A Probability Monad on Measure Spaces

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Outline

- Introduction: Finite Probability with $\mathcal{D}$.
- Difficulties with Conditional Probability.
- Previous work: Probabilistic Gelfand duality and the Radon monad.
- Gelfand duality for W*-algebras.
- Commutative W*-algebras and the double dual monad.
- A comonad on commutative W*-algebras and a monad on measure spaces.

Distribution Monad

- We start with the distribution monad $\mathcal{D}$.
- For a set X

$$\mathcal{D}(X) = \left\{ \phi : X \to [0,1] \;\middle|\; \phi \text{ finite support, } \sum_{x \in X} \phi(x) = 1 \right\}$$

- For each $\phi \in \mathcal{D}(X)$, we can extend it to sets. For $S \subseteq X$

$$\phi(S) = \sum_{x \in S} \phi(x)$$

- We can integrate functions $a : X \to \mathbb{R}$:

$$\int_X a \, \mathrm{d}\phi = \sum_{x \in X} a(x) \cdot \phi(x).$$

- For a function $f : X \to Y$

$$\mathcal{D}(f)(\phi)(y) = \phi(f^{-1}(y)) = \sum_{x \in f^{-1}(y)} \phi(x)$$

Distribution Monad II

- Unit $\eta_X : X \to \mathcal{D}(X)$

$$\eta_X(x)(x') = [\![x = x']\!] = \begin{cases} 0 & \text{if } x \neq x' \\ 1 & \text{if } x = x' \end{cases}$$

- Given $x \in X$, we define $\mathrm{ev}(x) : \mathcal{D}(X) \to [0, 1]$ by

$$\mathrm{ev}(x)(\phi) = \phi(x)$$

- Define $\mu_X : \mathcal{D}^2(X) \to \mathcal{D}(X)$

$$\mu_X(\Phi)(x) = \int_{\mathcal{D}(X)} \mathrm{ev}(x)\, \mathrm{d}\Phi = \sum_{\phi \in \mathcal{D}(X)} \phi(x) \cdot \Phi(\phi)$$

Relation to Stochastic Matrices

- For X, Y finite, a Kleisli map $X \to \mathcal{D}(Y)$ uncurries to a stochastic matrix $X \times Y \to [0,1]$.
- Kleisli composition agrees with matrix multiplication.
- $\mathcal{D}$ is the monad we get if we try to turn **FinStoch** into a Kleisli category.
- We can't have **FinStoch** itself be a Kleisli category over **FinSet** because **FinStoch**$(1,2) \cong [0,1]$ so is infinite, but hom sets of **FinSet** are finite.

Finite Probability Spaces

- A *probability space* is a pair (X, ϕ) where X finite and $\phi \in \mathcal{D}(X)$.
- A *measure-preserving map* $f : (X, \phi) \to (Y, \psi)$ is a function $f : X \to Y$ such that $\mathcal{D}(f)(\phi) = \psi$.
- A *nullset-reflecting map* f is one where $\psi(y) = 0$ implies $\mathcal{D}(f)(\phi)(y) = 0$.
- A *function* $f : X \to Y$ has its usual definition.
- A *random variable* $f : (X, \phi) \to Y$ is a function $f : X \to Y$.

What Sort of Thing are Random Variables?

- Functions form a category **FinSet**.
- Nullset-reflecting maps form a category **FinProbSp**, of which measure-preserving maps form a subcategory.
- Given a random variable $g : (X, \phi) \to Y$ and a nullset reflecting map $f : (W, \psi) \to (X, \phi)$

$$g \circ f : (W, \psi) \to Y$$

  is a random variable.
- Given a function $h : Y \to Z$

$$h \circ g : (X, \phi) \to Z$$

  is a random variable.
- Random variables are not a category, but a profunctor $\mathbf{FinProbSp}^{\mathrm{op}} \times \mathbf{FinSet} \to \mathbf{Set}$.

Example

- Let G_n be the set of graphs with n as their vertex set.
- Define $\phi \in \mathcal{D}(G_n)$ to give all graphs equal probability.
- Let $E : (G_n, \phi) \to \frac{1}{2}n(n-1)$ map a graph to its number of edges.
- Let $\chi : (G_n, \phi) \to n$ map a graph to its chromatic number.
- We can take the marginal probability, *e.g.*

$$i \mapsto \mathbb{P}(\chi = i) = \mathcal{D}(\chi)(\phi) \in \mathcal{D}(n)$$

- Conditional probabilities:

$$i \mapsto j \mapsto \mathbb{P}(E = j \mid \chi = i) = \frac{\mathbb{P}(E = j \wedge \chi = i)}{\mathbb{P}(\chi = i)} = \frac{\mathcal{D}(\langle E, \chi \rangle)(j, i)}{\mathcal{D}(\chi)(i)}$$

This gives a Kleisli map $n \to \mathcal{D}(\frac{1}{2}n(n-1))$.

Conditional Probabilities and Disintegrations

- For a random variable $\mathcal{Y} : (X, \phi) \to Y$, we can always consider conditional probabilities relative to $\mathrm{id}_X = \mathcal{X} : (X, \phi) \to X$, *i.e.*

$$y \mapsto x \mapsto \mathbb{P}(\mathcal{X} = x \mid \mathcal{Y} = y)$$

  defining a map $\mathbb{E}_{\mathcal{Y}} : Y \to \mathcal{D}(X)$.
- This is characterized by two properties:
  1. It is a *probabilistic section* of $\mathcal{Y}$, *i.e.* in $\mathcal{K}\ell(\mathcal{D})$ we have $\mathcal{Y} \diamond \mathbb{E}_{\mathcal{Y}} = \mathrm{id}_Y$, equivalently $\mathcal{D}(\mathcal{Y}) \circ \mathbb{E}_{\mathcal{Y}} = \eta_Y$ in **Set**.
  2. It is compatible with ϕ the other way, *i.e.* $(\mu_X \circ \mathcal{D}(\mathbb{E}_{\mathcal{Y}}) \circ \mathcal{D}(\mathcal{Y}))(\phi) = \phi$.
- If we have $\mathcal{Y} : (X, \phi) \to Y$ and $\mathcal{Z} : (X, \phi) \to Z$ then the map $\mathbb{P}(\mathcal{Y} \mid \mathcal{Z}) : Z \to \mathcal{D}(Y)$ can be recovered as:

$$\mathbb{P}(\mathcal{Y} \mid \mathcal{Z}) = \mathcal{Y} \diamond \mathbb{E}_{\mathcal{Z}} = \mathcal{D}(\mathcal{Y}) \circ \mathbb{E}_{\mathcal{Z}}$$

- So when generalizing we concentrate on disintegrations.

Towards the Radon Monad

- One way to generalize from finite sets is to use *profinite sets*, equivalently Stone spaces.
- This is successful with *e.g.* the finite power set monad, there is a monad on **Stone** extending it.
- This won't work with $\mathcal{D}$ because the usual topology on $\mathcal{D}(2) \cong [0,1]$ is not Stone.
- So use compact Hausdorff spaces.

C and the Radon functor $\mathcal{R}$

- For a compact Hausdorff space X, let $C(X)$ be the set of continuous functions $X \to \mathbb{C}$.
- Algebraic operations for $C(X)$ are defined pointwise.
- $a \leq b$ in $C(X)$ iff $\forall x \in X. a(x) \leq b(x)$.
- If $a \geq 0$ we say it is *positive*, $a \in C(X)_+$.
- We say $\phi : C(X) \to \mathbb{C}$ is *positive* if it preserves positivity, *i.e.* for all $a \in C(X)_+$, $\phi(a) \geq 0$. (Equivalent to being monotone)
- $\mathcal{R}(X) = \{\phi : C(X) \to \mathbb{C} \mid \phi \text{ positive}, \phi(1) = 1\}$.
- As a functor $\mathcal{R}$ is a composite $\mathcal{S} \circ C$ where C and $\mathcal{S}$ are both contravariant and defined by postcomposition.

The Radon Monad

- Unit is "Dirac δ functions":

$$\eta_X(x)(a) = a(x)$$

- For the multiplication, we need a function $\zeta_X : C(X) \to C(\mathcal{R}(X))$

$$\zeta_X(a)(\phi) = \phi(a)$$

- Multiplication is "barycentre":

$$\mu_X(\Phi)(a) = \Phi(\zeta_X(a))$$

- The Radon monad was originally defined by Świrszcz [Ś74], before the Giry monad.

Breaking Up is Hard to do

- Conditional probabilities don't always exist because not every continuous function has a probabilistic section.
- Example: the binary digits map $2^{\mathbb{N}} \to [0,1]$.
- On the dense set of numbers with a unique binary representation, $g : [0,1] \to \mathcal{R}(2^{\mathbb{N}})$ is contained in $\eta_X(2^{\mathbb{N}})$
- By continuity g maps into $\eta_X(2^{\mathbb{N}})$ which is a Stone space.
- A continuous map from $[0,1]$ to a Stone space is constant.

Using Measure Theory

- It's still not true, even for $[0,1]$, that every Borel measurable map has a probabilistic section (using the Giry monad $\mathcal{G}$)
- It is true that for standard Borel probability spaces (includes $[0,1]$ and $2^{\mathbb{N}}$) that disintegrations exist, but we must weaken the probabilistic section requirement to $\mathcal{G}(\mathcal{Y}) \circ \mathbb{E}_{\mathcal{Y}} = \eta_Y$ holding for *almost all* $y \in Y$.
- Define **StdBoProb** to have standard Borel probability spaces (X, Σ_X, ν_X) as objects and almost-everywhere equivalence classes of nullset-reflecting maps as morphisms.
- Want a monad T on **StdBoProb** such that **StdBoProb**$(1, T(X))$ is the set of measures absolutely continuous to ν_X (definable by a density function).
- Problem: **StdBoProb**$(1, Y)$ is always countable, **StdBoProb**$(1, T(2))$ should be $\mathcal{D}(2) \cong [0,1]$.

How to Proceed?

- We go beyond standard Borel spaces.
- We define the monad by defining a comonad on a dual category.
- We use Gelfand duality for commutative W^*-algebras, extending previous work [FJ15] on probabilistic Gelfand duality for the Radon monad.
- Under duality, disintegrations are *conditional expectations*, which are known to exist under the circumstances we want.

Gelfand Duality

- Algebraic Geometry with continuous functions.
- Stone duality : Stone spaces :: Gelfand duality : compact Hausdorff spaces
- $C(X)$ is a unital commutative *-algebra over $\mathbb{C}$, i.e. a ring, a $\mathbb{C}$-vector space and has an involution $-^*$, pointwise complex conjugation.
- The norm

$$\|a\| = \sup_{x \in X} |a(x)|$$

  makes $C(X)$ into a unital Banach *-algebra (an internal *-monoid in $\mathbf{Ban}_1$).
- Define a functor $C : \mathbf{CHaus} \to \mathbf{CBan^*Alg}^{\mathrm{op}}$ is defined for $f : X \to Y$ and $b \in C(Y)$ by

$$C(f)(b) = b \circ f \in C(X).$$

Gelfand Duality II

- For a Banach *-algebra A, the set $\mathbf{CBan^*Alg}(A, \mathbb{C})$ is called the *spectrum*, $\mathrm{Spec}(A)$.
- $\mathrm{Spec}(A)$ has a compact Hausdorff topology (the weak-* topology) and defines a functor $\mathrm{Spec} : \mathbf{CBan^*Alg}^{\mathrm{op}} \to \mathbf{CHaus}$, if $g : A \to B$ and $\psi \in \mathrm{Spec}(B)$ then

$$\mathrm{Spec}(g)(\psi) = \psi \circ g \in \mathrm{Spec}(A)$$

Gelfand Duality III

- Then $C \dashv \mathrm{Spec}$ with the unit and counit given by exchanging the role of function and argument:

$$\eta_X : X \to \mathrm{Spec}(C(X)) \qquad \epsilon_A : A \to C(\mathrm{Spec}(A))$$
$$\eta_X(x)(a) = a(x) \qquad \epsilon_A(a)(\phi) = \phi(a)$$

- η_X is always an isomorphism.
- ϵ_A is an isomorphism iff $\|a^*a\| = \|a\|^2$, in which case A is said to be a C^**-algebra*.
- So $C : \mathbf{CHaus} \to \mathbf{CC^*Alg}^{\mathrm{op}}$ and $\mathrm{Spec} : \mathbf{CC^*Alg}^{\mathrm{op}} \to \mathbf{CHaus}$ form an adjoint equivalence: *Gelfand duality*.

Positivity in C*-algebras

- In a C*-algebra A, the positive cone A_+ is defined to be the elements of the form $b^* b$.
- This recovers the definition for $C(X)$ seen earlier.
- A *positive map* $f : A \to B$ is a linear map such that $f(A_+) \subseteq B_+$.
- A *positive unital map* or *PU map* is a positive map that also preserves the unit element.
- These form a category $\mathbf{CC^*Alg}_{\mathrm{PU}}$ of which $\mathbf{CC^*Alg}$ is a subcategory (same objects).

States and Radon Measures

- A *state* on a C*-algebra is just a PU map to $\mathbb{C}$:

$$\mathcal{S}(A) = \mathbf{CC^*Alg}_{\mathrm{PU}}(A, \mathbb{C}).$$

- It defines a functor $\mathcal{S} : \mathbf{CC^*Alg}_{\mathrm{PU}} \to \mathbf{CHaus}$, and $\mathrm{Spec} \subseteq \mathcal{S}$.
- A probability measure ν defined on a σ-algebra Σ in which each $a \in C(X)$ is measurable defines a state on $C(X)$:

$$\phi_\nu(a) = \int_X a \,\mathrm{d}\nu$$

- The *Riesz representation theorem*[1] is that this gives a bijection between regular Borel probability measures on X and $\mathcal{S}(C(X)) = \mathcal{R}(X)$.

[1] Kakutani's version of it.

Probabilistic Gelfand Duality

- We have $\mathcal{K}\ell(\mathcal{R}) \simeq \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}}$. [FJ15]
- $C_{PU} : \mathcal{K}\ell(\mathcal{R}) \to \mathbf{CC^*Alg}_{\mathrm{PU}}$ defined by $C_{PU}(X) = C(X)$ and for a map $f : X \to \mathcal{R}(X)$ we define $C_{PU}(f) : C(Y) \to C(X)$

$$C_{PU}(f)(b)(x) = f(x)(b)$$

  i.e. curried swap of arguments.
- Fullness and faithfulness are easy, essential surjectivity follows from classical Gelfand duality.

Probabilistic Gelfand Duality II

- Since $\mathbf{CC^*Alg}^{\mathrm{op}} \simeq \mathbf{CHaus}$, there is a comonad T on $\mathbf{CC^*Alg}$ such that $\mathcal{K}\ell(T) \simeq \mathbf{CC^*Alg}_{\mathrm{PU}}$. In fact $T = C \circ \mathcal{S}$.
- Overall picture:

$$\begin{array}{ccc}
\mathcal{K}\ell(\mathcal{R}) & \xrightarrow{C_{PU}} & \mathbf{CC^*Alg}^{\mathrm{op}}_{\mathrm{PU}} \\
F_{\mathcal{R}} \uparrow \dashv \downarrow G_{\mathcal{R}} & & \hookuparrow \dashv \downarrow C \circ \mathcal{S} \\
\mathbf{CHaus} & \xrightarrow[C]{} & \mathbf{CC^*Alg}^{\mathrm{op}},
\end{array}$$

- Now we discuss how to do this with Gelfand duality for measure spaces.

Localizable Measure Spaces

- Let (X, Σ, ν) be a measure space. $L^1(X, \nu)$ is the absolutely integrable $\mathbb{C}$-valued random variables and $L^\infty(X)$ the bounded random variables, both up to equality *a.e.*.
- They are Banach spaces, $L^\infty(X)$ a C*-algebra.
- We define a bilinear pairing $L^\infty(X, \nu) \times L^1(X, \nu) \to \mathbb{C}$:

$$\langle a, \phi \rangle = \int_X a\phi \, \mathrm{d}\nu$$

- Currying gives a linear map $L^\infty(X, \nu) \to L^1(X, \nu)^*$.
- (X, Σ, ν) is called *localizable* iff this map is an isometric isomorphism. [Seg51]
- If ν is a probability measure or a σ-finite measure, (X, Σ, ν) is localizable.

W*-algebras

- A W*-algebra A is a C*-algebra with a *predual* A_*.
- A_* is a Banach space such that $(A_*)^* \cong A$ in $\mathbf{Ban}_1$.
- A_* is unique up to isometric isomorphism (Sakai in noncommutative case, Grothendieck in commutative case).
- So if (X, Σ, ν) is localizable, $L^\infty(X, \nu)$ is a W*-algebra with predual $L^1(X, \nu)$.
- Since W*-algebras are C*-algebras, we have *-homomorphisms and PU maps.
- To work with measure spaces, we need to strengthen these notions.

Morphisms of W*-algebras

- A W*-algebra is a bounded dcpo (if commutative a bounded-complete lattice): Every bounded monotone net has a least upper bound.
- A PU-map $f : A \to B$ is *normal* iff, equivalently:
  1. (i) f preserves least upper bounds of bounded directed nets. (So isomorphisms are normal)
  2. (ii) f is continuous in the weak-* topologies $\sigma(A, A_*)$ and $\sigma(B, B_*)$.
  3. (iii) There exists a linear map $f_* : B_* \to A_*$ such that $\langle f(a), \psi \rangle = \langle a, f_*(\psi) \rangle$ for all $a \in A$, $\psi \in B_*$.
- For *-homomorphisms, f is normal iff it defines a complete Boolean homomorphism on projections.
- **CW*Alg**$_{\mathrm{PU}}$ has normal PU-maps as homomorphisms, **CW*Alg** normal *-homomorphisms.

L^∞ as a functor

- For $f : (X, \Sigma_X, \nu_X) \to (Y, \Sigma_Y, \nu_Y)$ define

$$L^\infty(f) : L^\infty(Y, \nu_Y) \to L^\infty(X, \nu_X)$$
$$L^\infty(f)([b]) = [b \circ f]$$

- To be well-defined, we must require f to be nullset-reflecting, *i.e.* if $T \in \Sigma_Y$ with $\nu_Y(T) = 0$, then $\nu_X(f^{-1}(T)) = 0$.
- This makes a *-homomorphism, and it *is* normal if ν_Y is σ-finite.
- But what about the general case?

Counterexample?

- For the moment, drop the axiom of choice and suppose every subset of $[0,1]$ is Lebesgue measurable.
- So Lebesgue measure is a probability measure $\nu_X : \mathcal{P}([0,1]) \to [0,1]$.
- On $[0,1]$ let $\Sigma = \mathcal{P}([0,1])$ and ν_X be the Lebesgue measure, ν_Y the counting measure. Then $f = \mathrm{id} : ([0,1], \nu_X) \to ([0,1], \nu_Y)$ is measurable and nullset-reflecting.
- Define the monotone net $(\chi_F)_{F \in \mathcal{P}_{\mathrm{fin}}([0,1])}$. Then $\bigvee_{F \in \mathcal{P}_{\mathrm{fin}}([0,1])} \chi_F = 1$ in $L^\infty([0,1], \nu_Y) = \ell^\infty([0,1])$.
- But $[\chi_F] = 0$ for all $F \in \mathcal{P}_{\mathrm{fin}}([0,1])$ when considered in $L^\infty([0,1], \nu_X)$, so $\bigvee_{F \in \mathcal{P}_{\mathrm{fin}}([0,1])} L^\infty(f)(\chi_F) = \bigvee_{F \in \mathcal{P}_{\mathrm{fin}}([0,1])} 0 = 0$.
- So $L^\infty(f)$ is *not* normal.

Counterexample? II

- For this example, all we needed was a probability measure ν_Y on a set X such that $\nu_Y(\{x\}) = 0$ for all $x \in X$, which may be consistent with ZFC as well (if real-valued measurable cardinals are [Ula30]).
- For *localizable* measure spaces, this is a necessary assumption to get a non-normal map.
- Luckily this counterexample will disappear later once we add more assumptions about the measure spaces we use.
- Until we get to this, we will require f additionally to be *normal*, *i.e.* that $L^\infty(f)$ is normal in $\mathbf{CW^*Alg}$.

Gelfand Duality for W*-algebras

- First version: Adapt Gelfand duality for C*-algebras.
- Spec takes **CW*Alg** to a (non-full) subcategory of **CHaus**.
- The spaces we get are called *hyperstonean*.
- Two characterizations of when $C(f)$ is normal, for $f : X \to Y$ continuous:
  1. If $N \subseteq Y$ is a closed set with empty interior, so is $f^{-1}(N)$.
  2. If $U \subseteq X$ is open, so is $f(U)$ (*a.k.a.* f is open).
- So Spec : $\mathbf{CW^*Alg}^{\mathrm{op}} \to \mathbf{HypStonean}$ and C : $\mathbf{HypStonean} \to \mathbf{CW^*Alg}^{\mathrm{op}}$ form an adjoint equivalence by restricting Gelfand duality.
- How does this relate to measure theory?

Hyperstonean Spaces

- The nowhere dense sets form a σ-ideal.
- There is a localizable regular measure[2] ν on the Baire property σ-algebra such that nowhere dense sets are exactly the sets of measure zero.
- The inclusion map $C(X) \hookrightarrow L^\infty(X, \nu)$ is an isomorphism.
- So L^∞ is a duality between a subcategory of measure spaces and $\mathbf{CW^*Alg}$.
- Unfortunately the only hyperstonean spaces we ever start with are finite discrete spaces.
- $\mathbb{N}, 2^\mathbb{N}, \mathbb{N}^\mathbb{N}, [0,1], \mathbb{R}, \mathbb{R}^\mathbb{N}$ and so on are not hyperstonean.
- But hyperstonean spaces serve as a basis for generalization.

[2] Not unique, not in general a Radon measure

Duality for Measure Spaces

- Essential surjectivity of L^∞ is provided by hyperstonean spaces.
- Faithfulness doesn't hold for functions, we take equivalence classes.
- The equivalence relation is coarser than equality *a.e.*. For $f, g : (X, \nu_X) \to (Y, \nu_Y)$ we define

$$f \sim g \Leftrightarrow \forall T \in \Sigma_Y . \nu_X(f^{-1}(T) \triangle g^{-1}(T)) = 0$$

- This equivalence relation is compatible with composition so taking the quotient keeps L^∞ as a functor and makes it faithful.

Duality for Measure Spaces

- For a normal map $g : L^\infty(Y) \to L^\infty(X)$, by the naturality of the counit the following commutes:

$$\begin{array}{ccc}
L^\infty(Y) & \xrightarrow{\quad g \quad} & L^\infty(X) \\
\Big\downarrow{\scriptstyle \varepsilon_{L^\infty(Y)}} & & \Big\uparrow{\scriptstyle \varepsilon^{-1}_{L^\infty(X)}} \\
L^\infty(\mathrm{Spec}(L^\infty(Y))) & \xrightarrow[L^\infty(\mathrm{Spec}(g))]{} & L^\infty(\mathrm{Spec}(L^\infty(X)))
\end{array}$$

- We say X is *liftable* if there exists $\lambda_X : X \to \mathrm{Spec}(L^\infty(X))$ such that $L^\infty(\lambda_X) = \varepsilon^{-1}_{L^\infty(X)}$.
- We say Y is *coliftable* if there exists $\kappa_Y : \mathrm{Spec}(L^\infty(Y) \to Y$ such that $L^\infty(\kappa_Y) = \varepsilon_{L^\infty(Y)}$.
- In this case, define $f = \kappa_Y \circ \mathrm{Spec}(g) \circ \lambda_X$.

$$\begin{aligned}
L^\infty(f) &= L^\infty(\lambda_X) \circ L^\infty(\mathrm{Spec}(g)) \circ L^\infty(\kappa_Y) \\
&= \varepsilon^{-1}_{L^\infty(X)} \circ L^\infty(\mathrm{Spec}(g)) \circ \varepsilon_{L^\infty(Y)} = g.
\end{aligned}$$

Finally Choosing a Category of Measure Spaces

- So we could define **Meas** to have liftable coliftable localizable spaces as objects and classes of normal measurable maps.
- Fremlin [Fre02] showed that[3] these spaces are the compact[4] complete strictly localizable spaces.
- Furthermore, usual measures such as Lebesgue measure, counting measures, and the independent Bernoulli trial measures on 2^κ have these properties.
- A nullset-reflecting map from a compact measure space to a strictly localizable one is normal (essentially from [Fre03]), so we don't need to worry about that any more.
- We now have a **Meas** such that $L^\infty : \mathbf{Meas} \to \mathbf{CW^*Alg}^{\mathrm{op}}$ is an equivalence.
- Reference: [Pav22]

[3]Subject to a technical requirement of being complete and locally determined

[4]In the measure-theoretic sense, not topological

A Comonad on $\mathbf{CW^*Alg}$

- By analogy to C*-algebras, the probabilistic category of W*-algebras is $\mathbf{CW^*Alg}_{\mathrm{PU}}$ (normal positive unital maps).
- We want a monad T on **Meas** whose Kleisli category is equivalent to ${\mathbf{CW^*Alg}_{\mathrm{PU}}}^{\mathrm{op}}$. We can use W*-Gelfand duality to work on the W*-side first.
- So show that $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$ has a left adjoint F such that the comparison functor for the coKleisli category of the comonad is an equivalence.
- We need to boost up the left adjoint to $\mathbf{CC^*Alg} \hookrightarrow \mathbf{CC^*Alg}_{\mathrm{PU}}$ somehow.

Double Duals

- The forgetful functor $\mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CC^*Alg}_{\mathrm{PU}}$ has a left adjoint, the *enveloping W^*-algebra*. For $A \in \mathbf{CC^*Alg}$ it is the double dual A^{**}. This also produces a left adjoint to $\mathbf{CW^*Alg} \to \mathbf{CC^*Alg}$.
- Observe:

$$\begin{aligned}\mathbf{CW^*Alg}_{\mathrm{PU}}(A^{**}, B) \cong \mathbf{CC^*Alg}_{\mathrm{PU}}(A, B) &\cong \mathbf{CC^*Alg}(C(\mathcal{S}(A)), B) \\ &\cong \mathbf{CW^*Alg}(C(\mathcal{S}(A))^{**}, B).\end{aligned}$$

- It must be that $F(A^{**}) = C(\mathcal{S}(A))^{**}$.
- Not all W*-algebras are double duals!

Defining F

Lemma

CW*Alg *is monadic over* **CC*Alg**, *i.e.* $\mathbf{CW^*Alg} \simeq \mathcal{EM}(\text{-}^{**})$.

- Therefore
$$A^{****} \underset{\epsilon_A^{**}}{\overset{\epsilon_{A^{**}}}{\rightrightarrows}} A^{**} \xrightarrow{\epsilon_A} A$$
is a coequalizer (the *canonical presentation* of A).
- This coequalizer is preserved by the inclusion $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$ because it is reflexive [BW05].
- Since left adjoints preserve colimits and **CW*Alg** is cocomplete, this allows us to define $F : \mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CW^*Alg}$.
- The coKleisli comparison functor is an equivalence with $\mathbf{CW^*Alg}_{\mathrm{PU}}$ because $\mathbf{CW^*Alg}_{\mathrm{PU}}$ and **CW*Alg** have the same objects. [Wes17, Theorem 9]

Main Result

Theorem

There is a monad T on **Meas** *such that* $\mathcal{K}\ell(T) \simeq \mathbf{CW^*Alg}_{\mathrm{PU}}$.

- It seems the simplest way to realize $T(X)$ is just as $\mathrm{Spec}(F(L^\infty(X)))$.
- For a countable set X

$$T(X) \cong ([0,1], \mathcal{P}([0,1]), \nu_d) + ([0,1]^2, \mathcal{P}([0,1]) \otimes \widehat{\mathcal{B}o([0,1])}, \nu_d \otimes \nu_L)$$

  where ν_d is the counting measure and ν_L the Lebesgue measure.
- We only have that $\mathbf{Meas}(1, T(X))$ corresponds to the density functions on X, not that $T(X)$ does.

Final Remarks

- The need to use non-probabilistic, non-standard Borel spaces is analogous to the need to use **Set** instead of **FinSet** to define $\mathcal{D}$.
- I still have more work to finish with this.
- It should be that T is commutative so $\mathcal{K}\ell(T)$ and $\mathbf{CW^*Alg}_{\mathrm{PU}}{}^{\mathrm{op}}$ are Markov categories in the sense of [Fri20] (work in progress).
- Preprint available on `www.robertfurber.com`

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