positive* if it maps positive elements of A to positive elements of B. For linear maps, positivity is equivalent to monotonicity. It is easy to show that any *-homomorphism is positive.

Lemma 2.3. *Let $\mathcal{H}$ be a Hilbert space and $a \in B(\mathcal{H})$. The following are equivalent:*

(i) a is positive.

(ii) For all $\psi \in \mathcal{H}$, $\langle \psi, a(\psi) \rangle \geqslant 0$.

Proof. See [18, 1.6.7]. □

Therefore positive operators on finite-dimensional Hilbert spaces are positive semidefinite matrices by another name. The characterization above implies that $a \leqslant b$ iff for all $\psi \in \mathcal{H}$, $\langle \psi, a(\psi) \rangle \leqslant \langle \psi, b(\psi) \rangle$, and this is how the Löwner order was originally defined [20], rather than by using a cone.

Using the characterization of positive elements proved above, we can prove the following fact about self-adjoint elements. For ease of notation, we write $x \geqslant S$ for an element and a set S to mean x is greater than every element of S.

Lemma 2.4. *Let $a \in A$ be an element of a C*-algebra.*

(i) $\mathrm{sp}(-a) = -\mathrm{sp}(a)$.

(ii) Let $\alpha \in \mathbb{C}$. Then $\mathrm{sp}(a + \alpha 1) = \mathrm{sp}(a) + \alpha$, i.e. *shifting the operator by α shifts its spectrum by α.*

(iii) If a is self-adjoint and $\alpha \in \mathbb{R}$, $\alpha 1 \leqslant a$ iff $\alpha \leqslant \mathrm{sp}(a)$, and $\alpha 1 \geqslant a$ iff $\alpha \geqslant \mathrm{sp}(a)$.

(iv) If a is self-adjoint, there exist $\alpha, \beta \in \mathbb{R}$ such that $\beta 1 \leqslant a \leqslant \alpha 1$.

Proof. (i) First, observe that if a is invertible, with inverse a^{-1}, then $-a^{-1}$ is an inverse to $-a$. Therefore λ is outside the spectrum of a iff $a - \lambda 1$ is invertible iff $-a - (-\lambda)1$ is invertible iff $-\lambda$ is outside the spectrum of $-a$.

(ii) Let $\lambda \in \mathbb{C}$. We have that $(a + \alpha 1) - \lambda 1 = a - (\lambda - \alpha)1$, so $\lambda \in \mathrm{sp}(a + \alpha 1)$ iff $\lambda - \alpha \in \mathrm{sp}(a)$ iff $\lambda \in \mathrm{sp}(a) + \alpha$.

(iii) By part (ii) above and part (i) of Lemma 2.2, $\alpha 1 \leqslant a$ iff $a - \alpha 1$ is positive iff $\mathrm{sp}(a) - \alpha \geqslant 0$ iff $\alpha \leqslant \mathrm{sp}(a)$. So by part (i) above, $\alpha 1 \geqslant a$ iff $-\alpha 1 \leqslant -a$ iff $-\alpha \leqslant \mathrm{sp}(-a)$ iff $-\alpha \leqslant -\mathrm{sp}(a)$ iff $\alpha \geqslant \mathrm{sp}(a)$.

(iv) As a is self-adjoint, $\mathrm{sp}(a) \subseteq \mathbb{R}$, and as it is compact, it has an upper and a lower bound. So we pick $\alpha \geqslant \mathrm{sp}(a)$ and $\beta \leqslant \mathrm{sp}(a)$. By the previous part, $\beta 1 \leqslant \alpha \leqslant \alpha 1$. □

A function between posets $f : P \to Q$ is called an *order-embedding* if it is monotone and order-reflecting, *i.e.* for all $x, y \in P$, $x \leqslant y \Leftrightarrow f(x) \leqslant f(y)$. The antisymmetry axiom implies that order-embeddings are injective, but it is easy to find injective monotone maps that are not order embeddings (there is one involving 2-element posets). This problem does not occur with *-homomorphisms between C*-algebras.

Lemma 2.5. *Let A, B be C* algebras and $f : A \to B$ an injective *-homomorphism (not necessarily preserving the unit). Then f is an order-embedding.*

Proof. As f is a *-homomorphism, it is positive and therefore monotone. We show that f reflects positive elements, and deduce that it is an order-embedding from this. Let $a \in A$ and suppose that $f(a)$ is positive. Then $f(a^*) = f(a)^* = f(a)$, so by injectivity, a is self-adjoint. By [18, 1.3.10 (i)], $\mathrm{sp}'(f(a))$ in A is the same as $\mathrm{sp}'(f(a))$ in $f(A)$. As f is injective, it is an isomorphism onto its image [18, 1.8.3], and so $\mathrm{sp}'(f(a)) = \mathrm{sp}'(a)$. Therefore a is positive by Lemma 2.2 (i).

We can therefore show that f is order reflecting as follows. If $f(a) \leqslant f(b)$, then $f(b - a) = f(b) - f(a)$ is positive, so $b - a$ is positive, *i.e.* $a \leqslant b$. □

We will call a C*-algebra A *directed-complete*[4] if the set of self-adjoint elements A_{sa} is bounded directed-complete, *i.e.* for each directed set $(a_i)_{i \in I}$ in A_{sa} that has an upper bound (there exists a $b \in A_{\mathrm{sa}}$ such that for all $i \in I$, $b \geqslant a_i$), there is a *least* upper bound. If a C*-algebra is isometric to the dual space of a Banach space (in which case it is called a W*-algebra), for example $B(\mathcal{H})$, then it is directed-complete [21, 1.7.4]. There are also directed-complete C*-algebras that are not W*-algebras.

In many arguments, we need to use certain special elements of C*-algebras, called projections. A *projection*[5] in a C*-algebra A is a self-adjoint element p such that $p^2 = p$. We write $\mathrm{Proj}(A)$ for the set of projections. It is clear from the previous definition and Lemma 2.2 (iv) that $\mathrm{Proj}(A) \subseteq [0, 1]_A$ in any C*-algebra A, and this will be important later. For each projection $p \in B(\mathcal{H})$, the range of p is a closed subspace of $\mathcal{H}$. The mapping that takes a projection in $B(\mathcal{H})$ to its range is a poset isomorphism between $\mathrm{Proj}(B(\mathcal{H}))$ and the closed subspaces of $\mathcal{H}$, ordered by inclusion [22, §26 Theorem 4, §29 Theorem 2], and this is the reason for the name *projection.* The projections in a C*-algebra need not form a lattice, under the order coming from A [23, Lemma 2.1]. However, the projections do form a lattice in the finite-dimensional case, and we shall see that if a C*-algebra is directed-complete then its projections do form a lattice, and this is the case we are concerned with. We take this opportunity to summarize certain facts about projections. For any $p \in \mathrm{Proj}(A)$, we write $p^{\perp} = 1 - p$, because it projects onto the orthogonal complement in the Hilbert space case [22, §27 Theorem 3].

Lemma 2.6. *Let A be a C*-algebra.*

(i) If $p, q \in \mathrm{Proj}(A)$, then $p \leqslant q$ iff $q - p$ is a projection.

[4] Also known as *monotone-complete*.

[5] Also called a *projector*, such as in [7, §2.1.6].

(ii) *If* $p, q \in \mathrm{Proj}(A)$*, then* $p \leqslant q$ *iff* $pq = p$ *iff* $qp = p$*.*

(iii) *If* $p, q \in \mathrm{Proj}(A)$ *are commuting projections,* $pq = p \wedge q$*.*

(iv) *If* $p, q \in \mathrm{Proj}(A)$ *and* $p \leqslant q$*, then* $q - p = q \wedge p^{\perp}$*.*

(v) *The mapping* $a \mapsto 1 - a$ *is an isomorphism* $[0,1]_A \to [0,1]_A^{\mathrm{op}}$*.*

(vi) *For all* $q \in \mathrm{Proj}(A)$*, the mapping* $p \mapsto q - p$ *is an isomorphism* $\downarrow q \to (\downarrow q)^{\mathrm{op}}$*.*

Proof. Throughout, we use the fact that we can represent a C*-algebra in $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ to transfer facts about projections on Hilbert space.

(i) See [22, §29 Theorem 3].

(ii) See [22, §29 Theorem 2].

(iii) See [22, §30 Theorem 2].

(iv) As $p \leqslant q$, p commutes with q by (ii), so $1-p$ commutes with q by linearity. Therefore $q \wedge p^{\perp} = q(1-p) = q - qp = q - p$, by (iii) and (ii) in turn.

(v) It is a self-inverse bijection because $1-(1-a) = a$. It is an order-reversing isomorphism because

$$1 - b \leqslant 1 - a \Leftrightarrow 1 - a - 1 + b \in A_+ \Leftrightarrow b - a \in A_+ \Leftrightarrow a \leqslant b.$$

(vi) First we need to show that if $p \leqslant q$, then $q - p \in \downarrow q$. By (i) it is a projection, and as $q - (q - p) = p$, $q - p \leqslant q$. It is a self-inverse bijection because $q - (q - p) = p$. It is an order-reversing isomorphism because for all p, p' projections that are $\leqslant q$,

$$q - p \leqslant q - p' \Leftrightarrow q - p' - q + p \in A_+ \Leftrightarrow p - p' \in A_+ \Leftrightarrow p' \leqslant p.$$

□

Given a compact Hausdorff space X, the projections in $C(X)$ are continuous functions taking values in $\{0,1\}$, and therefore are indicator functions of clopen subsets of X, so form a Boolean algebra. By Gel'fand duality, this carries over to all commutative unital C*-algebras.

We need the notion of a product of C*-algebras. If $(A_i)_{i \in I}$ is an I-indexed family of C*-algebras, we define the product $\prod_{i \in I} A_i$ to have underlying set

$$\prod_{i \in I} A_i = \{(a_i)_{i \in I} \mid \forall i \in I. a_i \in A_i \text{ and } \exists \alpha \in \mathbb{R}_{\geqslant 0}. \forall i \in I. \|a_i\| < \alpha\},$$

i.e. it is the elements of the set-theoretic product for which the sequence of norms $\|a_i\|$ forms a sequence bounded uniformly in i. The unit is the constant 1 sequence, the vector space operations, multiplication and -* operation are defined pointwise, and the norm is defined by

$$\|(a_i)_{i \in I}\| = \sup_{i \in I} \|a_i\|.$$

This is sometimes called the *direct sum* of C*-algebras, because if the C*-algebras are all C*-subalgebras of $B(\mathcal{H}_i)$ one gets a C*-subalgebra of $B\left(\bigoplus_{i\in I}\mathcal{H}_i\right)$, but we find this name misleading because the reader might blithely expect it to be a biproduct of C*-algebras, which it is not, even in the case that I is finite. Dixmier [18, 1.3.3] calls it the product, and we do too, because it is the categorical product in **C*Alg**, the category of unital C*-algebras and unital *-homomorphisms.

Proposition 2.7. *The C*-algebra $\prod_{i\in I} A_i$, defined above, equipped with the projection *-homomorphisms $(\pi_i)_{i\in I}$ defined such that $\pi_j((a_i)) = a_j$, is the categorical product of $(A_i)_{i\in I}$ in* **C*Alg**.

Proof. The purely algebraic axioms of C*-algebras are easily verified for $\prod_{i\in I} A_i$ pointwise, and the axioms for the norm are verified using the universal property of the supremum. We have that $\|a_i\| \leqslant \sup_{i\in I}\|a_i\|$ for all $i \in I$, so if we have a Cauchy sequence $(a_{ij})_{i\in I, j\in\mathbb{N}}$ in $\prod_{i\in I} A_i$, for each $i \in I$ $(a_{ij})_{j\in\mathbb{N}}$ is a Cauchy sequence in A_i, so converges to an element b_i, but we still need to show that $\|(b_i)_{i\in I}\|$ is bounded to prove that $(b_i)_{i\in I}$ is an element of $\prod_{i\in I} A_i$. Given $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $j, k \geqslant \mathbb{N}$, $\|(a_{ij}) - (a_{ik})\| < 1$, *i.e.* for all $i \in I$, $\|a_{ij} - a_{ik}\| < 1$. Since $a_{ij} \to b_i$, for all $\epsilon' > 0$ there exists a $k_i \in \mathbb{N}$ such that $\|a_{ik} - b_i\| < \epsilon'$. By the triangle inequality, for all $i \in I$, all $j \geqslant N$ and all $\epsilon' > 0$, $\|a_{ij} - b_i\| < 1 + \epsilon'$, so $\|a_{ij} - b_i\| \leqslant 1$. So for all $i \in I$ and $j \geqslant N$

$$\|b_i\| = \|b_i - a_{ij} + a_{ij}\| \leqslant \|b_i - a_{ij}\| + \|a_{ij}\| = 1 + \|a_{ij}\|.$$

If we pick some $j \geqslant N$, there is a bound, uniform in I, $\alpha \geqslant \|a_{ij}\|$, so $1+\alpha \geqslant \|b_i\|$ for all $i \in I$. This proves $(b_i)_{i\in I} \in \prod_{i\in I} A_i$, so $\prod_{i\in I} A_i$ is complete in its norm, and a C*-algebra.

Because the C*-algebra operations are defined pointwise, $\pi_i : \prod_{i\in I} A_i \to A_i$ is easily seen to be a unital *-homomorphism for each $i \in I$. So we only have to prove the universal property of the product. Given a family of unital *-homomorphisms $(f_i)_{i\in I}$ where $f_i : B \to A_i$, B being a unital C*-algebra, we define $\langle f_i\rangle_{i\in I} : B \to \prod_{i\in I} A_i$ as follows, for each $b \in B$:

$$\langle f_i\rangle(b) = (f_i(b))_{i\in I}.$$

It is clear from the fact that the operations are defined pointwise that if this defines an element of $\prod_{i\in I} A_i$ for each $b \in B$, then $\langle f_i\rangle$ is a unital *-homomorphism, $\pi_i \circ \langle f_i\rangle = f_i$ for each $i \in I$ and $\langle f_i\rangle$ is the unique *-homomorphism with this property, so we only need to prove that $\langle f_i\rangle(b) \in \prod_{i\in I} A_i$.

As each f_i is a unital *-homomorphism, it has operator norm $\|f_i\| \leqslant 1$ [18, 1.3.7]. Therefore for all $i \in I$, $\|f_i(b)\| \leqslant \|b\|$, so we have proven that for each $b \in B$, $\langle f_i\rangle(b) = (f_i(b))_{i\in I}$ is uniformly bounded in I and therefore an element of $\prod_{i\in I} A_i$, as required. □

In categorical terms, the forgetful functor U : **C*Alg** $\to$ **Set** that takes a C*-algebra to its underlying set does *not* preserve products. However, the forgetful functor Ball : **C*Alg** $\to$ **Set**, taking a C*-algebra to its closed unit ball, not only preserves products, as seen above, but in fact has a left adjoint making **C*Alg** monadic over **Set** by this functor [24] [25, Lemma 3.1].

The following characterization of the order relation in products is useful, and we will use it later without explicit mention.

Lemma 2.8. *Let $A = \prod_{i \in I} A_i$ be a product C*-algebra. An element $(a_i)_{i \in I} \in A$ is positive iff for all $i \in I$, the element a_i is positive in A_i. Therefore $(a_i)_{i \in I} \leqslant (b_i)_{i \in I}$ iff for all $i \in I$, $a_i \leqslant b_i$.*

Proof. If $(a_i)_{i \in I}$ is positive, then there exists $(b_i)_{i \in I}$ such that $(b_i)^*_{i \in I}(b_i)_{i \in I} = (a_i)_{i \in I}$, which is equivalent to $b_i^* b_i = a_i$ for all $i \in I$, and therefore shows that $a_i \in A$ is positive for all $i \in I$. If $a_i \in A_i$ is positive for all $i \in I$, there exist $b_i \in A_i$ with $b_i^* b_i = a_i$, and therefore $(b_i)^*_{i \in I}(b_i)_{i \in I} = (a_i)_{i \in I}$, so $(a_i)_{i \in I}$ is positive in A.

It follows that

$$\begin{aligned}(a_i)_{i \in I} \leqslant (b_i)_{i \in I} &\Leftrightarrow (b_i)_{i \in I} - (a_i)_{i \in I} \in A_+ \Leftrightarrow \forall i \in I. b_i - a_i \in (A_i)_+ \\ &\Leftrightarrow \forall i \in I. a_i \leqslant b_i.\end{aligned}$$

□

An important fact is that a finite-dimensional C*-algebra A is always of the form $A \cong \prod_{i \in I} B(\mathcal{H}_i)$ where I is a finite set and $\mathcal{H}_i$ a finite-dimensional Hilbert space [19, Chapter I, Theorem 11.2].

For certain arguments we need the following notion, which is a restriction to positive operators of the more general notions of a left and right support projection.

Definition 2.9. *Let A be a C*-algebra and $a \in A_+$ a positive element. We say that a projection $p \in \mathrm{Proj}(A)$* supports *a iff $pa = a$. It is equivalent to say that $ap = a$ and $pap = a$. A* support projection *for a, written $\mathrm{supp}(a)$, is a projection $p \in \mathrm{Proj}(A)$ that is the smallest projection supporting a (if such a thing exists).*

Proof. We show that the alternative definitions for p supporting a are equivalent to each other. Since a is positive, it is self-adjoint, so $pa = a$ iff $(pa)^* = a^*$ iff $ap = a$. If $pap = a$, then $pa = ppap = pap = a$, and if $pa = a$ then since also $ap = a$, we have $a = pa = p(ap) = pap$. □

Since it is defined in terms of the C*-algebra structure, it is clear that if $i : A \to B$ is a *-isomorphism of C*-algebras, and p is the support projection of a in A, then $i(p)$ is the support of $i(a)$ in B.

Proposition 2.10. *Let $\mathcal{H}$ be a Hilbert space and $a \in B(\mathcal{H})$ an operator. Define its* null space *or* kernel *to be $\ker(a) = a^{-1}(0)$ and the* support *to be the orthogonal complement $\ker(a)^\perp$. Then the the projection onto $\ker(a)^\perp$ is the support projection $\mathrm{supp}(a)$ in $B(\mathcal{H})$.*

For a product C-algebra $A = \prod_{i \in I} B(\mathcal{H}_i)$, we have $\mathrm{supp}((a_i)_{i \in I}) = (\mathrm{supp}(a_i))_{i \in I}$.*

Proof. By linearity and continuity, $a^{-1}(0)$ is a closed linear subspace of $\mathcal{H}$ and so is it is an orthogonal complement, $\ker(a)^\perp$ is a closed subspace of $\mathcal{H}$ so we can define the projection $p \in B(\mathcal{H})$ that projects onto it. Each $\psi \in \mathcal{H}$ can be uniquely expressed as $\phi_1 + \phi_2$ with $\phi_1 \in \ker(a)$ and $\phi_2 \in \ker(a)^\perp$. So for all $\psi \in \mathcal{H}$:

$$ap(\psi) = ap(\phi_1 + \phi_2) = a(\phi_2) = a(\phi_1 + \phi_2) = a(\psi),$$

and so p supports a. Let q be a projection that supports a, *i.e.* $aq = a$. For all $\psi \in \mathcal{H}$, if $q(\psi) = 0$, then $a(\psi) = aq(\psi) = a(0) = 0$, so $\ker(q) \subseteq \ker(a)$, and therefore $\ker(a)^\perp \subseteq \ker(q)^\perp$, so $p \leqslant q$. This proves that p is the support of a.

Now consider a product $A = \prod_{i \in I} B(\mathcal{H}_i)$ and let $(a_i)_{i\in I} \in A_+$. Then each $a_i \in B(\mathcal{H}_i)_+$ and has a support p_i, as defined above. Then $(a_i)_{i\in I}(p_i)_{i\in I} = (a_i p_i)_{i\in I} = (a_i)_{i\in I}$, and if $(q_i)_{i\in I}$ is a projection such that $(a_i)_{i\in I}(q_i)_{i\in I} = (a_i)_{i\in I}$, then for each $i \in I$ we have $a_i q_i = a_i$, so $p_i \leqslant q_i$. It follows that $(p_i)_{i\in I} \leqslant (q_i)_{i\in I}$. □

An operator is injective iff its kernel is $\{0\}$, and therefore iff $\mathrm{supp}(a) = 1$. In the case that $\mathcal{H}$ is finite-dimensional, an operator $\mathcal{H} \to \mathcal{H}$ is injective iff it is invertible, so $\mathrm{supp}(a) = 1$ characterizes invertible operators. This does not hold if $\mathcal{H}$ is infinite-dimensional.

The following facts about the support are used in the next section.

Lemma 2.11. *Let $\mathcal{H}$ be a Hilbert space.*

(i) If $a \in B(\mathcal{H})_+$, then $\psi \in \ker(a)$ iff $\langle \psi, a(\psi) \rangle = 0$.

(ii) Let $a, b \in B(\mathcal{H})_+$. Then $\mathrm{supp}(a + b) = \mathrm{supp}(a) \vee \mathrm{supp}(b)$.

Proof. (i) If $\psi \in \ker(a)$, then $\langle \psi, a(\psi)\rangle = \langle \psi, 0\rangle = 0$. For the other direction, $0 = \langle \psi, a(\psi)\rangle = \langle a^{\frac{1}{2}}(\psi), a^{\frac{1}{2}}(\psi)\rangle = \|a^{\frac{1}{2}}(\psi)\|^2$, so $a^{\frac{1}{2}}(\psi) = 0$. Therefore $a(\psi) = a^{\frac{1}{2}}(a^{\frac{1}{2}}(\psi)) = a^{\frac{1}{2}}(0) = 0$, so $\psi \in \ker(a)$.

(ii) First we show that $\ker(a + b) = \ker(a) \wedge \ker(b)$, and then the statement follows from the fact that $\text{-}^{\perp}$ is an order-reversing bijection. If $\psi \in \ker(a) \wedge \ker(b)$, then $(a+b)(\psi) = a(\psi)+b(\psi) = 0+0 = 0$, so $\psi \in \ker(a+b)$. For the other direction, if $\psi \in \ker(a + b)$, then by part (i), $\langle \psi, (a + b)(\psi)\rangle = 0$, so $\langle \psi, a(\psi)\rangle + \langle \psi, b(\psi)\rangle = 0$. As a and b are positive, this implies $\langle \psi, a(\psi)\rangle = 0 = \langle \psi, b(\psi)\rangle$, which implies $\psi \in \ker(a) \wedge \ker(b)$ by part (i). □

Definition 2.12. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K} \subseteq \mathcal{H}$ a closed subspace. We write $B(\mathcal{K}|\mathcal{H})$ for the operators $a \in B(\mathcal{H})$ that are "defined on $\mathcal{K}$",* i.e. *$\mathrm{supp}(a) \subseteq \mathcal{K}$ and $a(\mathcal{H}) \subseteq \mathcal{K}$.*

*Letting $i : \mathcal{K} \to \mathcal{H}$ be the inclusion mapping and $i^* : \mathcal{H} \to \mathcal{K}$ its adjoint, the mapping $a \mapsto i \circ a \circ i^* : B(\mathcal{K}) \to B(\mathcal{K}|\mathcal{H})$ is a *-isomorphism with inverse the map $a \mapsto a|_{\mathcal{K}} : B(\mathcal{K}|\mathcal{H}) \to B(\mathcal{K})$.*

Proof. Let us write $f(a) = i \circ a \circ i^*$. This is linear by bilinearity of composition. We have

$$f(a)^* = (i \circ a \circ i^*)^* = i \circ a^* \circ i^* = f(a^*)$$

and

$$f(a) \circ f(b) = i \circ a \circ i^* \circ i \circ b \circ i^* = i \circ a \circ b \circ i^* = f(a \circ b),$$

so f is a *-homomorphism $B(\mathcal{K}) \to B(\mathcal{H})$. It is injective, because if $f(a) = f(b)$, then for all $\psi \in \mathcal{K}$ we have $i^*(\psi) = \psi$, so

$$f(a)(\psi) = i(a(i^*(\psi))) = a(\psi)$$

and $f(b)(\psi) = b(\psi)$, so $a = b$.

If $a \in B(\mathcal{K})$, then $f(a) \in B(\mathcal{K}|\mathcal{H})$, by the following reasoning. If $\psi \in \mathcal{H}$, we have $f(a)(\psi) = i(a(i^*(\psi)))$, which since the range of a is in $\mathcal{K}$, is in $\mathcal{K}$. So the range of $f(a)$ is $\mathcal{K}$. If $\psi \in \mathcal{K}\perp$, then for all $\phi \in \mathcal{H}$ we have $\langle \phi, i^*(\psi)\rangle_{\mathcal{K}} = \langle i(\phi), \psi\rangle_{\mathcal{H}} = 0$ because $i(\phi) \in \mathcal{K}$. Therefore $i^*(\psi) = 0$, so $f(a)(\psi) = 0$, which shows that $f(a)$ is supported on $\mathcal{K}$ and so is an element of $B(\mathcal{K}|\mathcal{H})$.

Finally, we show that f is surjective onto $B(\mathcal{K}|\mathcal{H})$. If $b \in B(\mathcal{K}|\mathcal{H})$, then define $a = b|_{\mathcal{K}}$. Since the range of b is in $\mathcal{K}$, $a \in B(\mathcal{K})$, and we aim to show $f(a) = b$. For all $\psi \in \mathcal{H}$, we can write it as $\psi = \phi_{\mathcal{K}} + \phi_{\mathcal{K}^\perp}$, where $\phi_{\mathcal{K}} \in \mathcal{K}$ and $\phi_{\mathcal{K}^\perp} \in \mathcal{K}^\perp$. Then since $b \in B(\mathcal{K}|\mathcal{H})$ we have $b(\psi) = b(\phi_{\mathcal{K}})$. So $f(a)(\psi) = i(a(i^*(\psi))) = a(\phi_{\mathcal{K}}) = b(\phi_{\mathcal{K}})$, and therefore $f(a) = b$. □

Lemma 2.13. *Let $a, b \in B(\mathcal{H})$ be positive, and $p = \mathrm{supp}(a)$, with $\mathcal{K}$ the corresponding subspace. If $b \leqslant a$, then $\ker(a) \subseteq \ker(b)$ and so $\mathrm{supp}(b) \leqslant \mathrm{supp}(a)$, and $b = bp = pb = pbp$, and $b \in B(\mathcal{K}|\mathcal{H})$.*

Proof. If $\psi \in \ker(a)$, then

$$0 \leqslant \langle \psi, b(\psi)\rangle \leqslant \langle \psi, a(\psi)\rangle = \langle \psi, 0\rangle = 0,$$

so by Lemma 2.11 (i), $\psi \in \ker(b)$. Therefore $\ker(a) \subseteq \ker(b)$, and it follows by the fact that $-^\perp$ is order reversing that $\mathrm{supp}(b) \leqslant \mathrm{supp}(a)$.

So for each $\phi \in \mathcal{H}$

$$b(\phi) = b((1-p)(\phi) + p(\phi)) = b((1-p)(\phi)) + b(p(\phi)) = b(p(\phi))$$

because $1 - p$ is the projection onto $\ker(a) = \mathcal{K}^\perp$. Therefore $b = bp$. Taking adjoints, $b = b^* = p^*b^* = pb$, and combining these two facts, $b = bp = pbp$. So b vanishes on $\mathcal{K}^\perp$, and its range lies in $\mathcal{K}$, so $b \in B(\mathcal{K}|\mathcal{H})$. □

We need some results about how directed suprema behave under multiplication and the relationship between different notions of directed completeness and continuity.

Lemma 2.14. *Let A be a C*-algebra, $(a_i)_{i\in I}$ a directed set that has a supremum a.*

(i) Let $\beta \in \mathbb{R}_{\geqslant 0}$. Then $\beta a = \bigvee_{i\in I} \beta a_i$.

(ii) Let $b \in A$. Then $a + b = \bigvee_{i\in I}(a_i + b)$.

Proof.

(i) If $\beta = 0$, then this is true because $0 = 0$. If $\beta \neq 0$, we reason as follows. We have $a_i \leqslant a$, so $a - a_i \in A_+$, so $\beta a - \beta a_i \in A_+$, as it is a cone, so $\beta a \geqslant \beta a_i$. Therefore βa is an upper bound for $(\beta a_i)_{i\in I}$. Suppose $b \geqslant \beta a_i$ for all $i \in I$. Then $\beta^{-1} b \geqslant a_i$ for all $i \in I$, so $\beta^{-1} b \geqslant a$, and therefore $b \geqslant \beta a$.

(ii) As $a_i \leqslant a$, $a - a_i \in A_+$, so $a + b - (a_i + b) \in A_+$, so $a_i + b \leqslant a + b$ for all $i \in I$, and therefore $a + b$ is an upper bound for $(a + b_i)_{i\in I}$. Suppose $c \geqslant a_i + b$ for all $i \in I$. Then $c - b \geqslant a_i$ for all $i \in I$, so $c - b \geqslant a$, and $c \geqslant a + b$. □

Lemma 2.15. *Let A be a C*-algebra, and $a, b \in A$, and $\beta \in \mathbb{R}_{>0}$. Then $a \ll b$ implies $\beta a \ll \beta b$.*

Proof. Let $a \ll b$, and let $(a_i)_{i\in I}$ be a directed set with supremum $\bigvee_{i\in I} a_i \geqslant \beta b$. By Lemma 2.14 (i), $\bigvee_{i\in I} \beta^{-1} a_i = \beta^{-1} \bigvee_{i\in I} a_i \geqslant b$, so there exists $i \in I$ such that $\beta^{-1} a_i \geqslant a$. Therefore $a_i \geqslant \beta a$ for this i. As this holds for any directed set with supremum exceeding βb, we have proved $\beta a \ll \beta b$. □

Proposition 2.16. *Let A be a C^*-algebra*

(a) The following are equivalent:

(i) A is bounded directed complete.

(ii) A_+ is bounded directed complete.

(iii) $[0,1]_A$ is a dcpo.

(b) The following are equivalent when A is a bounded directed complete C^*-algebra:

(i) A_+ is continuous.

(ii) $[0,1]_A$ is continuous.

Proof.

(a)

- (i) $\Rightarrow$ (ii):
Let $(a_i)_{i\in I}$ be a bounded directed set in A_+. Then it is a bounded directed set in A, so there exists $a = \bigvee_{i\in I} a_i$. Pick $j \in I$, and then $a \geqslant a_j \geqslant 0$, so $a \in A_+$. Therefore A_+ is bounded directed complete.

- (ii) $\Rightarrow$ (iii):
Let $(a_i)_{i\in I}$ be a directed set in $[0,1]_A$. As $1 \geqslant a_i$ for all $i \in I$, it is a bounded set in A_+ and so has a supremum $a \in A_+$. As 1 is an upper bound for $(a_i)_{i\in I}$, $a \leqslant 1$ so is the supremum in $[0,1]_A$. Therefore $[0,1]_A$ is directed complete.

- (iii) $\Rightarrow$ (i):
Let $(a_i)_{i\in I}$ be a directed set that is bounded above. Pick $i_0 \in I$, and define $J = \uparrow i_0$. Then $(a_j)_{j\in J}$ is cofinal in $(a_i)_{i\in I}$, because $(a_i)_{i\in I}$ is directed. Define $(b_j)_{j\in J}$ by $b_j = a_j - a_{i_0}$. Let $b \in A$ be an upper bound for $(a_i)_{i\in I}$ (equivalently for $(a_j)_{j\in J}$), and therefore $b - a_{i_0}$ is an upper bound for $(b_j)_{j\in J}$. By Lemma 2.4 (iv) there exists $n \in \mathbb{N}$ such that $n \cdot 1 \geqslant b - a_{i_0}$, so $b_j \leqslant n \cdot 1$ for all $j \in J$. We can therefore define $(c_j)_{j\in J}$ by $c_j = \frac{1}{n} b_j$, which is a directed set in $[0,1]_A$. Let $c = \bigvee_{j\in J} c_j$. By Lemma 2.14 (i), $nc = \bigvee_{j\in J} b_j$, and by Lemma 2.14 (ii), $nc + a_{i_0} = \bigvee_{j\in J} a_j = \bigvee_{i\in I} a_i$.

(b)

- (i) $\Rightarrow$ (ii):
Let $a \in [0,1]_A$. Then $a \in A_+$ and $\Downarrow a \cap [0,1]_A = \Downarrow a \cap A_+$ because $b \ll a$ implies $b \leqslant a \leqslant 1$. So $\Downarrow a \cap [0,1]_A$ is directed and $a = \bigvee \Downarrow a \cap [0,1]_A$, proving $[0,1]_A$ is continuous.

- (ii) $\Rightarrow$ (i):
Let $a \in A_+$. By Lemma 2.4 (iv), there exists $n \in \mathbb{N}$ such that $a \leqslant n \cdot 1$. Therefore $0 \leqslant \frac{1}{n} a \leqslant 1$. By the assumption that $[0,1]_A$ is continuous, $\Downarrow \frac{1}{n} a$ is directed and $\bigvee \Downarrow \frac{1}{n} a = \frac{1}{n} a$. If $b \in n \Downarrow \frac{1}{n} a$ then $\frac{1}{n} b \ll \frac{1}{n} a$ so $b \ll a$ (Lemma 2.15). Similarly if $b \ll a$, $b \in n \Downarrow \frac{1}{n} a$, so $n \Downarrow \frac{1}{n} a = \Downarrow a$. As $\Downarrow \frac{1}{n} a$ is directed, $\Downarrow a = n \Downarrow \frac{1}{n} a$ is directed, and $\bigvee \Downarrow a = \bigvee n \Downarrow \frac{1}{n} a = n \bigvee \Downarrow \frac{1}{n} a = n \frac{1}{n} a = a$, by Lemma 2.14 (i). $\square$

In view of the above, we will simply say a C*-algebra A is *directed complete* if we mean that A or A_+ is bounded directed complete or $[0,1]_A$ is directed complete, and we will say that A is continuous if we mean that A_+ or $[0,1]_A$ is continuous. For technical reasons, we are unable in the infinite-dimensional case to prove that if A is continuous as a poset under its natural order, then A_+ and $[0,1]_A$ are continuous, and we do not know of any counterexample either.

3 Domain Theory in Finite-Dimensional C*-Algebras

In this section *all Hilbert spaces are finite-dimensional.*

We provide some proofs of facts that are somewhat well known in the infinite-dimensional case, expressed in finite-dimensional terms that should be accessible for readers who understand [7, Chapter 2]. For $\mathcal{H}$ a Hilbert space, we write $\mathrm{SA}(\mathcal{H}) = B(\mathcal{H})_{\mathrm{sa}}$ for the $\mathbb{R}$-Banach space of self-adjoint operators. It is helpful to recall that $\mathrm{sp}(a)$ is simply the set of eigenvalues of a for a finite-dimensional $\mathcal{H}$.

Lemma 3.1. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. The following sets are the same.*

(i) The norm unit ball: $U = \{a \in \mathrm{SA}(\mathcal{H}) \mid \|a\| \leqslant 1\}$.

(ii) The interval from -1 to 1 in the Löwner order: $[-1,1]_{\mathcal{H}} = \{a \in \mathrm{SA}(\mathcal{H}) \mid -1 \leqslant a \leqslant 1\}$.

(iii) The set of self-adjoint operators with spectrum in $[-1,1]$: $\{a \in \mathrm{SA}(\mathcal{H}) \mid \mathrm{sp}(a) \subseteq [-1,1]\}$.

So for a finite-dimensional C-algebra A,* $\mathrm{Ball}(A_{\mathrm{sa}}) = [-1,1]_A$.

Proof. The equivalence of (ii) and (iii) follows directly from Lemma 2.4.

- $\|a\| \leqslant 1 \Rightarrow \mathrm{sp}(a) \subseteq [-1,1]$:

  Suppose for a contradiction that $\|a\| \leqslant 1$ and there exists an eigenvalue with $|\lambda| > 1$. Let ψ be an eigenvector of Hilbert norm 1 with eigenvalue λ. Then
  $$\|a(\psi)\|^2 = \langle \psi, a^2(\psi)\rangle = \lambda^2\langle \psi, \psi\rangle = \lambda^2 > 1.$$
  Taking square roots, $\|a(\psi)\| > 1$, which, as $\|\psi\| = 1$, contradicts $\|a\| \leqslant 1$.

- $\mathrm{sp}(a) \subseteq [-1,1] \Rightarrow \|a\| \leqslant 1$:

  Let $(\psi_i)_{i\in I}$ be an orthonormal basis of eigenvectors of a, $(\lambda_i)_{i\in I}$ the corresponding eigenvalues. Let $\phi \in \mathcal{H}$ with $\|\phi\| \leqslant 1$, which we write as $\sum_{i\in I} \alpha_i\psi_i$, so $\sum_{i\in I} \overline{\alpha}_i\alpha_i \leqslant 1$. By the defining property of eigenvectors
  $$\|a(\phi)\|^2 = \left\langle \sum_{i\in I} \lambda_i\alpha_i\psi_i, \sum_{i\in I} \lambda_i\alpha_i\psi_i \right\rangle = \sum_{i\in I} \lambda_i^2\overline{\alpha}_i\alpha_i \leqslant 1$$
  because $\sum_{i\in I} \overline{\alpha}_i\alpha_i \leqslant 1$ and $0 \leqslant \lambda_i^2 \leqslant 1$, by the initial assumption. Taking square roots, $\|a(\psi)\| \leqslant 1$, so $\|a\| \leqslant 1$ in the operator norm.

For a C*-algebra $A = \prod_{i \in I} B(\mathcal{H}_i)$, by the definition of the product norm we have $\mathrm{Ball}(A_{\mathrm{sa}}) = \prod_{i \in I} \mathrm{Ball}(\mathrm{SA}(\mathcal{H}_i))$, and since the order is pointwise, the above implies $\mathrm{Ball}(A_{\mathrm{sa}}) = [-1, 1]_A$. Therefore this statement holds for any finite-dimensional C*-algebra. $\square$

Lemma 3.2. *The following are equivalent, for an operator $a \in B(\mathcal{H})$:*

(i) $a \in \mathrm{int}\,(B(\mathcal{H})_+)$, i.e. *a is in the norm interior of the positive cone.*

(ii) There exists $\epsilon > 0$ such that $a \geqslant \epsilon \cdot 1$.

(iii) $a \in B(\mathcal{H})_+$ and a is invertible.

Proof. We have $a \in \mathrm{int}\,(B(\mathcal{H})_+)$ iff there exists $\epsilon > 0$ such that $[a - \epsilon 1, a + \epsilon 1] \subseteq B(\mathcal{H})_+$ by Lemma 3.1 (ii). Then $[a - \epsilon 1, a + \epsilon 1] \subseteq B(\mathcal{H})_+$ iff $a - \epsilon 1 \in B(\mathcal{H})_+$ (*i.e.* $a \geqslant \epsilon 1$), and this in turn holds iff $\mathrm{sp}(a) \geqslant \epsilon$, by Lemma 2.4 (iii). As a is positive and $\mathrm{sp}(a)$ is closed, the existence of an $\epsilon > 0$ such that $\mathrm{sp}(a) \geqslant \epsilon$ is equivalent to $0 \notin \mathrm{sp}(a)$, which by definition is equivalent to a being invertible. $\square$

In order to work with the set of completely positive subunital maps between C*-algebras, we will often let our C*-algebras have another norm $|||\text{-}|||$, consider the positive part of the unit ball with respect to this norm, as a subposet of A_+:

$$\mathrm{Ball}_+(|||\text{-}|||) = A_+ \cap \mathrm{Ball}(|||\text{-}|||) = \{a \in A_+ \mid |||a||| \leqslant 1\}.$$

We reserve the notation $\|\text{-}\|$ for the C*-norm.

Lemma 3.3. *Let A be a finite dimensional C*-algebra, and $(a_i)_{i \in I}$ a directed set in A_{sa}. The following are equivalent for an element $a \in A$.*

(i) $a = \bigvee_{i \in I} a_i$.

(ii) a_i converges to a.

If $|||\text{-}|||$ is a norm on A, the same characterization holds for a directed set $(a_i)_{i \in I}$ contained in $\mathrm{Ball}_+(|||\text{-}|||)$ *and makes it into a dcpo.*

Proof.

- (i) $\Rightarrow$ (ii):

  As a finite-dimensional C*-algebra is reflexive and therefore a W*-algebra, we can apply [21, 1.7.4], which shows that (i) $\Rightarrow$ (ii) for weak-* convergence. Since all vector space topologies agree on a finite-dimensional vector space [26, I.3.2], this is the same as norm convergence.

- (ii) $\Rightarrow$ (i):

  First we show that a is an upper bound for $(a_i)_{i \in I}$, so let $i \in I$, aiming to prove that $a_i \leqslant a$. By the definition of convergence, for all $\epsilon > 0$, there exists $j \in I$ such that for all $k \geqslant j$, $\|a_k - a\| \leqslant \epsilon$. Since I is directed, there exists $k \geqslant i, j$, and we have $a_i \leqslant a_k$. We have $a_k - a \in \epsilon \mathrm{Ball}(A)$, so by Lemma 3.1, $-\epsilon 1 \leqslant a_k - a \leqslant \epsilon 1$, so $a_i \leqslant a_k \leqslant a + \epsilon 1$. It follows that $(a + \epsilon 1) - a_i \in A_+$, holds for all $\epsilon > 0$. Since $2^{-n} 1 \to 1$ and the cone A_+ is norm-closed we have

  $$a - a_i = \lim_{n \to \infty} a + 2^{-n} 1 - a_i \in A_+,$$

so $a_i \leqslant a$ for all $i \in I$, as required to prove a an upper bound.

Now suppose b is an upper bound for $(a_i)_{i\in I}$. Then $b - a_i \in A_+$ for all $i \in I$, and since A_+ is norm-closed, $b - a = \lim_{i\in I}(b - a_i) \in A_+$, making $a \leqslant b$.

Now let $|||\text{-}|||$ be a norm on A. All norms on finite-dimensional vector spaces are equivalent, so there is a constant $\alpha \in \mathbb{R}_{\geqslant 0}$ such that for all $a \in A$ we have $|||a||| \leqslant \alpha\|a\|$. Let $(a_i)_{i\in I}$ be a directed set in $B = \mathrm{Ball}_+(|||\text{-}|||)$. Then $B \subseteq \alpha[-1,1]_A$ (Lemma 3.1) and therefore $\alpha \cdot 1$ is an upper bound for $(a_i)_{i\in I}$ in A_+. So a least upper bound a exists in A_+. Since all norms are equivalent, $\mathrm{Ball}(|||\text{-}|||)$ is $\|\text{-}\|$-closed, as is A_+, and therefore so is B. Since $a_i \to a$, we have $a \in B$, from which it follows that $a = \bigvee_{i\in I} a_i$ as calculated in B.

In the other direction, if $(a_i)_{i\in I}$ is directed and converges to a, with everything happening in B, then a is the least upper bound of $(a_i)_{i\in I}$ in A_+, and so is also the least upper bound in B. □

Lemma 3.4. *Let a be a non-zero positive operator on $\mathcal{H}$ and $p = \mathrm{supp}(a)$. Then there exists an $N \in \mathbb{N}$ such that for all $i \geqslant N$, $a - 2^{-i}p$ is positive, and*

$$\bigvee_{i=N}^{\infty} (a - 2^{-i}p) = a. \tag{3.5}$$

in the positive cone of $B(\mathcal{H})$.

If A is a finite-dimensional C-algebra equipped with a norm $|||\text{-}|||$, the same holds inside $B = \mathrm{Ball}_+(|||\text{-}|||)$,* i.e. *there exists $N \in \mathbb{N}$ such that for all $i \geqslant N$, $a - 2^{-i}p \in B$ and* (3.5) *holds inside B.*

Proof. As a is positive, it is self-adjoint, and so as it is non-zero, it has a non-zero eigenvalue. Let $(\psi_j)_{j\in J}$ be an orthonormal basis of eigenvectors for a (J a finite set), $(\lambda_j)_{j\in J}$ their corresponding eigenvalues, and let $K \subseteq J$ be the indices such that $\lambda_j \neq 0$. Then $(\psi_k)_{k\in K}$ spans the support of a, because each ψ_k is orthogonal to the null space of a, and every vector in the support of a is expressible in terms of $(\psi_j)_{j\in J}$, but cannot use any of the ψ_j with $\lambda_j = 0$.

Let $\lambda > 0$ be the smallest nonzero eigenvalue of a. Let N be the smallest $N \in \mathbb{N}$ such that $2^{-N} \leqslant \lambda$, so for all $i \geqslant N$ and $k \in K$, $2^{-i} \leqslant \lambda_k$. Let $\phi \in \mathcal{H}$, and express it in terms of eigenvectors as $\sum_{j \in J} \alpha_j \psi_j$. Then

$$\begin{aligned}
\langle \phi, a(\phi) \rangle &= \left\langle \phi, a\left(\sum_{j \in J} \alpha_j \psi_j\right)\right\rangle \\
&= \left\langle \phi, \sum_{j \in J} \alpha_j \lambda_j \psi_j \right\rangle \\
&= \left\langle \phi, \sum_{k \in K} \alpha_k \lambda_k \psi_k \right\rangle \\
&= \sum_{k \in K} \lambda_k \langle \phi, \alpha_k \psi_k \rangle \\
&\geqslant \sum_{k \in K} 2^{-i} \langle \phi, \alpha_k \psi_k \rangle \\
&= \left\langle \phi, 2^{-i} \sum_{k \in K} \alpha_k \psi_k \right\rangle \\
&= \langle \phi, 2^{-i} p(\phi) \rangle,
\end{aligned}$$

so $a \geqslant 2^{-i}p$, *i.e.* $a - 2^{-i}p$ is positive.

Since the sequence $(a - 2^{-i}p)_{i=N}^{\infty}$ is directed and converges to a, we have $\bigvee_{i=N}^{\infty}(a - 2^{-i}p) = a$ by Lemma 3.3.

Now consider an algebra $A = \prod_{j \in J} B(\mathcal{H}_j)$ where J is finite and each $\mathcal{H}_j$ finite-dimensional, and let $a = (a_j)_{j \in J} \in A_+$ and $p_j = \mathrm{supp}(a_j)$ so that $p = (p_j)_{j \in J}$ is the support of a (Proposition 2.10). By what we have just proved, for each $j \in J$, there exist $N_j \in \mathbb{N}$ such that for all $i \geqslant N_j$ the element $a_j - 2^{-i}p_j \in B(\mathcal{H}_j)_+$. Since J is finite, we can pick N' to be the largest N_j, and then for all $i \geqslant N'$ we have $(a_j)_{j \in J} - 2^{-i}\mathrm{supp}((a_j)_{j \in J}) \in A_+$. Then by Lemma 3.3 we have $\bigvee_{i=N'} a - 2^{-i}p = a$.

If A is equipped with a norm $|||\text{-}|||$, and defining $B = \mathrm{Ball}_+(|||\text{-}|||)$ we have $a \in B$, then since $a - 2^{-i}p \to a$, there exists $N \in \mathbb{N}$ such that for all $i \geqslant N$ we have $a - 2^{-i}p \in \mathrm{Ball}(|||\text{-}|||)$. So if we define $N = \max M, N'$, for all $i \geqslant N$ we have $(a_j)_{j \in J} - 2^{-i}(p_j)_{j \in J} \in B$ and so by Lemma 3.3 (3.5) holds in B. □

As a warm-up, and for later comparison, we characterize the way-below relation in $\mathrm{SA}(\mathcal{H})$.

Lemma 3.6. *Let a, b be self-adjoint operators on a finite-dimensional Hilbert space $\mathcal{H}$. The following are equivalent:*

(i) $b \ll a$ *in* $\mathrm{SA}(\mathcal{H})$

(ii) $a - b$ *is in the interior of* $B(\mathcal{H})_+$

(iii) There exists $\epsilon > 0$ *such that* $a - b \geqslant \epsilon \cdot 1$.

Proof. The equivalence of (ii) and (iii) follows from Lemma 3.2. Suppose (iii) holds, and let $(c_i)_{i \in I}$ be a directed set with supremum $c \geqslant a$. Then $c \geqslant b + \epsilon \cdot 1$, so $c - b$ is in the interior of $B(\mathcal{H})_+$, by the same lemma. Since $c_i \to c$ (Lemma 3.3), we have $c_i - b \to c - b$, so there exists $i \in I$ such that $c_i - b \in B(\mathcal{H}_+)$, *i.e.* $c_i \geqslant b$. This proves (iii) implies (i).

Suppose that (i) holds. Observe that $(a - 2^{-i}1))_{i \in \mathbb{N}}$ is a monotone sequence in $\mathrm{SA}(\mathcal{H})$ converging to a, so a is its least upper bound (Lemma 3.3). It follows that there exists $i \in \mathbb{N}$ such that $a - 2^{-i}1 \geqslant b$, and therefore $a - b \geqslant 2^{-i}1$, proving (iii). $\square$

We now characterize the way-below relation on positive operators.

Lemma 3.7. *Let a, b be positive operators on $\mathcal{H}$, where $\mathcal{H}$ is finite-dimensional. The following are equivalent:*

(i) $b \ll a$ in $B(\mathcal{H})_+$

(ii) There exists $\epsilon > 0$ such that $b \leqslant a - \epsilon \cdot \mathrm{supp}(a)$.

For A a finite-dimensional C-algebra equipped with a norm $|||\text{-}|||$, the same characterization holds in $B = \mathrm{Ball}_+(|||\text{-}|||)$.*

Proof. Throughout, define $p = \mathrm{supp}(a)$ for short. We start with the Hilbert space case.

- (i) $\Rightarrow$ (ii):

  By Lemma 3.4 and the fact that $b \ll a$, there exists $i \in \mathbb{N}$ such that $b \leqslant a - 2^{-i}p$, so we can take $\epsilon = 2^{-i}$.

- (ii) $\Rightarrow$ (i):

  Suppose that $b \leqslant a - \epsilon p$ for some $\epsilon > 0$. Let $(c_i)_{i \in I}$ be a directed set of positive operators with supremum $c \geqslant a$. Let $\mathcal{K}$ be the support of c. If $\mathcal{K} = \{0\}$, then $c = 0$, so $a = b = 0$ and therefore $b \ll a$. So we now assume that $\mathcal{K} \neq \{0\}$ and therefore $c \neq 0$. As $c_i \leqslant c$ for all $i \in I$ and $a, b \leqslant c$, all these operators can be restricted to elements of $B(\mathcal{K})$ by Lemma 2.13, and by Lemma 2.5, $\bigvee_{i \in I} c_i = c$ in $B(\mathcal{K})$ and all other order relations that hold in $B(\mathcal{H})$ continue to hold in $B(\mathcal{K})$. In $B(\mathcal{K})$, we have $\mathrm{supp}(c) = 1$.

  As $a \leqslant c$, $c - a$ is positive, and since $(c-a)+a = c$, we have, by Lemma 2.11 (ii), $\mathrm{supp}(c-a) \vee \mathrm{supp}(a) = \mathrm{supp}(c) = 1$, these supports being calculated in $B(\mathcal{K})$. Then

  $$\begin{aligned}\mathrm{supp}(c - (a - \epsilon p)) &= \mathrm{supp}((c-a) + \epsilon p) = \mathrm{supp}(c-a) \vee \mathrm{supp}(\epsilon p) \\ &= \mathrm{supp}(c-a) \vee \mathrm{supp}(a) = 1,\end{aligned}$$

  using Lemma 2.11 (ii) again. Therefore $c - (a - \epsilon p)$ is invertible, by the finite-dimensionality of $\mathcal{K}$, and so $c - (a - \epsilon p)$ is in the norm interior of $B(\mathcal{K})_+$ (Lemma 3.2).

  By Lemma 2.14 (ii), $c - (a - \epsilon p) = \bigvee_{i \in I} c_i - (a - \epsilon p)$, and so $(c_i - (a - \epsilon p))_{i \in I}$ converges to $c - (a - \epsilon p)$ (Lemma 3.3) so there exists $i \in I$ such that $c_i - (a - \epsilon p) \in B(\mathcal{K})_+$. Therefore $c_i \geqslant a - \epsilon p \geqslant b$ in $B(\mathcal{K})$, so $b \leqslant c_i$ in $B(\mathcal{H})$ by Lemma 2.5. This proves $b \ll a$.

Let $A = \prod_{j \in J} B(\mathcal{H}_i)$ with J finite and $\mathcal{H}_i$ finite-dimensional Hilbert spaces, $|||\text{-}|||$ a norm on A, and let $a, b \in B = \mathrm{Ball}_+(|||\text{-}|||)$ such that $b \ll a$. The argument that (i) implies (ii) is essentially unchanged, using Lemma 3.4. For (ii) implies (i), we expand $a = (a_j)_{j \in J}$ and $b = (b_j)_{j \in J}$ and observe that $p = \mathrm{supp}(a)$ is of the form $(p_j)_{j \in J}$ with $p_j = \mathrm{supp}(a_j)$. Then by (ii) $\Rightarrow$ (i) of the Hilbert space case, we have $b_j \ll a_j$ in each $B(\mathcal{H}_j)_+$. If we have a directed set $(c_i)_{i\in I}$ in B with $\bigvee_{i\in I} c_i = c \geqslant b$, then for each $j \in J$ we have $\bigvee_{i\in I} c_{i,j} = c_j \geqslant b_j$ (in $B(\mathcal{H}_j)_+$, by Lemma 3.3), so there exists $n_j \in I$ such that $c_{n_j,j} \geqslant a_j$. Using directedness of I and finiteness of J, there exists $n \in I$ such that for all $j \in J$ we have $c_{n,j} \geqslant b_j$, and so $c_n \geqslant b$. This proves that $b \ll a$ in B. □

In [3, Example 2.7] Selinger characterized the way-below relation on $B(\mathcal{H})$ in a different way, and Keimel has yet another characterization in [27, Proposition 5.1]. We can now reprove what was first proven by Selinger in [3, Example 2.7, §5.1 Example], at a slightly different level of generality as we do not require that the norm be monotone (see [3, §2.3 Definition]), though this makes no difference for our application.

Theorem 3.8 (Selinger)**.** *Let A be a finite-dimensional C^*-algebra equipped with a norm $|\!|\!|\text{-}|\!|\!|$. Then $\mathrm{Ball}_+(|\!|\!|\text{-}|\!|\!|)$ is a continuous dcpo.*

Proof. We have already seen that $B = \mathrm{Ball}_+(|\!|\!|\text{-}|\!|\!|)$ is a dcpo in Lemma 3.3. So we need to show that $\twoheaddownarrow a$ is directed and $a = \bigvee \twoheaddownarrow a$ for all $a \in B$. Let $p = \mathrm{supp}(a)$ throughout. If $b_1, b_2 \in B$ and $b_1, b_2 \ll a$, then by Lemma 3.7 there exist $\epsilon_1, \epsilon_2 > 0$ such that $b_i \leqslant a - \epsilon_i p$ for $i \in \{1,2\}$. Taking $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, we have $b_i \leqslant a - \epsilon p \ll a$ (by Lemma 3.7).

The sequence $(a - 2^{-i}p)_{i=N}^{\infty}$ from Lemma 3.4 is a subset of $\twoheaddownarrow a$ by Lemma 3.7, so

$$a = \bigvee_{i=N}^{\infty} (a - 2^{-i}p) \leqslant \bigvee \twoheaddownarrow a \leqslant a,$$

the last inequality being because a is an upper bound for $\twoheaddownarrow a$. □

We note the following consequence of Theorem 3.8 that we use later.

Theorem 3.9. *Let X be a set (no longer required to be finite), and $(\mathcal{H}_x)_{x\in X}$ a family of finite-dimensional Hilbert spaces. Then $\prod\limits_{x\in X} B(\mathcal{H}_x)$ is a continuous directed-complete C^*-algebra. The way-below relation is characterized by $(a_x)_{x\in X} \ll (b_x)_{x\in X}$ iff there exists a finite subset $S \subseteq X$ such that $a_x = 0$ for all $x \in X\backslash S$, and for all $x \in S$ $a_x \ll b_x$.*

Proof. For convenience, we write $A_x = B(\mathcal{H}_x)$ and $A = \prod_{x\in X} A_x$. By Theorem 3.8, if we take $|\!|\!|\text{-}|\!|\!| = \|\text{-}\|$, the unit interval $[0,1]_{A_x}$ is a continuous dcpo for all $x \in X$. The order on $[0,1]_A$ is the product ordering and $[0,1]_A$ is the poset product $\prod_{x\in X}[0,1]_{A_x}$.

By [14, Proposition I-2.1 (ii)], $\prod_{x\in X}[0,1]_{A_x}$ is a continuous dcpo, and $(a_x)_{x\in X} \ll (b_x)_{x\in X}$ iff there exists a finite set $S \subseteq X$ such that $a_x = 0$ except when $x \in S$, and $a_x \ll b_x$ for all $x \in X$, which, as $0 \ll b_x$, is satisfied iff $a_x \ll b_x$ for all $x \in S$.

Therefore, by Proposition 2.16 (b), A is continuous. □

We now show why we cannot approximate arbitrary completely positive maps using directed joins from a fixed countable set. For this we need the notion of a *basis* for a continuous dcpo [14, Definition III-4.1]. To avoid confusion with the linear notion of basis for a vector space, we will refer to this as a *base* instead. A *base* of a dcpo D is a set $B \subseteq D$ such that for all $d \in D$, $\twoheaddownarrow d \cap B$ is directed, and $d = \bigvee \twoheaddownarrow d \cap B$. A dcpo D has a base iff it is continuous, and it is immediate from the definition that if D is continuous, D is a base.

Lemma 3.10. *Let A be a noncommutative finite-dimensional C*-algebra, so $A \cong \prod_{j\in J} B(\mathcal{H}_j)$ where for at least one j, $\dim(\mathcal{H}_j) \geqslant 2$. Let $|||\text{-}|||$ be a norm on A. Then any base of $\mathrm{Ball}_+(|||\text{-}|||)$ has cardinality $2^{\aleph_0}$.*

Proof. Let $D \subseteq B = \mathrm{Ball}_+(|||\text{-}|||)$ be a base. Let $k \in J$ be such that $\dim(\mathcal{H}_k) \geqslant 2$, and define P to be the set of elements $(a_j)_{j\in J} \in A$ such that a_k is a 1-dimensional projection and $a_j = 0$ for all $j \neq k$.

For each $p \in P$, $|||p||| > 0$ so we can define q_p to be equal to p if $|||p||| \leqslant 1$, and otherwise $q_p = \frac{p}{|||p|||}$. Then $q_p \leqslant p$ and $q_p \in B$. Since D is a base, there must exist $d_p \in D$ such that $d_p \ll q_p$, and therefore $d_p \leqslant q_p$. Fix such a d_p for each $p \in P$, and we show that the map $p \mapsto d_p$ is injective. Let $p, p' \in P$ such that $d_p = d_{p'}$. Define $\mathcal{K}, \mathcal{K}'$ to be the respective 1-dimensional subspaces of $\mathcal{H}_k$ corresponding to p_k, p'_k. Since $d_p \leqslant q_p \leqslant p$, we have $d_{p,k} \leqslant p_k$ and so $d_{p,k} \in B(\mathcal{K}|\mathcal{H})$, and likewise $d_{p',k} \in B(\mathcal{K}'|\mathcal{H})$. Since $d_{p,k} = d_{p',k}$ are non-zero and $\mathcal{K}, \mathcal{K}'$ are 1-dimensional, we have $\mathcal{K} = \mathcal{K}'$ and therefore $p_k = p'_k$. Since $p_j, p'_j = 0$ for all other $j \in J$, we have $p = p'$. All together, we have defined an injective map $P \to D$.

Since $\mathcal{H}_k$ is finite-dimensional and at least 2-dimensional the cardinality of P is that of the continuum, $2^{\aleph_0}$. Therefore this is true of D as well. □

We write $\mathbf{CP}(A, B)$ for the set of completely positive maps $A \to B$ and $\mathbf{CPSU}(A, B)$ for the set of completely positive subunital maps $A \to B$. By subunital, we mean that $f : A \to B$ has $f(1) \leqslant 1$. These are superoperators in the Heisenberg picture. In the Schrödinger picture they correspond to maps $f : B \to A$ that are completely positive and trace-reducing (*a.k.a.* trace nonincreasing).

Theorem 3.11. *Let A, B be finite-dimensional C*-algebras, one of which is noncommutative. If a set $D \subseteq \mathbf{CPSU}(A, B)$ of complete positive subunital maps is such that for all $f \in \mathbf{CPSU}(A, B)$ there exists a directed set $(f_i)_{i\in I}$ in D with $f = \bigvee_{i\in I} f_i$, then D is uncountable.*

Proof. Choi's version [28] of the Choi-Jamiołkowski isomorphism [29] gives, for each $n, m \in \mathbb{N}$, an isomorphism between the matrix algebra M_{nm} and the set of linear maps $M_n \to M_m$ that is itself linear and restricts to an isomorphism between $(M_{nm})_+$ and $\mathbf{CP}(M_n, M_m)$. This extends to an isomorphism between $(A \otimes B)_+$ and $\mathbf{CP}(A, B)$ for finite-dimensional C*-algebras A, B. We have that if at least one of A, B is noncommutative, then so is $A \otimes B$.

A map $f \in \mathbf{CP}(A, B)$ is subunital iff its operator norm[6] is $\leqslant 1$ [30, Lemma 5.3]. So under the Choi-Jamiołkowski isomorphism we have an isomorphism of $\mathbf{CPSU}(A, B)$ with $\mathrm{Ball}_+(|||\text{-}|||)$ for some norm $|||\text{-}|||$ on $A \otimes B$, and $\mathbf{CPSU}(A, B)$ is therefore a continuous dcpo (Theorem 3.8). If $D \subseteq \mathbf{CPSU}(A, B)$ is a set as described in the statement of the theorem, then it follows from continuity of $\mathbf{CPSU}(A, B)$ as a dcpo that D is a base [14, Proposition III-4.2 (5) $\Rightarrow$ (1)]. So Lemma 3.10 implies D is uncountable. □

At this point we note that Theorem 3.8 together with Lemma 3.7 the above shows that Keimel's claim [27, Proposition 5.3] that the Lawson topology on the "kegelspitz", the set of positive operators of trace $\leqslant 1$, agrees with the Euclidean topology is false. If p is a 1-dimensional projection in $B(\mathbb{C}^2)$, for instance, the

[6] As an operator $A \to B$ with respect to their C*-norms.

way-up set $\twoheaduparrow \frac{p}{2}$ is Scott open [14, Theorem II-1.14 (1) $\Rightarrow$ (2)] and therefore Lawson open, and contains p, but we can find a sequence of 1-dimensional projections $p_i \to p$ in the Euclidean topology such that $p_i \neq p$ for all $i \in \mathbb{N}$ and therefore $\frac{p}{2} \not\ll p_i$ for any $i \in \mathbb{N}$, which shows that $\twoheaduparrow \frac{p}{2}$ is not open in the restriction of the Euclidean topology of $B(\mathbb{C}^2)$ to the kegelspitz.

Theorem 3.11 can also be applied to the domain theory of spacetimes (see [31, 32]), because $3+1$-dimensional Minkowski spacetime is isomorphic to $\mathrm{SA}(\mathbb{C}^2)$, the self-adjoint 2×2 matrices. We have to be careful with the way-below relation – the one used in [31] is for the whole space as a poset, not the positive cone, and so corresponds to Lemma 3.6 rather than Lemma 3.7.

Theorem 3.12. *Let M be $3+1$-dimensional Minkowski space, and we write $\leqslant$ for the causal order. If $D \subseteq M$ is a set such that for each pair of events $x, y \in M$ with $x \leqslant y$, there exists a causally directed set $(z_i)_{i \in I}$ in D with $x \leqslant z_i$ for all $i \in I$ and $\bigvee_{i \in I} z_i = y$, then D is uncountable.*

Proof. The map f that takes $(t, x, y, z) \in M$ to

$$t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

is a linear isomorphism of M with $\mathrm{SA}(\mathbb{C}^2)$ that maps the causal forward cone of the origin in M to the positive matrices.

So if D is a set with the required property, then $f(D) \cap [0,1]_{M_2}$ is a set with the property required in Theorem 3.11, so is uncountable. It follows that D is uncountable. $\square$

The above could be generalized to $1+n$-dimensional Minkowski space by using JB-algebras instead of C*-algebras, but we omit this for reasons of space.

4 Characterization of Continuous C*-algebras

In this section, we show that if the unit interval of a C*-algebra A is a continuous dcpo, then A is a product of finite-dimensional matrix algebras. This includes the case that A is a W*-algebra, but there are directed-complete C*-algebras that are not W*-algebras[7].

In general, an induced sub-dcpo of a continuous dcpo is not necessarily continuous [14, Exercise I-2.19]. So we need an extra condition to pass continuity to a sub-dcpo. This is provided by the following lemma, an extension of [14, Theorem I-2.7], better adapted to directed-complete C*-algebras because it does not require that the ambient poset be a lattice.

For clarity, for each join or meet we take, we write the name of the poset in which it is interpreted, so $\bigvee^E_{i \in I} x_i$ is the least upper bound of $(x_i)_{i \in I}$ in E. We need such a notation because if E is contained in a larger poset F there may be a smaller upper bound in F.

Lemma 4.1. *Let D be a continuous dcpo, and $E \subseteq D$ a complete lattice in the induced ordering, such that the inclusion mapping preserves all non-empty meets, and directed joins. Then E is a continuous lattice.*

[7] An example of one is the bounded Borel-measurable functions on $[0,1]$ modulo meagre sets (if it were modulo sets of Lebesgue measure 0, this would be a W*-algebra).

Proof. We use condition (DD) of [14, Theorem I-2.7], which is to say, let J be a non-empty set, $\{K_j\}_{j \in J}$ a J-indexed family of posets, $\{x_{j,k}\}_{j \in J, k \in K_j}$ be a family of elements in E such that for all $j \in J$, $\{x_{j,k}\}_{k \in K_j}$ is directed, then we want to show

$$\overset{E}{\bigwedge_{j \in J}} \overset{E}{\bigvee_{k \in K_j}} x_{j,k} = \overset{E}{\bigvee_{f \in \prod\limits_{j \in J} K_j}} \overset{E}{\bigwedge_{j \in J}} x_{j,f(j)}, \tag{4.2}$$

where we have written E above the lattice operations to emphasize that they should be calculated in E, rather than D. This is a kind of distributivity property that holds iff E is a continuous lattice by [14, Theorem I-2.7].

The proof is by showing the inequality in each direction.

- $\geqslant$:

  This holds in any complete lattice, so the proof does not depend on D, so we do not need to use the notation above that emphasizes which poset the joins and meets are calculated in, as they will all be calculated in E. We have

$$\begin{aligned} \forall f \in \prod_{j \in J} K_j, j \in J.\ \bigwedge_{j' \in J} x_{j',f(j')} \leqslant x_{j,f(j)} \leqslant \bigvee_{k \in K_j} x_{j,k} \qquad & \text{so} \\ \forall j \in J.\ \bigvee_{f \in \prod\limits_{j \in J} K_j} \bigwedge_{j' \in J} x_{j',f(j')} \leqslant \bigvee_{k \in K_j} x_{j,k} \qquad & \text{so} \\ \bigvee_{f \in \prod\limits_{j \in J} K_j} \bigwedge_{j \in J} x_{j,f(j)} \leqslant \bigwedge_{j \in J} \bigvee_{k \in K_j} x_{j,k}. & \end{aligned}$$

- $\leqslant$:

  We use the continuity of D in the following way. If we want to show that $x \leqslant y$ in a continuous dcpo, we can show that for all $z \ll x$, we have $z \leqslant y$. Then $x = \bigvee {\downarrow}x \leqslant y$. Therefore what we want to show is that if $y \in D$ and $y \ll \overset{E}{\bigwedge_{j \in J}} \overset{E}{\bigvee_{k \in K_j}} x_{j,k}$, then $y \leqslant \overset{E}{\bigvee_{f \in \prod\limits_{j \in J} K_j}} \overset{E}{\bigwedge_{j \in J}} x_{j,f(j)}$. We start with

$$y \ll \overset{E}{\bigwedge_{j \in J}} \overset{E}{\bigvee_{k \in K_j}} x_{j,k} \leqslant \overset{E}{\bigvee_{k \in K_j}} x_{j,k} = \overset{D}{\bigvee_{k \in K_j}} x_{j,k}$$

  for all $j \in J$, by the assumption that the inclusion of E into D preserves directed joins. So by the definition of way below, for all $j \in J$ there exists $g(j) \in K_j$ such that $y \leqslant x_{j,g(j)}$, which defines a function $g \in \prod_{j \in J} K_j$. By the assumption that E is a complete lattice, $\bigwedge^E_{j \in J} x_{j,g(j)}$ exists, and by the assumption that the inclusion of E into D preserves non-empty meets $\bigwedge^D_{j \in J} x_{j,g(j)}$ exists[8] and is equal to $\bigwedge^E_{j \in J} x_{j,g(j)}$. Therefore:

$$y \leqslant \overset{E}{\bigwedge_{j \in J}} x_{j,g(j)} \leqslant \overset{E}{\bigvee_{f \in \prod\limits_{j \in J} K_j}} \overset{E}{\bigwedge_{j \in J}} x_{j,f(j)}.$$

□

[8]This is needed because we cannot assume that $y \in E$, only that it is in D.

In order to continue the proof, will need to use the fact that directed-complete C*-algebras are AW*-algebras[9]. This fact is known to experts, but does not seem to have made its way into textbooks, so we give a proof here. First we must define AW*-algebras. To do this, we need some definitions. If A is a *-algebra, the *right annihilator* of a subset $S \subseteq A$ is defined to be

$$R(S) = \{a \in A \mid \forall s \in S. sa = 0\}.$$

In order to imagine what $R(S)$ is, it may help to consider the case of $C(X)$. If $a \in C(X)$ is a complex-valued function, $R(\{a\})$ is the set of functions that vanish wherever a is nonzero.

We can also define the *commutant* of S, written S':

$$S' = \{a \in A \mid \forall b \in S. ab = ba\},$$

i.e. S' is the set of elements that commute with everything in S.

Definition 4.3. *Let A be a C*-algebra. The following four conditions are equivalent and define what it is for A to be an AW*-algebra.*

(i) A is a Baer **-ring,* i.e. *for all $S \subseteq A$, there exists a projection $p \in A$ such that $R(S) = pA$.*

(ii) The projections of A form a complete lattice and A is a Rickart **-ring,* i.e. *for all $a \in A$, there exists a projection $p \in A$ such that $R(\{a\}) = pA$.*

*(iii) Every set of orthogonal projections in A has a supremum and A is a Rickart *-ring.*

*(iv) Every set of orthogonal projections in A has a supremum and every maximal commutative *-subalgebra of A is generated by its projections.*

Proof. The equivalence of the first three is shown in [17, §4 Proposition 1]. Part (iv) is actually the original definition of an AW*-algebra and is proved to imply (i) in [33, Theorem 2.3]. To show (iii) implies (iv), we only need to show the second part holds. So let A be a C*-algebra satisfying (iii), and let $B \subseteq A$ be a maximal commutative *-subalgebra. By [17, §3 Proposition 9 (5)], A'' is commutative and contains A, so as A is maximal, $A'' = A$. By [17, §4 Proposition 8 (iv)], this implies A is a commutative AW*-algebra (in the sense of (i)). Commutative AW*-algebras are isomorphic to $C(X)$ for a stonean space X [17, §7 Theorem 1], so are generated by their projections. □

Proposition 4.4. *Every directed-complete C*-algebra is an AW*-algebra.*

Proof. Let A be a directed-complete C*-algebra. To show that A is an AW*-algebra, it suffices to show that every maximal commutative *-subalgebra B is directed-complete [34, Proposition 1.4]. Let $(a_i)_{i \in I}$ be a bounded directed set of self-adjoint elements of B, and $b = \bigvee_{i \in I} a_i$, as calculated in A. By [34, Lemma 1.6], $b \in B$. As B is order-embedded in A (Lemma 2.5), b is also the supremum of $(a_i)_{i \in I}$ in B. Therefore B is directed-complete. □

The following standard lemma characterizes when products of positive elements are positive.

[9] Whether the converse is true is an open problem.

Lemma 4.5. *Let A be a C*-algebra and $a, b \in A$ positive. Then ab is positive iff $ab = ba$.*

Proof. If ab is positive, then it is self-adjoint, so $ab = (ab)^* = b^*a^* = ba$. For the other direction, suppose that $ab = ba$. Then a and b generate a commutative C*-subalgebra of A, which, by Gel'fand duality, is isomorphic to $C(X)$ for some X. Since positive elements of $C(X)$ correspond to functions taking values in $\mathbb{R}_{\geqslant 0}$, ab is positive. □

We need the following purely technical lemma about positive operators and projections on a Hilbert space.

Lemma 4.6. *Let A be a C*-algebra, $a \in [0,1]_A$, and $p \in \mathrm{Proj}(A)$. Then the following are equivalent:*

(i) $a \leqslant p$

(ii) $a = ap$

(iii) $a = pa$

(iv) $a = pap$

Proof. In the proof, we observe that all these relations are preserved and reflected by isomorphisms. As for every C*-algebra A there exists a Hilbert space $\mathcal{H}$ such that A is isomorphic to a C*-subalgebra of $B(\mathcal{H})$, we can reduce to proving the equivalence of (i)-(iv) for A a C*-subalgebra of $B(\mathcal{H})$.

- (i) $\Rightarrow$ (ii), (ii) $\Leftrightarrow$ (iii), (iii) $\Rightarrow$ (iv):

  If we apply Lemma 2.13 to a and p, using the fact that $p = \mathrm{supp}(p)$, we get (i) $\Rightarrow$ (ii), and the proof used in that Lemma to show that (ii) $\Rightarrow$ (iii) actually also shows (iii) $\Rightarrow$ (ii), and it is clear that (ii) and (iii) together imply (iv).

- (iv) $\Rightarrow$ (ii): We have $a = pap$ and therefore $ap = pap^2 = pap = a$.

- (ii),(iii) $\Rightarrow$ (i): We want to show that $a \leqslant p$, *i.e.* $p - a \in A_+$. By (ii), we have
  $$p - a = p - ap = (1 - a)p$$
  and similarly by (iii), $p - a = p(1 - a)$, so $(1 - a)$ commutes with p. Then p is positive, and $(1 - a)$ is positive because $a \in [0,1]_A$, so by Lemma 4.5, $p - a = (1 - a)p$ is positive. □

Care is required in interpreting the following proposition, for two reasons. Firstly, in general the map $\mathrm{Proj}(A) \hookrightarrow A_{\mathrm{sa}}$ does not even preserve finite joins or meets. This happens even in the case when A is the 2×2 matrix algebra[35, Lemma 7]. This is why the restriction in the codomain to the sub-poset $[0,1]_A \subseteq A_{\mathrm{sa}}$ is needed. The second is that $[0,1]_A$ is not a lattice if A is non-commutative, so the natural way of phrasing the following, "$\mathrm{Proj}(A)$ is a sublattice of $[0,1]_A$", isn't strictly right.

Proposition 4.7. *Let A be an AW*-algebra. Then the inclusion map $\mathrm{Proj}(A) \to [0,1]_A$ preserves all lattice operations.*

Proof. Let $(p_i)_{i\in I}$ be a family of projections, and let $p = \bigwedge_{i\in I}^{\mathrm{Proj}(A)} p_i$. We want to show that $\bigwedge_{i\in I}^{[0,1]_A} p_i = p$. As p is a lower bound for $(p_i)_{i\in I}$ in $[0,1]_A$, it suffices to show that p is greater than any lower bound $a \in [0,1]_A$ for $(p_i)_{i\in I}$.

So let $a \in [0,1]_A$ such that for all $i \in I$, $a \leqslant p_i$. By Lemma 4.6, $a = ap_i$ for all $i \in I$. If we define $q_i = 1 - p_i$ for all $i \in I$ and $q = 1 - p$, we have $q = \bigvee_{i\in I}^{\mathrm{Proj}(A)} q_i$ by the fact that the map $p \mapsto 1 - p$ is an isomorphism of $\mathrm{Proj}(A)$ with its opposite (Lemma 2.6 (vi)).

As $ap_i = a$, we have $aq_i = 0$ for all $i \in I$. By [17, §3 Proposition 6], this implies $aq = 0$, and therefore $ap = a$, which by Lemma 4.6 implies $a \leqslant p$. Therefore $p = \bigwedge_{i\in I}^{[0,1]_A} p_i$.

It then follows from the fact that $a \mapsto 1 - a$ is an isomorphism of $[0,1]_A$ with its opposite (Lemma 2.6 (v)), and restricts to a such a map on $\mathrm{Proj}(A)$ as well, that the inclusion morphism $\mathrm{Proj}(A) \to [0,1]_A$ preserves joins as well, and so $\mathrm{Proj}(A)$ is a complete sublattice of $[0,1]_A$. □

We can now make full use of Lemma 4.1's extra generality.

Proposition 4.8. *If A is a continuous directed complete C*-algebra, A is an AW*-algebra with* $\mathrm{Proj}(A)$ *a continuous lattice.*

Proof. By Proposition 4.4, A is an AW*-algebra. We also have that $[0,1]_A$ is a continuous dcpo (Proposition 2.16), so by Proposition 4.7, the inclusion $\mathrm{Proj}(A) \hookrightarrow [0,1]_A$ satisfies the conditions of Lemma 4.1, and therefore $\mathrm{Proj}(A)$ is a continuous lattice. □

We can now prove that certain projection lattices are not continuous. An *atom* in a poset P with a bottom element 0 is an element $a \in P$ such that there is no element strictly between a and 0. We say a poset is *atomic* if for each non-zero $b \in P$, there is an atom $a \leqslant b$. A poset is *atomless* if it has no atoms.

The following is due to Nik Weaver [15]. Although it is stated there for von Neumann algebras, the same proof works for AW*-algebras.

Lemma 4.9 (Weaver)**.** *Let A be an AW*-algebra. If* $\mathrm{Proj}(A)$ *is continuous, then it is atomic.*

Proof. We prove the contrapositive, *i.e.* that if $\mathrm{Proj}(A)$ is not atomic, it is not continuous. To clarify, in the following we say an element p "has no atoms below it" to mean there is no atom $a \in \mathrm{Proj}(A)$ such that $a \leqslant p$. So let $p \in P$ be an element with no atoms below it, which must exist if $\mathrm{Proj}(A)$ is not atomic. We will show that $\twoheaddownarrow p = \{0\}$, so $p \neq \bigvee \twoheaddownarrow p$. Let $q \leqslant p$ and $q \neq 0$. If q had an atom below it, so would p, so q has no atoms below it. Therefore we can construct a decreasing sequence such that $q_1 = q$, $q_{i+1} \leqslant q_i$ and $q_{i+1} \neq q_i$ and $q_{i+1} \neq 0$ inductively.

Define $q' = \bigwedge_{i=1}^{\infty} q_i$ and $p_i = p - (q_i - q')$. Using Lemma 2.6 (i), we see that $q' \leqslant q_i$ implies $q - q_i$ is a projection, and $q_i - q' \leqslant q_i \leqslant q \leqslant p$ implies $p - (q_i - q')$ is a projection, so p_i is a projection. Now, as for all $i \in \mathbb{N}$, $q^{\perp} \wedge q_i = 0$, we have

$(p \wedge q^\perp) \wedge (q_i \wedge q'^\perp) = 0$, and therefore, as $q_i \neq 0$, $q_i - q' \not\leqslant p \wedge q^\perp = p - q$ (Lemma 2.6 (iv)), so $q \not\leqslant p - q_i + q' = p_i$. But

$$\begin{aligned}
\bigvee_{i=1}^{\infty} p_i &= \bigvee_{i=1}^{\infty} (p - (q_i - q')) \\
&= p - \bigwedge_{i=1}^{\infty} (q_i - q') && \text{Lemma 2.6 (vi)} \\
&= p - \bigwedge_{i=1}^{\infty} q_i \wedge (q')^\perp && \text{Lemma 2.6 (iv)} \\
&= p - \left(\bigwedge_{i=1}^{\infty} q_i \right) \wedge (q')^\perp \\
&= p - q' \wedge (q')^\perp \\
&= p.
\end{aligned}$$

Therefore $(p_i)_{i \in \mathbb{N}}$ shows that $q \not\ll p$. So $\downarrow p = \{0\}$, and as $p \neq 0$, $p \neq \bigvee \downarrow p$, proving that $\mathrm{Proj}(A)$ is not continuous. □

In the commutative case, we have the following.

Lemma 4.10. *Let A be a commutative AW*-algebra. If $\mathrm{Proj}(A)$ is continuous then $\mathrm{Proj}(A) \cong \mathcal{P}(X)$ for some set X.*

Proof. The projection lattice of a commutative C*-algebra AW*-algebra is a complete Boolean algebra, because the commutativity implies that it is a Boolean algebra, and Definition 4.3 (ii) implies that it is a complete lattice. The fact that $\mathrm{Proj}(A) \cong \mathcal{P}(X)$ then follows from [14, Theorem I-4.20]. However, we can prove it directly from Lemma 4.9. That lemma implies that $\mathrm{Proj}(A)$ is atomic. Then take X to be the set of atoms of $\mathrm{Proj}(A)$, and define $f : \mathcal{P}(X) \to \mathrm{Proj}(A)$ by $f(S) = \bigvee S$. It is then easy to prove that f is an isomorphism of Boolean algebras. □

We will require the notion of an AW*-subalgebra. For the benefit of the reader, we condense [17, §4 Definitions 3 and 4] and [17, §3 Definition 4]. Given an AW*-algebra and an element $a \in A$, and taking p to be the unique projection such that $pA = R(\{a\})$ (recall Definition 4.3 (ii)), we define the *right projection* $RP(a)$ to be $1 - p$ [17, §3 Proposition 3, Definition 4].

Definition 4.11. *Let A be an AW*-algebra and $B \subseteq A$ a *-subalgebra. We say that it is an AW*-subalgebra if*

(i) B is norm-closed, i.e. *B is a C*-subalgebra.*

(ii) If $x \in B$ then $RP(x) \in B$ (as calculated in A).

(iii) If $(p_i)_{i \in I}$ is a nonempty family of projections in B, $\bigvee_{i \in I} p_i \in B$ (the join being calculated in $\mathrm{Proj}(A)$).

By [17, §4 Proposition 8 (i)], if $B \subseteq A$ is an AW*-subalgebra of an AW*-algebra A, then B is an AW*-algebra.

Lemma 4.12. *Let A be an AW*-algebra and $B \subseteq A$ an AW*-subalgebra. Then* $\mathrm{Proj}(B) \subseteq \mathrm{Proj}(A)$ *has the induced ordering and the inclusion map* $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ *preserves arbitrary joins and nonempty meets. It preserves all meets iff the unit element of A is contained in B.*

Proof. First, observe that $\mathrm{Proj}(A)$ is order-embedded in A and $\mathrm{Proj}(B)$ is order-embedded in B, and B is order-embedded in A by Lemma 2.5, so $\mathrm{Proj}(B)$ is order-embedded in $\mathrm{Proj}(A)$.

By Definition 4.11 (iii), nonempty suprema are preserved by the inclusion map, and as B is a *-subalgebra, 0 is preserved as well, showing all suprema are preserved.

To show that non-empty meets are preserved, it helps to factorize the inclusion map into two maps. Let $u \in B$ be the unit element of B (which exists because B is an AW*-algebra). Now $\mathrm{Proj}(B) \subseteq {\downarrow} u \subseteq \mathrm{Proj}(A)$. If $(p_i)_{i \in I}$ be a non-empty family of projections in ${\downarrow} u$. By the nonemptiness, if $q \in A$ and $q \leqslant p_i$ for all $i \in I$, then $q \in {\downarrow} u$. Therefore the inclusion map ${\downarrow} u \to \mathrm{Proj}(A)$ preserves non-empty meets. Now, as the complement of an element $a \in \mathrm{Proj}(B)$ is $u - a$, and this is also true for ${\downarrow} u$, the inclusion map $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ preserves complements. As it preserves joins, it preserves meets. Therefore the composite inclusion map $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ preserves non-empty meets.

As the unit element is the empty meet, the inclusion map preserves all joins iff $u \in A$. □

The following is the combination of the previous lemma with Lemma 4.1 that we will use twice.

Corollary 4.13. *Let A be an AW*-algebra such that* $\mathrm{Proj}(A)$ *is continuous, and $B \subseteq A$ an AW*-subalgebra. Then* $\mathrm{Proj}(B)$ *is continuous.*

Proof. The inclusion $\mathrm{Proj}(B) \subseteq \mathrm{Proj}(A)$ satisfies the hypotheses of Lemma 4.1[10] by Lemma 4.12. □

Recall that for a C*-algebra A, the *centre* $Z(A)$ is defined to be the set of elements that commute with every element of A. For an AW*-algebra A, if $Z(A)$ is as small as possible, consisting only of multiples of the identity element, we say that A is a *factor*, or *AW*-factor* to emphasize the fact that it is an AW*-algebra. Contrariwise, $Z(A) = A$ iff A is commutative. The projections in the centre $\mathrm{Proj}(Z(A))$ are called the *central projections* of A.

Lemma 4.14. *Let A be an AW*-algebra and p a central projection,* i.e. $p \in \mathrm{Proj}(Z(A))$.

(i) Let $a \in A$. Then $pa = ap = pap$.

(ii) The element $a \in A$ is in $pAp = pA = Ap$ iff $a = pa$ (and therefore iff $a = ap$ or $a = pap$).

(iii) pAp is an AW-subalgebra of A, with unit element p.*

*(iv) The map $\pi_p : A \to pAp$ defined by $\pi_p(a) = pap$ (equivalently pa or ap) is a unital *-homomorphism.*

[10]In fact [14, Theorem I-2.7] would work unaltered in this specific case, but we used (and needed) the extra generality in Proposition 4.8.

(v) $Z(pAp) = pZ(A)p$, i.e. *the centre of* pAp *is the image of the centre of* A.

Proof.

(i) $pa = ap$ follows from $p \in Z(A)$, and therefore $pap = p^2a = pa$.

(ii) First, by (i), $pAp = pA = Ap$. By definition, $a \in pA$ iff there is some $b \in A$ such that $a = pb$. So $a \in pA$ implies $pa = p^2b = pb = a$. Conversely, if $pa = a$, then immediately $a \in pA$. By (i) these statements hold equally well for $ap = a$ and $pap = a$.

(iii) See [17, §4 Proposition 8 (iii)] for the proof that pAp is an AW*-subalgebra of A. It is then easy to see that p is the unit element, because $ppap = pap = papp$ for all $a \in A$. It follows that the inclusion morphism $pAp \to A$ is a *-homomorphism, but is not unital unless $p = 1$.

(iv) Since (i) shows that $\pi_p(a) = pa$, we will work with this definition, as it is slightly simpler. If $\alpha a + \beta b$ is a $\mathbb{C}$-linear combination in A, then $\pi_p(\alpha a+\beta b)+p(\alpha a+\beta b) = \alpha pa+\beta pb = \alpha\pi_p(a)+\beta\pi_p(b)$, proving linearity. For all $a \in A$, we have $\pi_p(a^*) = pa^* = a^*p = (pa)^* = \pi_p(a)^*$, so π_p preserves the -* operation. If $a, b \in A$, then $\pi_p(ab) = pab = p^2ab = papb = \pi_p(a)\pi_p(b)$. Finally, $\pi_b(1) = p1 = p$, which is the unit element of pAp by (iii), so $\pi_b : A \to pAp$ is a unital *-homomorphism.

(v) If $a \in Z(A)$, then as $p \in Z(A)$ and the centre is a *-subalgebra of A, $pap \in Z(A)$, so $pZ(A)p \subseteq Z(A)$. As every element of pAp is an element of A, $pZ(A)p \subseteq Z(pAp)$. For the opposite inclusion, suppose that $a \in Z(pAp)$, *i.e.* $pap = a$ and a commutes with all elements of pAp. We show that $a \in Z(A)$, and therefore $a \in pZ(A)p$ (because $a = pap$). It follows from $a \in pAp$ that $(1 - p)a = 0$. Let $b \in A$, and

$$\begin{aligned} ba &= (bp + b(1-p))a \\ &= bpa + b(1-p)a \\ &= bpa && \text{because } (1-p)a = 0 \\ &= (pbp)a && \text{part (i)} \\ &= a(pbp) && \text{because } a \in Z(pAp) \\ &= apb && \text{part (i)} \\ &= ab && \text{part (ii).} \end{aligned}$$

Therefore $Z(pAp) = pZ(A)p$. ☐

Proposition 4.15. *Let* A *be an AW*-algebra such that* $\mathrm{Proj}(A)$ *is continuous. Then* $\mathrm{Proj}(Z(A)) \cong \mathcal{P}(X)$ *for some set* X*, and* $A \cong \prod_{x\in X} xAx$ *where each* xAx *is an AW*-factor such that* $\mathrm{Proj}(xAx)$ *is continuous.*

Proof. By [17, §4 Proposition 8 (v)] $Z(A)$ is an AW*-subalgebra of A. By the continuity of $\mathrm{Proj}(A)$, $\mathrm{Proj}(Z(A))$ is continuous (Corollary 4.13). Therefore $\mathrm{Proj}(Z(A))$ is isomorphic to $\mathcal{P}(X)$, where X is the set of atoms of $\mathrm{Proj}(Z(A))$ (Lemma 4.10).

The atoms of $\mathrm{Proj}(Z(A))$ form a disjoint family of central projections whose join is 1, so we can apply [17, §10 Proposition 2] to conclude that the mapping $\phi : A \to \prod_{x\in X} xAx$ defined by $\phi(a) = (xax)_{x\in X}$ is an isomorphism.

By Lemma 4.14 (iii), xAx is an AW*-algebra for all $x \in X$. If p is a central projection in xAx, then $p \in Z(A)$ and $px = xp = x$ by Lemma 4.14 (v) and (ii). By Lemma 2.6 (ii), $p \leqslant x$, so as x is an atom, either $p = x$ or $p = 0$. Since commutative AW*-algebras are the closed $\mathbb{C}$-linear span of their projections [17, Proposition 1 (3)], it follows that $Z(xAx)$ is the linear span of x, and therefore xAx is a factor.

Finally, as xAx is an AW*-subalgebra of A, $\mathrm{Proj}(xAx)$ is continuous (Corollary 4.13). □

As in Lemma 4.9, the von Neumann algebra version of the following is due to Nik Weaver [15]. The non-continuity of $\mathrm{Proj}(B(\mathcal{H}))$ for infinite-dimensional $\mathcal{H}$ was also shown independently by Keimel [36, Proposition 3.1], and independently of both of these by the author, who based the proof on Birkhoff and von Neumann's proof that $\mathrm{Proj}(\mathcal{H})$ is not modular [37, §11].

Proposition 4.16. *An AW*-factor A has* $\mathrm{Proj}(A)$ *continuous iff there exists a finite-dimensional Hilbert space $\mathcal{H}$ such that $A \cong B(\mathcal{H})$.*

Proof. The fact that $\mathrm{Proj}(B(\mathcal{H}))$ is continuous if $\mathcal{H}$ is finite-dimensional follows from Theorem 3.9 and Proposition 4.8. So we only need to show that if $\mathrm{Proj}(A)$ is continuous, $A \cong B(\mathcal{H})$ for $\mathcal{H}$ finite-dimensional. By Lemma 4.9, $\mathrm{Proj}(A)$ must be atomic. By [17, §15, Theorem 1, (4)], there is a central projection h_4 such that h_4A is a *discrete AW*-algebra* and $(1 - h_4)A$ is a *continuous*[11] *AW*-algebra* (see [17, §15, Definition 3] for the definitions of these). Since A is a factor, the only central projections are 0 and 1, so either A is a discrete AW*-algebra or a continuous AW*-algebra. If A were continuous, then the only abelian projection (see [17, §15, Definition 2] for the definition of this) is 0. So by [17, §19 Lemma 1] every projection other than 0 contains a strictly smaller non-zero projection (see [17, §14, Proposition 2 and Corollary 1] for why an AW*-algebra "has PC"). As $\mathrm{Proj}(A)$ is atomic, this is false, so A must be discrete, or a *type I AW*-algebra* [17, §15, Definition 4].

Therefore, by [38, Lemma 1], there exists a Hilbert space $\mathcal{H}$ such that $A \cong B(\mathcal{H})$. So all we need to do is show that $\mathcal{H}$ cannot be infinite dimensional. The counterexample to $\mathrm{Proj}(B(\mathcal{H}))$ being continuous for infinite-dimensional $\mathcal{H}$ is related to Birkhoff and von Neumann's counterexample to $\mathrm{Proj}(B(\mathcal{H}))$ being modular for infinite-dimensional $\mathcal{H}$ [37, §11].

By taking an orthonormal basis, identify $\mathcal{H}$ with $\ell^2(\kappa)$ for some cardinal κ. We use $(e_\alpha)_{\alpha \in \kappa}$ for the basis vectors, the functions taking the value 0 everywhere except for at α, where they take the value 1. Define $p = |e_0\rangle\langle e_0|$ and for $i \in \omega$, define

$$\psi_n = \frac{e_0 + \frac{1}{n+1} e_{n+1}}{\sqrt{\frac{n+2}{n+1}}}$$

so that $p_n = |\psi_n\rangle\langle\psi_n|$ is the projection onto the span of $e_0 + \frac{1}{n+1} e_{n+1}$. So $\bigvee_{i=0}^n p_i$ is the projection onto the span of $\{e_0 + e_1, e_0 + \frac{1}{2} e_2, \ldots, e_0 + \frac{1}{n+1} e_{n+1}\}$. It is clear that e_0 is not in this subspace for any $n \in \omega$, so $p \not\leqslant \bigvee_{i=0}^n p_i$ for any $n \in \omega$. However, as

$$\left\| e_0 + \frac{1}{n} e_n - e_0 \right\| = \left\| \frac{1}{n} e_n \right\| = \frac{1}{n} \to 0$$

[11] As will soon be apparent, it is important not to confuse this notion with the notion of continuity for dcpos.

we have that e_0 is in the closure of the span $\{e_0+e_1, e_0+\frac{1}{2}e_2, \ldots\}$, so $p \leqslant \bigvee_{i=0}^{\infty} p_i$. We can then define $q_0 = \bigvee_{\alpha\in\kappa\setminus\omega} |e_\alpha\rangle\langle e_\alpha|$, $q_n = q_{n-1} \vee p_{n-1}$, for $n > 1$ in ω, and we have a chain of projections such that $\bigvee_{i=0}^{\infty} q_i = 1$, and $p \nleqslant q_i$ for any $i \in \omega$. So $p \not\ll 1$.

We can re-run this argument for any projection onto a 1-dimensional subspace, by extending a unit vector ψ contained in that subspace to an orthonormal basis and identifying ψ with e_0. Therefore no projection onto a 1-dimensional subspace is way below 1. As every non-zero projection contains a 1-dimensional subspace, this shows that the only projection that is way below 1 is 0, so 1 is not the supremum of elements way below it, and $\mathrm{Proj}(B(\mathcal{H}))$ is not continuous. □

We can now state and prove the precise characterization of C*-algebras whose effect algebra is a continuous dcpo.

Theorem 4.17. *(i) For a C*-algebra A, the unit interval $[0,1]_A$ is a continuous dcpo iff A is of the form $\prod_{x\in X} B(\mathcal{H}_x)$ where $\mathcal{H}_x$ is finite-dimensional.*

(ii) The projection lattice $\mathrm{Proj}(A)$ of an AW-algebra A is continuous iff A is of the form $\prod_{x\in X} B(\mathcal{H}_x)$ where $\mathcal{H}_x$ is finite-dimensional.*

Proof. Let A be a C*-algebra such that $[0,1]$ is a continuous dcpo. Since it is a dcpo, A is bounded directed complete (Proposition 2.16 (a) (iii) ⇒ (i)), so by Proposition 4.8, A is an AW*-algebra and $\mathrm{Proj}(A)$ a continuous lattice.

Therefore we are in the situation of (ii). If A is an AW*-algebra with $\mathrm{Proj}(A)$ a continuous lattice, then $A \cong \prod_{x\in X} xAx$ (Proposition 4.15) and by Proposition 4.16 $xAx \cong B(\mathcal{H}_x)$ for a finite-dimensional Hilbert space $\mathcal{H}_x$ for all $x \in X$. So we have proved the forward implication of both (i) and (ii).

The backward implication of (i) follows from Theorem 3.9. The backward implication of (ii) then follows by Proposition 4.8. □

We can now combine the results from this section with the previous one. Recall that the C*-algebra $\ell^\infty(X)$ is the categorical product[12] of $\mathbb{C}$ indexed by X in **C*Alg**. It consists of bounded functions $a : X \to \mathbb{C}$, *i.e.* functions that have a bound $\mathbb{R}_{\geqslant 0} \ni \alpha \geqslant |a(x)|$ that depends on a but does not depend on $x \in X$.

Proposition 4.18. *Let A be a C*-algebra such that $[0,1]_A$ is a dcpo with a countable base. Then $A \cong \ell^\infty(X)$, where X is a countable set. In particular, A is commutative.*

Proof. Since $[0,1]_A$ has a countable base B, it is continuous, so $A \cong \prod_{i\in I} B(\mathcal{H}_i)$, with $\mathcal{H}_i$ finite-dimensional, by Theorem 4.17 (i). If there were an $i \in I$ such that $\dim(\mathcal{H}_i) \geqslant 2$, then $\pi_i : [0,1]_A \to [0,1]_{B(\mathcal{H}_i)}$ is a Scott continuous surjective map. So $\pi_i(B)$ would be a countable base for $[0,1]_{B(\mathcal{H}_i)}$ by [14, Proposition III-4.12], which contradicts Lemma 3.10. Therefore for all $i \in I$, $\mathcal{H}_i$ is 0 or 1-dimensional.

If $\dim(\mathcal{H}) = 0$, then $B(\mathcal{H}) \cong \{0\}$, the ring with $0 = 1$, and if $\dim(\mathcal{H}) = 1$, then $B(\mathcal{H}) \cong \mathbb{C}$, mapping the identity map to $1 \in \mathbb{C}$. So, defining $X = \{i \in I \mid \dim(\mathcal{H}_i) = 1\}$. Then $A \cong \ell^\infty(X)$, including the case when $X = \varnothing$. So all

[12] Which in this case never agrees with the set-theoretic product if X is infinite.

that remains is to prove that X is countable. We do this by constructing an injection $X \to B$.

Let $\delta_x : X \to \mathbb{C}$ be the function that takes the value 1 at x and 0 everywhere else. We reuse B for the image of the countable base in $\ell^\infty(X)$, which is a countable base for the dcpo $[0,1]^X$. For each $x \in X$, $\delta_x = \bigvee \mathord{\downarrow\!\!\downarrow}\delta_x \cap B$, so there exists $b_x \in B$ such that $b_x \neq 0$ and $b_x \leqslant \delta_x$. Fix such a b_x for each $x \in X$. We show $x \mapsto b_x$ is injective as follows. If $x, x' \in X$ such that $b_x = b_{x'}$, if $x \neq x'$ then $0 \leqslant b_x(x') \leqslant \delta_x(x') = 0$, so $0 = b_x(x') = b_{x'}(x')$. As $b_{x'} \leqslant \delta_{x'}$, this shows $b_{x'} = 0$, which is a contradiction. Therefore $x = x'$, and so $x \mapsto b_x$ is injective. Since B is countable, X is countable. □

5 Conclusion

We have shown that domain theory and the Löwner order do not combine in a way that is suitable for building up completely positive maps from a fixed set of quantum gates, and that they also do not combine well with infinite dimensions. A suitable quantum domain theory is left to future work, based on approaches to domain theory that include topology from the start [39, 40, 41].

Acknowledgements

Many thanks to Mathys Rennela, Nik Weaver and Samson Abramsky.

Robert Furber has been financially supported by the Danish Council for Independent Research, Project 4181-00360.

References

[1] Glynn Winskel. *The Formal Semantics of Programming Languages: An Introduction*. Foundations of Computing Series. MIT Press, 1993. 1

[2] Peter Selinger. Towards a Quantum Programming Language. *Mathematical Structures in Computer Science*, 14(4):527–586, 2004. 1

[3] Peter Selinger. Towards a Semantics for Higher-Order Quantum Computation. In *Proceedings of the 2nd International Workshop on Quantum Programming Languages*, volume 33 of *TUCS General Publications*, pages 127–143. Turku Centre for Computer Science, 2004. 1, 19

[4] Ellie d'Hondt and Prakash Panangaden. Quantum Weakest Preconditions. *Mathematical Structures in Computer Science*, 16(3):429–451, 2006. 1

[5] Alan M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, s2-42(1):230–265, 1937. 1

[6] Alan M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. A Correction. *Proceedings of the London Mathematical Society*, s2-43(1):544–546, 01 1938. 1

[7] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2011. 1, 7, 14

[8] Abbas Edalat. Domain Theory and Integration. *Theoretical Computer Science*, 151(1):163–193, 1995. Selected Papers of the Workshop on Topology and Completion in Semantics. 2

[9] Abbas Edalat. Domains for Computation in Mathematics, Physics and Exact Real Arithmetic. *Bulletin of Symbolic Logic*, 3(4):401–452, 1997. 2

[10] Mathys Rennela. Operator Algebras in Quantum Computation. Master's thesis, Université Paris 7 Diderot, 2013. `https://arxiv.org/abs/1510.06649`. 2

[11] Mathys Rennela. *Enrichment in Categorical Quantum Foundations*. PhD thesis, Radboud University, Nijmegen, 2018. 2

[12] Kenta Cho. Semantics for a Quantum Programming Language by Operator Algebras. *New Generation Computing*, 34(1):25–68, 2016. 2

[13] Mathys Rennela. Towards a Quantum Domain Theory: Order-enrichment and Fixpoints in W*-algebras. *Electronic Notes in Theoretical Computer Science*, 308:289–307, 2014. Proceedings of the 30th Conference on the Mathematical Foundations of Programming Semantics (MFPS XXX). 2

[14] Gerhard Gierz, Karl H. Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael W. Mislove, and Dana S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003. 2, 19, 20, 21, 22, 26, 27, 30

[15] Nik Weaver. Projections in a W*-algebra as a continuous lattice? `https://mathoverflow.net/q/133037`, 2013. 2, 25, 29

[16] John B. Conway. *A Course In Functional Analysis, Second Edition*, volume 96 of *Graduate Texts in Mathematics*. Springer Verlag, 1990. 3

[17] Sterling K. Berberian. *Baer *-Rings*, volume 195 of *Grundlehren der mathematischen Wissenschaften*. Springer, 1972. 3, 23, 25, 26, 28, 29

[18] Jacques Dixmier. *C*-algebras*, volume 15 of *North-Holland Mathematical Library*. North-Holland Publishing Company, 1977. 4, 5, 6, 7, 9

[19] Masamichi Takesaki. *Theory of Operator Algebra I*. Springer Verlag, 1979. 4, 10

[20] Karl Löwner. Über monotone Matrixfunktionen. *Mathematische Zeitschrift*, 38(1):177–216, Dec 1934. 6

[21] Shôichirô Sakai. *C*-algebras and W*-algebras*, volume 60 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1971. 7, 15

[22] Paul R. Halmos. *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*. Chelsea Publishing Company, 1951. 7, 8

[23] Aldo J. Lazar. AF Algebras with a Lattice of Projections. *Mathematica Scandinavica*, 50:135–144, 1982. 7

[24] Donovan van Osdol. C*-algebras and Cohomology. In H. Lamar Bentley, Horst Herrlich, Minakshisundaram Rajagopalan, and Harvey Wolff, editors, *Categorical Topology: Proceedings of the International Conference held at the University of Toledo, Ohio, U.S.A. August 1–5, 1983*, volume 5 of *Sigma Series in Pure Mathematics*, pages 582–587. Heldermann Verlag, 1983. 9

[25] Joan Wick Pelletier and Jiři Rosický. On the Equational Theory of C*-algebras. *Algebra Universalis*, 30:275–284, 1993. 9

[26] Helmut H. Schaefer. *Topological Vector Spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer Verlag, 1966. 15

[27] Klaus Keimel. Bicontinuous Domains and Some Old Problems in Domain Theory. *Electronic Notes in Theoretical Computer Science*, 257:35–54, 2009. Proceedings of the Fifth International Symposium on Domain Theory (ISDT 2009). 19, 20

[28] Man-Duen Choi. Completely Positive Linear Maps on Complex Matrices. *Linear Algebra and its Applications*, 10(3):285 – 290, 1975. 20

[29] Andrzej Jamiołkowski. Linear Transformations which Preserve Trace and Positive Semidefiniteness of Operators. *Reports on Mathematical Physics*, 3(4):275 – 278, 1972. 20

[30] E. Christopher Lance. *Hilbert C*-Modules: A toolkit for operator algebraists*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1995. 20

[31] Keye Martin and Prakash Panangaden. Spacetime Topology from Causality. `https://arxiv.org/abs/gr-qc/0407093`, 2004. 21

[32] Keye Martin and Prakash Panangaden. A Domain of Spacetime Intervals in General Relativity. `https://arxiv.org/abs/gr-qc/0407094`, 2004. 21

[33] Irving Kaplansky. Projections in Banach Algebras. *Annals of Mathematics*, 53(2):235–249, 1951. 23

[34] Kazuyuki Saitô and J. D. Maitland Wright. On Defining AW*-algebras and Rickart C*-algebras. *The Quarterly Journal of Mathematics*, 2015. 23

[35] Richard V. Kadison. Order Properties of Bounded Self-Adjoint Operators. *Proceedings of the American Mathematical Society*, 2(3):505–510, 1951. 24

[36] Eike Neumann, Martin Pape, and Thomas Streicher. Computability in Basic Quantum Mechanics. *Logical Methods in Computer Science*, 14(2), 2018. 29

[37] Garrett Birkhoff and John von Neumann. The Logic of Quantum Mechanics. *Annals of Mathematics*, 37(4):823–843, 1936. 29

[38] Irving Kaplansky. Algebras of Type I. *Annals of Mathematics*, 56(3):460–472, 1952. 29

[39] Achim Jung. Stably Compact Spaces and the Probabilistic Powerspace Construction. *Electronic Notes in Theoretical Computer Science*, 87:5–20, 2004. 31

[40] Jimmie Lawson. Stably Compact Spaces. *Mathematical Structures in Computer Science*, 21(1):125–169, 2011. 31

[41] Ingo Battenfeld, Matthias Schröder, and Alex Simpson. A Convenient Category of Domains. *Electronic Notes in Theoretical Computer Science*, 172:69–99, 2007. Computation, Meaning, and Logic: Articles dedicated to Gordon Plotkin. 31