A Probability Monad on Measure Spaces

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FinStock

- **FinStoch** is the category with finite sets as objects and stochastic matrices as morphisms.
- **FinSet** $\hookrightarrow$ **FinStoch**.
- Since **FinStoch** consists of "modified functions" we look for a monad $\mathcal{D}$ on **FinSet** such that **FinStoch** $\simeq \mathcal{K}\ell(\mathcal{D})$.
- Rows summing to 1 indicates $\mathcal{D}(X)$ consists of nonnegative real functions on X that sum to 1.
- If $|X| \geq 2$, $\mathcal{D}(X)$ is infinite, so it has to be defined on **Set**. Then **FinStoch** $\hookrightarrow \mathcal{K}\ell(\mathcal{D})$ is the full subcategory on finite sets.
- We cannot handle probabilities such as sequences of independent coin flips on $2^{\mathbb{N}}$ or Lebesgue measure on $[0, 1]$ this way. We need a different category to play the role of **Set**.

Compact Hausdorff spaces and C*-algebras

- First attempt: $2^{\mathbb{N}}$ and $[0,1]$ are examples of *compact Hausdorff spaces*.
- Why concentrate on them? They have a good duality theory.
- If X is compact Hausdorff space $C(X) = \mathbf{Top}(X, \mathbb{C})$ is a (commutative unital) C*-algebra.
- A unital C*-algebra is an internal *-monoid in $\mathbf{Ban}_1$ with the (nontrivial) extra condition that $\|a^*a\| = \|a\|^2$.
- But the important part is $C : \mathbf{CHaus} \to \mathbf{CC^*Alg}^{\mathrm{op}}$ is an equivalence, where morphisms in $\mathbf{CC^*Alg}^{\mathrm{op}}$ are unital *-homomorphisms. (Gel'fand Duality).
- $\mathrm{Spec} : \mathbf{CC^*Alg}^{\mathrm{op}} \to \mathbf{CHaus}$ is the inverse where $\mathrm{Spec}(A) = \mathbf{CC^*Alg}(A, \mathbb{C})$.

Positive Unital Maps

- C*-algebras have a positive cone, on $C(X)$ it is the set of functions with values in $[0, \infty) \subseteq \mathbb{C}$.
- A positive unital map is a linear map that preserves the positive cone (equivalent to monotonicity w.r.t. the order) and unit.
- $\mathbf{CC^*Alg}_{\mathrm{PU}}$ has positive unital maps as morphisms, $\mathbf{CC^*Alg}$ is a subcategory.
- The state space $\mathcal{S}(A) = \mathbf{CC^*Alg}_{\mathrm{PU}}(A, \mathbb{C})$.

The Radon Monad

- $\mathcal{R}(X) = \mathcal{S}(C(X)) = \mathbf{CC^{*}Alg}_{\mathrm{PU}}(C(X), \mathbb{C})$ is a compact Hausdorff space (the space of Radon probability measures). It is a monad on **CHaus**.
- Example: On $[0,1]$ define $\phi : C([0,1]) \to \mathbb{C}$

$$\phi(a) = \int_0^1 a(x)\,\mathrm{d}x$$

- The Riesz representation theorem puts regular probability measures on X in bijection with elements $\phi \in \mathcal{R}(X)$.
- $\mathcal{K}\ell(\mathcal{R})$ is like $\mathcal{K}\ell(\mathcal{D})$ but with continuity.

Probabilistic Gel'fand Duality

- We can extend C to a functor $C_{PU} : \mathcal{K}\ell(\mathcal{R}) \to \mathbf{CC^*Alg}^{\mathrm{op}}_{\mathrm{PU}}$.
- On $f : X \to \mathcal{R}(Y)$ we define $C_{PU}(f) : C(Y) \to C(X)$ by

$$C(f)(b)(x) = f(x)(b)$$

  i.e. swapping the arguments of a curried function.
- C_{PU} is an equivalence. [FJ15]

$$\begin{array}{ccc} \mathcal{K}\ell(\mathcal{R}) & \xrightarrow{C_{PU}} & \mathbf{CC^{*}Alg}_{\mathrm{PU}}^{\mathrm{op}} \\ F_{\mathcal{R}} \uparrow \dashv \downarrow G_{\mathcal{R}} & & \uparrow \dashv \downarrow C \circ \mathcal{S} \\ \mathbf{CHaus} & \xrightarrow[C]{} & \mathbf{CC^{*}Alg}^{\mathrm{op}}, \end{array}$$

$\mathbf{CC^{*}Alg}_{\mathrm{PU}}$ is therefore the coKleisli category of a comonad on $\mathbf{CC^{*}Alg}$.

Conditional Probability Maps for $\mathcal{D}$

Given a finite set X and $\phi \in \mathcal{D}(X)$, and a function $\mathcal{Y} : X \to Y$ we can define a function $e : Y \to \mathcal{D}(X)$

$$e(y)(x) = \mathbb{P}(\mathcal{X} = x \mid \mathcal{Y} = y) = \frac{\mathbb{P}(\mathcal{X} = x, \mathcal{Y} = y)}{\mathbb{P}(\mathcal{Y} = y)}$$

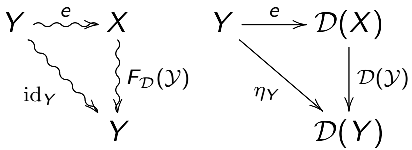
$$= \frac{\phi(x)[\mathcal{Y}(x) = y]}{\sum_{x' \in \mathcal{Y}^{-1}(y)} \phi(x')}$$

(where $\mathcal{X} : X \to X$ is the identity function)

Conditional Probability Maps in General

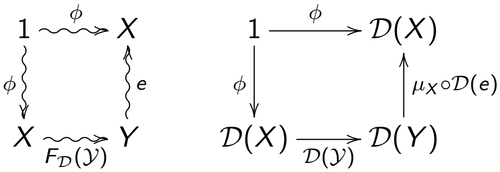
This conditional probability map satisfies two properties:

1. e is a "probabilistic section" of $\mathcal{Y}$:



   (or $e(y)$ is supported on $\mathcal{Y}^{-1}(y)$)

2. ϕ is mapped back to itself by the maps the other way



   (marginal probability and conditional probability reproduce joint probability)

Problems

- We can use this to define what a conditional probability map in $\mathcal{K}\ell(\mathcal{R})$ should be.
- But there are surjective maps with no probabilistic section, *e.g.* the binary digits map $2^{\mathbb{N}} \to [0, 1]$.
- We might try using the Giry monad $\mathcal{G}$ on measurable spaces. But even on standard Borel spaces there are surjective maps with no probabilistic section.
- A modification of this notion where we only require a probabilistic section "almost everywhere" exists for standard Borel spaces and is known as a *regular conditional probability*.

Idea

How about working in a category of measure spaces that ignores null sets to begin with?

Approach

- When trying to make this work, it helps to use probabilistic Gel'fand duality.
- Under probabilistic Gel'fand duality, a conditional probability map corresponds to the notion of a *conditional expectation* from operator algebra [Tom57, Tak72].
- This is not a coincidence (but no Kleisli categories were used in defining it originally).
- We need the measure theoretic analogue of C, which is L^∞.

L^∞ the W*-algebra

Let (X, ν) be a probability space:

- $L^\infty(X, \nu)$ is the space of bounded measurable functions modulo equality ν-almost everywhere. It is a commutative C*-algebra.
- $L^1(X, \nu)$ is the space of (absolutely) ν-integrable functions modulo equality ν-a.e.
- The pairing $\langle \text{-}, \text{-} \rangle : L^\infty(X, \nu) \times L^1(X, \nu) \to \mathbb{C}$ defined by integration
$$\langle a, \phi \rangle = \int_X a\phi \, \mathrm{d}\nu$$
defines an isometry $L^\infty(X, \nu) \to L^1(X, \nu)^*$. This makes $L^\infty(X, \nu)$ a commutative *W*-algebra*, $L^1(X, \nu)$ is the *predual*.
- In fact we cannot stay confined to probability spaces, but we cannot be too general because $L^\infty(X, \nu) \not\cong L^1(X, \nu)$ for all measure spaces.

Gel'fand Duality for W*-algebras

- The objects of **Meas** are compact complete strictly localizable measure spaces, the morphisms equivalence classes of nullset-reflecting measurable maps.
- This class of measure spaces was singled out by Fremlin in [Fre02] for duality (between measure spaces and a full subcategory of complete Boolean algebras).
- **CW*Alg** is a non-full subcategory of **CC*Alg** – the morphisms are *normal* *-homomorphisms, which are maps that are equivalently weak-* continuous or Scott continuous.
- L^∞ : **Meas** $\to$ **CW*Alg**$^{\mathrm{op}}$ is an equivalence.

Gel'fand Duality for W*-algebras II

- An inverse to L^∞ is given by $\mathrm{Spec} : \mathbf{CW^*Alg}^{\mathrm{op}} \to \mathbf{Meas}$ (hyperstonean spaces).
- Every object of **Meas** is isomorphic to

$$\coprod_{i \in I} (2^{\kappa_i}, \nu_{2^{\kappa_i}})$$

  for some family of cardinals $(\kappa_i)_{i \in I}$ (Maharam's theorem).
- Reference for W*-algebra Gel'fand duality: [Pav22].

A Monad for Conditional Expectations?

- By analogy to C*-algebras, the probabilistic category of W*-algebras is $\mathbf{CW^*Alg}_{\mathrm{PU}}$ (normal positive unital maps).
- Nonexistence problems are over: Conditional expectations exist in $\mathbf{CW^*Alg}_{\mathrm{PU}}$ for $L^\infty(f)$ if f is between probability spaces.
- We want a monad T on **Meas** whose Kleisli category is equivalent to ${\mathbf{CW^*Alg}_{\mathrm{PU}}}^{\mathrm{op}}$. We can use W*-Gel'fand duality to work on the W*-side first.
- So show that $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$ has a left adjoint F such that the comparison functor for the coKleisli category of the comonad is an equivalence.

Double Duals

- The forgetful functor $\mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CC^*Alg}_{\mathrm{PU}}$ has a left adjoint, the *enveloping W^*-algebra*. For $A \in \mathbf{CC^*Alg}$ it is the double dual A^{**}. This also produces a left adjoint to $\mathbf{CW^*Alg} \to \mathbf{CC^*Alg}$.
- Observe:

$$\begin{aligned}\mathbf{CW^*Alg}_{\mathrm{PU}}(A^{**}, B) \cong \mathbf{CC^*Alg}_{\mathrm{PU}}(A, B) &\cong \mathbf{CC^*Alg}(C(\mathcal{S}(A)), B) \\ &\cong \mathbf{CW^*Alg}(C(\mathcal{S}(A))^{**}, B).\end{aligned}$$

- It must be that $F(A^{**}) = C(\mathcal{S}(A))^{**}$.
- Not all W*-algebras are double duals!

Defining F

Lemma

$\mathbf{CW^*Alg}$ *is monadic over* $\mathbf{CC^*Alg}$, *i.e.* $\mathbf{CW^*Alg} \simeq \mathcal{EM}(\text{-}^{**})$.

- Therefore

$$A^{****} \underset{\epsilon_A^{**}}{\overset{\epsilon_{A^{**}}}{\rightrightarrows}} A^{**} \xrightarrow{\epsilon_A} A$$

  is a coequalizer (the *canonical presentation* of A).
- This coequalizer is preserved by the inclusion $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$.
- Since left adjoints preserve colimits and $\mathbf{CW^*Alg}$ is cocomplete, this allows us to define $F : \mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CW^*Alg}$.
- The coKleisli comparison functor is an equivalence with $\mathbf{CW^*Alg}_{\mathrm{PU}}$ because $\mathbf{CW^*Alg}_{\mathrm{PU}}$ and $\mathbf{CW^*Alg}$ have the same objects. [Wes17, Theorem 9]

Main Result

Theorem

There is a monad T on **Meas** *such that* $\mathcal{K}\ell(T) \simeq \mathbf{CW^*Alg}_{\mathrm{PU}}$.

- It seems the simplest way to realize $T(X)$ is to take the Gel'fand spectrum of $F(L^\infty(X))$.

Final Remarks

- For a countable set X

  $$T(X) \cong ([0,1], \mathcal{P}([0,1]), \nu_d) + ([0,1]^2, \mathcal{P}([0,1]) \otimes \widehat{\mathcal{B}o([0,1])}, \nu_d \otimes \nu_L)$$

  where ν_d is the counting measure and ν_L the Lebesgue measure.
- The need to use non-probabilistic spaces is analogous to the need to use **Set** instead of **FinSet** to define $\mathcal{D}$.
- We only have that **Meas**$(1, T(X))$ corresponds to the density functions on X, not that $T(X)$ does.
- It should be that $\mathcal{K}\ell(T)$ and $\mathbf{CW^*Alg}_{\mathrm{PU}}{}^{\mathrm{op}}$ are Markov categories in the sense of [Fri20] (work in progress).

References I

- Robert Furber and Bart Jacobs, *From Kleisli Categories to Commutative C^*-algebras: Probabilistic Gelfand Duality*, Logical Methods in Computer Science **11** (2015), no. 2, 1–28.
- David H. Fremlin, *Measure Theory, Volume 3*, `https://www.essex.ac.uk/maths/people/fremlin/mt.htm`, 2002.
- Tobias Fritz, *A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics*, Advances in Mathematics **370** (2020), 107239.
- Dmitri Pavlov, *Gelfand-type duality for commutative von Neumann algebras*, Journal of Pure and Applied Algebra **226** (2022), no. 4, 106884.

References II

- Masamichi Takesaki, *Conditional expectations in von Neumann algebras*, Journal of Functional Analysis **9** (1972), no. 3, 306–321.
- Jun Tomiyama, *On the Projection of Norm One in W*-algebras*, Proceedings of the Japan Academy **33** (1957), no. 10, 608–612.
- Bram Westerbaan, *Quantum Programs as Kleisli Maps*, Electronic Proceedings in Computer Science (EPTCS) **236** (2017), 215–228.