The Foundations: Logic and Proofs

Chapter 1, Part III: Proofs

Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
- Rules of Inference for Quantified Statements
- Building Arguments for Quantified Statements

Revisiting the Socrates Example

- We have the two premises:
  - “All men are mortal.”
  - “Socrates is a man.”
- And the conclusion:
  - “Socrates is mortal.”
- How do we get the conclusion from the premises?

The Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\begin{array}{l} \forall x(Man(x) \rightarrow Mortal(x)) \\ Man(Socrates) \\ \hline \therefore \quad Mortal(Socrates) \end{array}$$

- We will see shortly that this is a valid argument.

Valid Arguments

- We will show how to construct valid arguments in two stages; first for propositional logic and then for predicate logic. The rules of inference are the essential building block in the construction of valid arguments.

1. Propositional Logic
    2. Inference Rules
3. Predicate Logic
    4. Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Arguments in Propositional Logic

- A *argument* in propositional logic is a sequence of propositions. All but the final proposition are called *premises*. The last statement is the *conclusion*.
- The argument is valid if the premises imply the conclusion. An *argument form* is an argument that is valid no matter what propositions are substituted into its propositional variables.
- If the premises are $p1, p2, \ldots, pn$ and the conclusion is q then

  $(p1 \land p2 \land \ldots \land pn) \rightarrow q$ is a tautology.
- Inference rules are all argument simple argument forms that will be used to construct more complex argument forms.

Rules of Inference for Propositional Logic: Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Example:

Let p be "It is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$

Example:
Let p be "it is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:
$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example:
Let *p* be "it snows."
Let *q* be "I will study discrete math."
Let *r* be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$p \rightarrow (p \vee q)$

Example:
Let *p* be “I will study discrete math.”
Let *q* be “I will visit Las Vegas.”

“I will study discrete math.”

“Therefore, I will study discrete math or I will visit
Las Vegas.”

Simplification

$$\frac{p \wedge q}{\therefore q}$$

Corresponding Tautology:
$(p \wedge q) \rightarrow p$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:
$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:
Let p be “I will study discrete math.”
Let q be “I will study English literature.”

“I will study discrete math.”
“I will study English literature.”

“Therefore, I will study discrete math and I will study English literature.”

Resolution

Resolution plays an important role in AI and is used in Prolog.

$$\begin{array}{l} \neg p \lor r \\ p \lor q \\ \hline \therefore q \lor r \end{array}$$

Corresponding Tautology:
$((\neg p \lor r) \land (p \lor q)) \rightarrow (q \lor r)$

Example:
Let *p* be "I will study discrete math."
Let *r* be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature."

Using the Rules of Inference to Build Valid Arguments

- A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$\begin{array}{l} S_1 \\ S_2 \\ \vdots \\ S_n \\ \hline \therefore C \end{array}$$

Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Conjunction using (1)
3. $p \rightarrow q$	Conjunction using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments

Example 2:

- With these hypotheses:

"It is not sunny this afternoon and it is colder than yesterday."

"We will go swimming only if it is sunny."

"If we do not go swimming, then we will take a canoe trip."

"If we take a canoe trip, then we will be home by sunset."

- Using the inference rules, construct a valid argument for the conclusion:

"We will be home by sunset."

Solution:

1. Choose propositional variables:

p : "It is sunny this afternoon." *r* : "We will go swimming." *t :* "We will be home by sunset."

q : "It is colder than yesterday." *s :* "We will take a canoe trip."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$$
$$\text{Conclusion: } t$$

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Valid Arguments

3. Construct the Valid Argument

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

- Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

“All dogs are cuddly.”

“Therefore, Fido is cuddly.”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Using Rules of Inference

Example 1: Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

"Every man has two legs." "John Smith is a man."

Solution: Let $M(x)$ denote "x is a man" and $L(x)$ " x has two legs" and let John Smith be a member of the domain.

Valid Argument:

Step	**Reason**
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let *C*(*x*) denote "*x* is in this class," *B*(*x*) denote " *x* has read the book," and *P*(*x*) denote "*x* passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \to P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

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Using Rules of Inference

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	EG from (8)

Returning to the Socrates Example

$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$

$$\therefore \quad Mortal(Socrates)$$

Solution for Socrates Example

Valid Argument

Step	**Reason**
1. $\forall x(Man(x) \rightarrow Mortal(x))$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (4)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$
\begin{array}{l}
\forall x(P(x) \rightarrow Q(x)) \\
P(a), \text{where } a \text{ is a particular} \\
\quad \text{element in the domain} \\
\hline
\therefore Q(a)
\end{array}
$$

This rule could be used in the Socrates example.

Introduction to Proofs

Section 1.7

Proofs of Mathematical Statements

- A *proof* is a valid argument that establishes the truth of a statement.
- In math, CS, and other disciplines, informal proofs which are generally shorter, are generally used.
  - More than one rule of inference are often used in a step.
  - Steps may be skipped.
  - The rules of inference used are not explicitly stated.
  - Easier for to understand and to explain to people.
  - But it is also easier to introduce errors.
- Proofs have many practical applications:
  - verification that computer programs are correct
  - establishing that operating systems are secure
  - enabling programs to make inferences in artificial intelligence
  - showing that system specifications are consistent

Definitions

- A *theorem* is a statement that can be shown to be true using:
  - definitions
  - other theorems
  - *axioms* (statements which are given as true)
  - rules of inference
- A *lemma* is a ‘helping theorem’ or a result which is needed to prove a theorem.
- A *corollary* is a result which follows directly from a theorem.
- Less important theorems are sometimes called *propositions*.
- A *conjecture* is a statement that is being proposed to be true. Once a proof of a conjecture is found, it becomes a theorem. It may turn out to be false.

Forms of Theorems

- Many theorems assert that a property holds for all elements in a domain, such as the integers, the real numbers, or some of the discrete structures that we will study in this class.
- Often the universal quantifier (needed for a precise statement of a theorem) is omitted by standard mathematical convention.

For example, the statement:

“If $x > y$, where x and y are positive real numbers, then $x2 > y2$ ”

really means

“For all positive real numbers x and y, if $x > y$, then $x2 > y2$.”

Proving Theorems

- Many theorems have the form:
$$\forall x(P(x) \rightarrow Q(x))$$

- To prove them, we show that where *c* is an arbitrary element of the domain, $P(c) \rightarrow Q(c)$

- By universal generalization the truth of the original formula follows.

- So, we must prove something of the form: $p \rightarrow q$

Proving Conditional Statements: $p \rightarrow q$

- *Trivial Proof*: If we know q is true, then

  $p \rightarrow q$ is true as well.

"If it is raining then 1=1."

- *Vacuous Proof*: If we know p is false then

  $p \rightarrow q$ is true as well.

"If I am both rich and poor then 2 + 2 = 5."

[Even though these examples seem silly, both trivial and vacuous proofs are often used in mathematical induction, as we will see in Chapter 5)]

Even and Odd Integers

Definition: The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k, such that $n = 2k + 1$. Note that every integer is either even or odd and no integer is both even and odd.

We will need this basic fact about the integers in some of the example proofs to follow. We will learn more about the integers in Chapter 4.

Proving Conditional Statements: $p \rightarrow q$

- *Direct Proof*: Assume that p is true. Use rules of inference, axioms, and logical equivalences to show that q must also be true.

Example: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Assume that n is odd. Then $n = 2k + 1$ for an integer k. Squaring both sides of the equation, we get:

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2r + 1$, where $r = 2k^2 + 2k$, an integer.

We have proved that if n is an odd integer, then n^2 is an odd integer. ▶

(▶ marks the end of the proof. Sometimes **QED** is used instead.)

Proving Conditional Statements: $p \rightarrow q$

Example: Prove that for an integer *n*, if n^2 is odd, then *n* is odd.

Solution: Use proof by contraposition. Assume *n* is even (i.e., not odd). Therefore, there exists an integer *k* such that $n = 2k$. Hence,

$$n^2 = 4k^2 = 2(2k^2)$$

and n^2 is even (i.e., not odd).

We have shown that if *n* is an even integer, then n^2 is even. Therefore by contraposition, for an integer *n*, if n^2 is odd, then *n* is odd. ▶

Proving Conditional Statements: $p \rightarrow q$

- *Proof by Contradiction*: (AKA *reductio ad absurdum*).

To prove p, assume $\neg p$ and derive a contradiction such as $p \wedge \neg p$. (an indirect form of proof). Since we have shown that $\neg p \rightarrow \mathbf{F}$ is true , it follows that the contrapositive $\mathbf{T} \rightarrow p$ also holds.

Example: Prove that if you pick 22 days from the calendar, at least 4 must fall on the same day of the week.

Solution: Assume that no more than 3 of the 22 days fall on the same day of the week. Because there are 7 days of the week, we could only have picked 21 days. This contradicts the assumption that we have picked 22 days.

Proof by Contradiction

- A preview of Chapter 4.

Example: Use a proof by contradiction to give a proof that √2 is irrational.

Solution: Suppose √2 is rational. Then there exists integers *a* and *b* with √2 = *a/b*, where *b*≠ *0* and *a* and *b* have no common factors (see Chapter 4). Then a^2=2b^2

Therefore *a^2* must be even. If *a^2* is even then *a* must be even (an exercise). Since *a* is even, *a* = 2*c* for some integer *c*. Thus,

$$2 = \frac{a^2}{b^2} \qquad 2b^2 = a^2 \qquad 2b^2 = 4c^2 \qquad b^2 = 2c^2$$

Therefore *b*^2 is even. Again then *b* must be even as well.

But then 2 must divide both *a* and *b*. This contradicts our assumption that *a* and *b* have no common factors. We have proved by contradiction that our initial assumption must be false and therefore √2 is irrational .

Proof by Contradiction

- A preview of Chapter 4.

  Example: Prove that there is no largest prime number.

  Solution: Assume that there is a largest prime number. Call it *pn*. Hence, we can list all the primes 2,3,.., *pn*. Form

$$r = p_1 \times p_2 \times \ldots \times p_n + 1$$

None of the prime numbers on the list divides *r*. Therefore, by a theorem in Chapter 4, either *r* is prime or there is a smaller prime that divides *r*. This contradicts the assumption that there is a largest prime. Therefore, there is no largest prime.

Theorems that are Biconditional Statements

- To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \rightarrow q$ and $q \rightarrow p$ are both true.

Example: Prove the theorem: "If n is an integer, then n is odd if and only if n^2 is odd."

Solution: We have already shown (previous slides) that both $p \rightarrow q$ and $q \rightarrow p$. Therefore we can conclude $p \leftrightarrow q$.

Sometimes *iff* is used as an abbreviation for "if an only if," as in

"If n is an integer, then n is odd iff n^2 is odd."

What is wrong with this?

"Proof" that *1 = 2*

Step	**Reason**
1. $a = b$	Premise
2. $a^2 = a \times b$	Multiply both sides of (1) by a
3. $a^2 - b^2 = a \times b - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Algebra on (3)
5. $a + b = b$	Divide both sides by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$
7. $2 = 1$	Divide both sides of (6) by b

Solution: Step 5. a - b = 0 by the premise and division by 0 is undefined.

Proof Methods and Strategy

Section 1.8

Proof by Cases

- To prove a conditional statement of the form:

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$$

- Use the tautology

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q] \leftrightarrow$$
$$[(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$$

- Each of the implications $p_i \rightarrow q$ is a *case*.

Proof by Cases

Example: Let a @ b = max{a, b} = a if $a \geq b$, otherwise a @ b = max{a, b} = b.

Show that for all real numbers a, b, c

(a @b) @ c = a @ (b @ c)

(This means the operation @ is associative.)

Proof: Let a, b, and c be arbitrary real numbers.

Then one of the following 6 cases must hold.

1. $a \geq b \geq c$
2. $a \geq c \geq b$
3. $b \geq a \geq c$
4. $b \geq c \geq a$
5. $c \geq a \geq b$
6. $c \geq b \geq a$

Continued on next slide ▭

Proof by Cases

Case 1: $a \geq b \geq c$

$(a @ b) = a$, $a @ c = a$, $b @ c = b$

Hence $(a @ b) @ c = a = a @ (b @ c)$

Therefore the equality holds for the first case.

A complete proof requires that the equality be shown to hold for all 6 cases. But the proofs of the remaining cases are similar. Try them.

Without Loss of Generality

Example: Show that if x and y are integers and both $x \bullet y$ *and* $x + y$ are even, then both x and y are even.

Proof: Use a proof by contraposition. Suppose x and y are not both even. Then, one or both are odd. Without loss of generality, assume that x is odd. Then $x = 2m + 1$ for some integer k.

Case 1: y is even. Then $y = 2n$ for some integer n, so

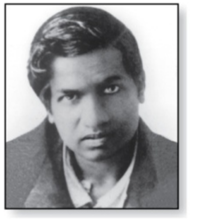
$$x + y = (2m + 1) + 2n = 2(m + n) + 1 \text{ is odd.}$$

Case 2: y is odd. Then $y = 2n + 1$ for some integer n, so

$$x \bullet y = (2m + 1)(2n + 1) = 2(2m \bullet n + m + n) + 1 \text{ is odd.}$$

▶

We only cover the case where x is odd because the case where y is odd is similar. The use phrase *without loss of generality* (WLOG) indicates this.



Existence Proofs

Srinivasa Ramanujan
(1887-1920)

- Proof of theorems of the form $\exists x P(x)$.
- **Constructive** existence proof:
  - Find an explicit value of *c*, for which *P(c)* is true.
  - Then $\exists x P(x)$ is true by Existential Generalization (EG).

Example: Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways:

Proof: 1729 is such a number since

1729 = 10^3 + 9^3 = 12^3 + 1^3

Godfrey Harold Hardy
(1877-1947)

Nonconstructive Existence Proofs

- In a *nonconstructive* existence proof, we assume no *c* exists which makes *P(c)* true and derive a contradiction.

Example: Show that there exist irrational numbers *x* and *y* such that x^y is rational.

Proof: We know that √2 is irrational. Consider the number $(\sqrt{2})^{\sqrt{2}}$. If it is rational, we have two irrational numbers x and y with x^y rational, namely $x = \sqrt{2}$ and $y = \sqrt{2}$.

But if $(\sqrt{2})^{\sqrt{2}}$ is irrational, then we can let $x = (\sqrt{2})^{\sqrt{2}}$ and $y = \sqrt{2}$ so that $x^y = ((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}\sqrt{2}} = (\sqrt{2})^2 = 2$, which is rational.

Counterexamples

- Recall $\exists x \neg P(x) \equiv \neg \forall x P(x)$.
- To establish that $\neg \forall x P(x)$ is true (or $\forall x P(x)$ is false) find a *c* such that $\neg P(c)$ is true or $P(c)$ is false.
- In this case *c* is called a *counterexample* to the assertion $\forall x P(x)$.

Example: "Every positive integer is the sum of the squares of 3 integers." The integer 7 is a counterexample. So the claim is false.

Uniqueness Proofs

- Some theorems asset the existence of a unique element with a particular property, $\exists! x\, P(x)$. The two parts of a *uniqueness proof* are
  - *Existence*: We show that an element x with the property exists.
  - *Uniqueness*: We show that if $y \neq x$, then y does not have the property.

Example: Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Solution:

- Existence: The real number $r = -b/a$ is a solution of $ar + b = 0$ because $a(-b/a) + b = -b + b = 0$.
- Uniqueness: Suppose that s is a real number such that $as + b = 0$. Then $ar + b = as + b$, where $r = -b/a$. Subtracting b from both sides and dividing by a shows that $r = s$.

Proof Strategies for proving $p \to q$

- Choose a method.
  1. First try a direct method of proof.
  2. If this does not work, try an indirect method (e.g., try to prove the contrapositive).
- For whichever method you are trying, choose a strategy.
  1. First try *forward reasoning.* Start with the axioms and known theorems and construct a sequence of steps that end in the conclusion. Start with p and prove q, or start with $\neg q$ and prove $\neg p$.
  2. If this doesn't work, try *backward reasoning*. When trying to prove q, find a statement p that we can prove with the property $p \to q$.

Universally Quantified Assertions

- To prove theorems of the form $\forall x P(x)$,assume *x* is an arbitrary member of the domain and show that *P*(*x*) must be true. Using UG it follows that $\forall x P(x)$.

Example: An integer *x* is even if and only if *x^2* is even.

Solution: The quantified assertion is

$\forall$*x* [*x* is even $\leftrightarrow$ *x^2* is even]

We assume *x* is arbitrary.

Recall that $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

So, we have two cases to consider. These are considered in turn.

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Universally Quantified Assertions

Case 1. We show that if x is even then x^2 *is* even using a direct proof (the *only if* part or *necessity*).

If x is even then $x = 2k$ for some integer k.

Hence $x^2 = 4k^2 = 2(2k^2)$ which is even since it is an integer divisible by 2.

This completes the proof of case 1.

Case 2 on next slide

Universally Quantified Assertions

Case 2. We show that if x^2 is even then x must be even (the *if* part or *sufficiency*). We use a proof by contraposition.

Assume x is not even and then show that x^2 is not even.

If x is not even then it must be odd. So, $x = 2k + 1$ for some k. Then $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

which is odd and hence not even. This completes the proof of case 2.

Since x was arbitrary, the result follows by UG.

Therefore we have shown that x is even if and only if x^2 is even.