Decidability of Zenoness, Syntactic Boundedness and Token-Liveness for Dense-Timed Petri Nets

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Abstract. We consider *Timed Petri Nets (TPNs)* : extensions of Petri nets in which each token is equipped with a real-valued clock. We consider the following three verification problems for TPNs.
(i) *Zenoness:* whether there is an infinite computation from a given marking which takes only a finite amount of time. We show decidability of zenoness for TPNs, thus solving an open problem from [dFERA00].
(ii) *Token liveness:* whether a token is *alive* in a marking, i.e., whether there is a computation from the marking which eventually consumes the token. We show decidability of the problem by reducing it to the *coverability problem*, which is decidable for TPNs.
(iii) *Boundedness:* whether the size of the reachable markings is bounded. We consider two versions of the problem; namely *semantic boundedness* where only live tokens are taken into consideration in the markings, and *syntactic boundedness* where also dead tokens are considered. We show undecidability of semantic boundedness, while we prove that syntactic boundedness is decidable through an extension of the Karp-Miller algorithm.

1 Introduction

Petri nets are one of the most widely used models for analysis and verification of concurrent systems. *Timed Petri nets (TPNs)* are extensions of Petri nets in the sense that each token has an "age" which is represented by a real valued clock (see [Bow96] for a survey). TPNs are computationally more powerful than timed automata [AD90], since they operate on a potentially unbounded number of clocks. This implies that TPNs can, among other things, model parameterized timed systems (systems consisting of an unbounded number of timed processes) [AN01]. Recently, several verification problems have been studied for TPNs (see e.g. [RGdFE99, dFERA00, AN01, AN02]). These problems are both extensions of classical problems previously studied for standard (untimed) Petri nets, and problems which are related to the timed behavior of TPNs. In this paper, we consider three verification problems for TPNs.

Zenoness A fundamental progress property for timed systems is that it should be possible for time to *diverge* [Tri99]. This requirement is justified by the fact that timed processes cannot be infinitely fast. Computations violating this property are called *zeno*. Given a TPN and a marking M, we check whether M is a *zeno-marking*, i.e., whether there is an infinite computation from M with a finite duration. The zenoness problem is solved in [Alu91] for timed automata using the region graph construction. Since region graphs only deal with a finite number of clocks, the algorithm of [Alu91] cannot be extended to check zenoness for TPNs. The zenoness problem was left open in [dFERA00]

both for dense TPNs (the model we consider in this paper) and for discrete TPNs (where behavior is interpreted over the discrete time domain).
We show that even for dense-timed Petri nets, a decidable symbolic representation of all zeno-markings can be effectively constructed. The construction can easily be modified (in fact simplified) to deal with the discrete-time case.
Token Liveness Markings in TPNs may contain tokens which cannot be used by any future computations of the TPN. Such tokens do not affect the behavior of the TPN and are therefore called *dead tokens*. We give an algorithm to check, given a token and a marking, whether the token is dead (or alive). We do this by reducing the problem to the problem of *coverability* in TPNs. An algorithm to solve the coverability problem is given in [AN01]. Token liveness for dense TPNs was left open in [dFERA00].
Boundedness We consider the *boundedness problem* for TPNs: given a TPN and an initial marking, check whether the size of reachable markings is bounded. The decidability of this problem depends on whether we take dead tokens into consideration. In *syntactic boundedness* one considers dead tokens as part of the (size of the) marking, while in *semantic boundedness* we disregard dead tokens; that is we check whether we can reach markings with unboundedly many live tokens. Using techniques similar to [RGdFE99] it can be shown that semantic boundedness is undecidable. On the other hand we show decidability of syntactic boundedness. This is achieved through an extension of the Karp-Miller algorithm where each node represents a region (rather than a single marking). The underlying ordering on the nodes (regions) inside the Karp-Miller tree is a *well quasi-ordering* [Hig52]. This guarantees termination of the procedure.
Decidability of syntactic boundedness was shown for the simpler discrete-time case in [dFERA00], while the problem was left open for the dense-time case.

2 Definitions

Timed Petri Nets We consider *Timed Petri Nets* (*TPN*s) where each token is equipped with a real-valued clock representing the "age" of the token. The firing conditions of a transition include the usual ones for Petri nets. Additionally, each arc between a place and a transition is labeled with a subinterval of natural numbers. When firing a transition, tokens which are removed (added) from (to) places should have their ages lying in the intervals of corresponding arcs.
We use $\mathcal{N}, \mathcal{R}^{\geq 0}$ to denote the sets of natural numbers and nonnegative reals, respectively. For a natural number k, we let $\mathcal{N}^k$ and $\mathcal{N}_\omega^k$ denote the set of vectors of dimension k over $\mathcal{N}$ and $\mathcal{N} \cup \{\omega\}$, respectively.
We use a set *Intrv* of intervals. An open interval is written as $(w : z)$ where $w \in \mathcal{N}$ and $z \in \mathcal{N} \cup \{\infty\}$. Intervals can also be closed in one or both directions, e.g. $[w : z]$ is closed in both directions and $[w : z)$ is closed to the left and open to the right.
For a set A, we use A^* and $A^\odot$ to denote the set of finite words and finite multisets over A, respectively. We view a multiset over A as a mapping from A to $\mathcal{N}$. Sometimes, we write multisets as lists, so $b = [2.4\ ,\ 5.1\ ,\ 5.1\ ,\ 2.4\ ,\ 2.4]$ represents a multiset b over $\mathcal{R}^{\geq 0}$ where $b(2.4) = 3$, $b(5.1) = 2$ and $b(x) = 0$ for $x \neq 2.4, 5.1$. For multisets b_1 and b_2 over $\mathcal{N}$, we say that $b_1 \leq b_2$ if $b_1(a) \leq b_2(a)$ for each $a \in A$. We define $b_1 + b_2$ to be the multiset b where $b(a) = b_1(a) + b_2(a)$, and (assuming $b_1 \leq b_2$) we define $b_2 - b_1$ to be the multiset b where $b(a) = b_2(a) - b_1(a)$, for each $a \in A$. We write $|b|$ for the number of elements in b. We use $\emptyset$ to denote the empty multiset and ϵ to denote the empty word.

Given a set A with an ordering $\preceq$ and a subset $B \subseteq A$, B is said to be *upward closed* if $a_1 \in B, a_2 \in A$ and $a_1 \preceq a_2$ implies $a_2 \in B$. Given a set $B \subseteq A$, we define the *upward closure* $B\uparrow$ to be the set $\{a \in A | \exists a' \in B : a' \preceq a\}$. A *downward closed* set B and the *downward closure* $B\downarrow$ are defined in a similar manner. We also use $a\uparrow$, $a\downarrow$, a instead of $\{a\}\uparrow$, $\{a\}\downarrow$, $\{a\}$, respectively.

Definition 1. *[AN01] A* Timed Petri Net (TPN) *is a tuple $N = (P, T, In, Out)$ where P is a finite set of places, T is a finite set of transitions and In, Out are partial functions from $T \times P$ to $Intrv$.*

If $In(t,p)$ (resp. $Out(t,p)$) is defined, then p is an *input (resp. output) place* of t. We let max denote the maximum integer appearing on the arcs of TPN. A *marking* M of N is a multiset over $P \times \mathcal{R}^{\geq 0}$. The marking M defines the numbers and ages of tokens in each place in the net. We identify a token in a marking M by the pair (p, x) representing its place and age in M. Then, $M((p,x))$ defines the number of tokens with age x in place p. Abusing notation again, we define, for each place p, a multiset $M(p)$ over $\mathcal{R}^{\geq 0}$, where $M(p)(x) = M((p,x))$. For a set M of markings, we define $|\mathsf{M}| = max(|M| : M \in \mathsf{M})$. For a marking M of the form $[(p_1, x_1)\ ,\ \ldots\ ,\ (p_n, x_n)]$ and $x \in \mathcal{R}^{\geq 0}$, we use M^{+x} to denote the marking $[(p_1, x_1 + x)\ ,\ \ldots\ ,\ (p_n, x_n + x)]$.
Transitions: We define two transition relations on the set of markings : timed transition and discrete transition. A *timed transition* increases the age of each token by the same real number. Formally, for $x \in \mathcal{R}^{\geq 0}$, $M_1 \longrightarrow_x M_2$ if $M_2 = M_1^{+x}$. We use $M_1 \longrightarrow_\delta M_2$ to denote that $M_1 \longrightarrow_x M_2$ for some $x \in \mathcal{R}^{\geq 0}$.
We define the set of *discrete transitions* $\longrightarrow_D$ as $\bigcup_{t \in T} \longrightarrow_t$, where $\longrightarrow_t$ represents the effect of *firing* the discrete transition t. More precisely, $M_1 \longrightarrow_t M_2$ if the set of input arcs $\{(p, \mathcal{I}) |\ In(t,p) = \mathcal{I}\}$ is of the form $\{(p_1, \mathcal{I}_1), \ldots, (p_k, \mathcal{I}_k)\}$, the set of output arcs $\{(p, \mathcal{I}) |\ Out(t,p) = \mathcal{I}\}$ is of the form $\{(q_1, \mathcal{J}_1), \ldots, (q_\ell, \mathcal{J}_\ell)\}$, and there are multisets $b_1 = [(p_1, x_1)\ , \ldots\ , (p_k, x_k)]$ and $b_2 = [(q_1, y_1)\ , \ldots\ , (q_\ell, y_\ell)]$ over $P \times \mathcal{R}^{\geq 0}$ such that the following holds: $b_1 \leq M_1$, $x_i \in \mathcal{I}_i$, for $i : 1 \leq i \leq k$, $y_i \in \mathcal{J}_i$, for $i : 1 \leq i \leq \ell$, $M_2 = (M_1 - b_1) + b_2$. We say that t is *enabled* in M if there is a b_1 such that the first two conditions are satisfied. A transition t may be fired only if for each incoming arc, there is a token with the "right" age in the corresponding input place. These tokens will be removed when the transition is fired. The newly produced tokens may only have ages which belong to the relevant intervals.
We write $\longrightarrow = \longrightarrow_\delta \ \cup \ \longrightarrow_D$ to denote all transitions and use $\xrightarrow{*}$ to denote the reflexive transitive closure of $\longrightarrow$. It is easy to extend $\xrightarrow{*}$ for sets of markings. We define $Reach(M) := \{M' \,|\, M \xrightarrow{*} M'\}$ as the set of markings reachable from M.
A *M-computation* π from a marking M is a sequence $M_0, M_1, \ldots$ of markings s.t $M_0 = M$ and $M_i \longrightarrow M_{i+1}$ for $i \geq 0$.
For a computation π from M, the *delay* $\Delta(\pi)$ of the computation is defined as follows.

- $\Delta(M_0) = 0$.
- $\Delta(M_0, \ldots, M_{i+1}) = \Delta(M_0, \ldots, M_i)$ if $M_i \longrightarrow_D M_{i+1}$.
- $\Delta(M_0, \ldots, M_{i+1}) = \Delta(M_0, \ldots, M_i) + x$ if $M_i \longrightarrow_x M_{i+1}$.

Intuitively, the delay is the total amount of time passed in all timed transitions in the sequence. Notice that untimed Petri nets are a special case in our model where all intervals are of the form $[0 : \infty)$. Next, we recall a constraint system called *regions* defined for *Timed automata* [AD90].

Regions A *region* defines the integral parts of clock values up to max (the exact age of a token is irrelevant if it is greater than max), and also the ordering of the fractional parts. For TPNs, we need to use a variant which also defines the place in which each token (clock) resides. Following Godskesen [God94] we represent a region in the following manner.

Definition 2. *A region is a triple* (b_0, w, b_{max}) *where*

- $b_0 \in (P \times \{0, \ldots, max\})^{\odot}$. *$b_0$ is a multiset of pairs. A pair of the form (p, n) represents a token with age exactly n in place p.*
- $w \in \left((P \times \{0, \ldots, max-1\})^{\odot}\right)^{*}$. *$w$ is a word over the set $(P \times \{0, \ldots, max-1\})^{\odot}$, i.e., w is a word where each element in the word is a multiset over $P \times \{0, \ldots, max-1\}$. Pair (p, n) represents a token in place p with age x such that $x \in (n : n+1)$. Pairs in the same multiset represent tokens whose ages have equal fractional parts. The order of the multisets in w corresponds to the order of the fractional parts.*
- $b_{max} \in P^{\odot}$. *b_{max} is a multiset over P representing tokens with ages strictly greater than max. Since the actual ages of these tokens are irrelevant, the information about their ages is omitted in the representation.*

Formally, each region characterizes an infinite set of markings as follows. Assume a marking $M = [(p_1, x_1), \ldots, (p_n, x_n)]$ and a region $R = (b_0, b_1 b_2 \cdots b_m, b_{m+1})$. Let b_j be of the form $[(q_{j1}, y_{j1}), \ldots, (q_{j\ell_j}, y_{j\ell_j})]$ for $j : 0 \leq j \leq m$ and b_{m+1} is of the form $[q_{m+1\ 1}, \ldots, q_{m+1\ l_{m+1}}]$. We say that M *satisfies* R, written $M \in R$, if there is a bijection h from the set $\{1, \ldots, n\}$ to the set of pairs $\{(j,k) \mid (0 \leq j \leq m+1) \wedge (1 \leq k \leq \ell_j)\}$ such that the following conditions hold:

- $p_i = q_{h(i)}$. Each token should have the same place as that required by the corresponding element in R.
- If $h(i) = (j,k)$ then $j = m+1$ iff $x_i > max$. Tokens older than max should correspond to elements in multiset b_{m+1}. The actual ages of these tokens are not relevant.
- If $x_i \leq max$ and $h(i) = (j,k)$ then $\lfloor x_i \rfloor = y_{j\ k}$. The integral part of the age of tokens should agree with the natural number specified by the corresponding elements in w.
- If $x_i \leq max$ and $h(i) = (j,k)$ then $frac(x_i) = 0$ iff $j = 0$. Tokens with zero fractional parts correspond to elements in multiset b_0.
- If $x_{i_1}, x_{i_2} < max$, $h(i_1) = (j_1, k_1)$ and $h(i_2) = (j_2, k_2)$ then $frac(x_{i_1}) \leq frac(x_{i_2})$ iff $j_1 \leq j_2$. Tokens with equal fractional parts correspond to elements in the same multiset (unless they belong to b_{m+1}). The ordering among multisets inside R reflects the ordering among fractional parts in clock values.

Notice that given a marking M, it is easy to compute the unique region R_M satisfied by M. We let $[\![R]\!] = \{M \mid M \in R\}$. Given a marking M and a set $\mathbf{R}$ of regions, we say that $M \in \mathbf{R}$ if there is a region $R \in \mathbf{R}$ such that $M \in R$. We also let $[\![\mathbf{R}]\!] = \{M \mid \exists R \in \mathbf{R}.\ M \in R\}$. The region construction defines an equivalence relation $\equiv$ on the set of markings such that $M_1 \equiv M_2$ if, for each region R, it is the case that $M_1 \in [\![R]\!]$ iff $M_2 \in [\![R]\!]$. It is easy to show (similar to [AD90]) that $\equiv$ is a congruence on the

set of markings. In other words, if $M_1 \longrightarrow M_2$ and $M_1 \equiv M_3$ then there is an M_4 such that $M_2 \equiv M_4$ and $M_3 \longrightarrow M_4$.

Ordering We define an ordering $\preceq$ on the set of markings s.t. $M_1 \preceq M_2$ if there is an M_2' with $M_1 \equiv M_2'$ and $M_2' \leq M_2$. In other words, $M_1 \preceq M_2$ if we can delete a number of tokens from M_2 and as a result obtain a new marking equivalent to M_1. We let $M_1 \prec M_2$ denote that $M_1 \preceq M_2$ and $M_1 \not\equiv M_2$. Notice that $\longrightarrow$ is *monotonic* with respect to the ordering $\preceq$, i.e, if $M_1 \longrightarrow M_2$ and $M_1 \preceq M_3$ then there is an M_4 such that $M_2 \preceq M_4$ and $M_3 \longrightarrow M_4$. We identify a region R with the language $[\![R]\!]$ it represents (we write R instead of $[\![R]\!]$). Next we show how to compute the language entailment of two regions. We lift $\preceq$ on the set of markings to the set of regions.

Definition 3. *Let $R = (b_0, b_1 \dots b_m, b_{m+1})$ and $R' = (c_0, c_1 \dots c_l, c_{l+1})$ be regions. Then, $R \preceq R'$ iff there is a strict monotone injection $g : \{0, \dots, m+1\} \to \{0, \dots, l+1\}$ with $g(0) = 0$ and $g(m+1) = l+1$ and $b_i \leq c_{g(i)}$ for each $i : 0 \leq i \leq m+1$.*

For regions R and R', if $R \preceq R'$ then for each $M \in [\![R]\!], M' \in [\![R']\!]$, we have $M \preceq M'$. We let $R \prec R'$ denote that $R \preceq R'$ and $R \neq R'$.

Lemma 4. *Union and intersection of (sets of) regions are computable as finite sets of regions.*

We define a function *Post* such that for a region R, *Post*(R) is a finite set of regions where a marking M' satisfies a region in *Post*(R) iff there is a marking $M \in [\![R]\!]$ such that $M \longrightarrow M'$. Thus *Post*(R) characterizes the set of markings we can reach from a marking satisfying R by a single transition.

Lemma 5. *The set Post(R) is effectively constructible.*

For a set $\mathbf{R}_1$ of regions, we define *Pre*$(\mathbf{R}_1) := \{R |\ \exists M_1 \in \mathbf{R}_1, M \in R.\ M \longrightarrow M_1\}$. We use $\min$ to denote a function which, given a set $\mathbf{R}_1$ of regions, returns the minimal elements of $\mathbf{R}_1$. We use $minpre([\![R]\!]\uparrow)$ to denote the set $\min($*Pre*$(\{R\}\uparrow))$. We define *Pre** to be the reflexive, transitive closure of $minpre$.

Lemma 6. *Given a region R, the set Pre$^*([\![R]\!]\uparrow)$ is effectively constructible as a finite set of regions [AN01].*

3 Zenoness

A zeno-run of a timed Petri net is an infinite run that takes only finite time.

Definition 7. *Let $N = (P, T, In, Out)$ be a TPN. A marking M is called a* zeno-marking *iff there exists an infinite M-computation π and a finite number m s.t. $\Delta(\pi) < m$. Let $ZENO$ denote the set of all zeno-markings.*

The computability of the zenoness-problem for timed Petri nets (i.e., the problem if $M \in ZENO$ for a given marking M, or, more generally, constructing $ZENO$) was mentioned in [dFERA00] as an open problem for discrete-time Petri nets. Here we solve this problem even for the more general class of dense-time Petri nets. We show that, for any TPN, a decidable symbolic representation of the set $ZENO$ can be effectively computed. This implies the computability of $ZENO$ also for discrete-time Petri nets, since they can be encoded into dense-time ones. An alternative solution for discrete-time Petri nets is discussed in Remark 23.

The following brief outline explains the main steps of our proof.

1. We translate the original timed Petri net N into an untimed simultaneous-disjoint-transfer net N'. Simultaneous-disjoint-transfer nets are a subclass of transfer nets where all transfers happen at the same time and don't affect each other. The computations of N' represent, in a symbolic way, the computations of N that can be done in time $< 1 - \delta$ for some predefined $\delta > 0$.
2. Consider the set INF of markings of N' where infinite computations start. Since INF is upward-closed, it is characterized by its finitely many minimal elements; let INF_{min} be this set of minimal elements. While INF_{min} is not computable for general transfer nets [DJS99, May03], it is computable for simultaneous-disjoint-transfer nets, as shown in Lemma 21.
3. We re-interpret the set INF (resp. INF_{min}) of N' markings in the context of the timed Petri net N and construct from it $ZENO$ as a finite set of regions.

Definition 8. A simultaneous-disjoint-transfer net (short SD-TN) N is described by a tuple $(P, T, Input, Output, Trans)$ where P is a set of places, T is a set of ordinary transitions, $Input, Output : T \to 2^P$ are functions that describe the input and output places of every transition, respectively. $Trans$ describes the simultaneous and disjoint transfer transition. We have $Trans = (I, O, S)$ where $I \subseteq P$, $O \subseteq P$, and $S \subseteq P \times P$. I and O describe the normal input and output places of the transfer transition and the pairs in S describe the source and target place of the transfer. The following restrictions on $Trans$ must be satisfied: If $(s,t),(s',t') \in S$ then s, s', t, t' are all different and $\{s,t\} \cap (I \cup O) = \emptyset$. Let $M : P \to \mathcal{N}$ be a marking of N. The firing of normal transitions $t \in T$ is defined just as for ordinary Petri nets. The transfer transition $Trans$ is enabled at M iff $\forall p \in I.\, M(p) \geq 1$. Firing $Trans$ yields the new marking M' where

$M'(p) = M(p)$	if $p \in I \cap O$	$M'(p) = M(p) - 1$	if $p \in I - O$
$M'(p) = M(p) + 1$	if $p \in O - I$	$M'(p) = 0$	if $\exists p'.\, (p,p') \in S$
$M'(p) = M(p) + M(p')$	if $(p',p) \in S$	$M'(p) = M(p)$	otherwise

(The restrictions above ensure that these cases are disjoint.)

Definition 9. For a given TPN $N = (P, T, In, Out)$ we now construct a SD-TN $N' = (P', T', Input, Output, Trans)$. The rough intuition is that N' simulates symbolically all computations of N which can happen in time $< 1 - \delta$ for some predefined $\delta > 0$.

Let $\max$ be the maximal finite constant that appears in the arcs of the TPN. We define a finite set of symbols $Sym := \{k \mid k \in \mathcal{N}, 0 \leq k \leq \max\} \cup \{k+ \mid k \in \mathcal{N}, 0 \leq k \leq \max\} \cup \{k- \mid k \in \mathcal{N}, 1 \leq k \leq \max\}$ and a total order on Sym by $k < k+ < (k+1)- < (k+1)$ for every k.

For every place $p \in P$ of N we have several corresponding places $p(s)$ for all $s \in Sym$ in P' of the SD-TN N'. Thus $P' = \{p(s) \mid p \in P, s \in Sym\}$. The intuition is as follows. A token on place $p(k)$ encodes a token of age exactly k on place p. A token on $p(k+)$ encodes a token on place p of an age a which satisfies $k < a \leq k + \delta$ for some a-priori defined $\delta > 0$. This means that the age of this token cannot reach $k+1$ in any run taking time $< 1 - \delta$. A token on $p(k-)$ encodes a token on p whose age a satisfies $k - 1 + \delta < a < k$ and which may or may not reach age k during a run taking time $1 - \delta$. The SD-TN tokens $p(k), p(k+)$ and $p(k-)$ are called *symbolic encodings* of the corresponding TPN token (p, a).

In particular, the age of such a $p(k-)$ token could be chosen arbitrarily close to k, such that its age could reach (or even exceed) k in runs taking an arbitrarily small time. For

example, consider a run taking time $\epsilon > 0$ and a transition with an outgoing arc labeled with the interval $[0:1)$. This transition could produce a token with age $1 - \epsilon/2$ which could reach age 1 or even $1 + \epsilon/2$ later.
We define a function $enc : Intrv \to 2^{Sym}$ as follows.

$$enc([x:y]) := \{s \in Sym \mid x \leq s \leq y\} \qquad enc((x:y]) := \{s \in Sym \mid x < s \leq y\}$$
$$enc([x:y)) := \{s \in Sym \mid x \leq s < y\} \qquad enc((x:y)) := \{s \in Sym \mid x < s < y\}$$

For every transition $t \in T$ in the TPN N we have a set $T'(t)$ of new transitions in N'. The intuition is that the transitions in $T'(t)$ encode all possibilities of the age intervals of input and output tokens. For example, a discrete transition t with an input arc from place p labeled $[0:1]$ would yield 4 different transitions in $T'(t)$ with input arcs from places $p(0), p(0+), p(1-)$ and $p(1)$, respectively.
We denote the transitions with their sets of input and output places A and B by $t'(A,B)$, i.e., $Input(t'(A,B)) = A$ and $Output(t'(A,B)) = B$. For every $t \in T$ we define $A(t) \subseteq 2^{P'}$ as follows.

$$A(t) := \{A \subseteq P' \mid In(t,p) \neq \bot \Leftrightarrow \exists_1 s \in enc(In(t,p)).\, p(s) \in A\}$$

For every $t \in T$ we define $B(t) \subseteq 2^{P'}$ as follows.

$$B(t) := \{B \subseteq P' \mid Out(t,p) \neq \bot \Leftrightarrow \exists_1 s \in enc(Out(t,p)).\, p(s) \in B\}$$

We use $\exists_1$ to mean there exists 'exactly one' in the above.
We define $T'(t) := \{t'(A,B) \mid A \in A(t), B \in B(t)\}$ and finally $T' := \bigcup_{t \in T} T'(t)$. Note that in transition $t'(A,B)$ of N', the set of input places A contains only one input place $p(s)$ for each non-empty interval $In(t,p)$ s.t. $s \in enc(In(t,p))$. A similar remark holds for the set of output places B.
So far, the transitions in T' only encode the discrete transitions of N. The transfer arc will be used to encode the passing of time. However, since we need to keep discreet transitions and time-passing separate, we must first modify the net to obtain alternating discrete phases and time-passing phases.
First we add two extra places p_{disc} and p_{time} to P' which act as control-states for the different phases. Then we modify all transitions $t \in T'$ by adding p_{disc} to $Input(t)$ and $Output(t)$. Thus normal transitions can fire iff p_{disc} is marked.
The transitions which encode the passing of time include both the transfer transition and several normal transitions. After an arbitrarily small amount of time < 1 passes, all tokens of age k have an age $> k$. This is encoded by the simultaneous-disjoint transfer arc, which moves all tokens from places $p(k)$ to places $p(k+)$. Formally, $Trans := (I, O, S)$ where $I := \{p_{disc}\}$, $O := \{p_{time}\}$, and $S := \{(p(k), p(k+)) \mid 0 \leq k \leq \max\}$. Note that $Trans$ starts the time-passing phase and switches the control-state from p_{disc} to p_{time}. Now we add a new set of transitions which encode what happens to tokens of age $k-$ when (a small amount of) time passes. Their age might either stay below k, reach k or exceed k. For every $k \in \{1, \ldots, \max\}$ we have a transition with input places p_{time} and $p(k-)$ and output places p_{time} and $p(k)$. Furthermore, for every $k \in \{1, \ldots, \max\}$ we have a transition with input places p_{time} and $p(k-)$ and output places p_{time} and $p(k+)$. Finally, we add an extra transition t_{switch} with input place p_{time} and output place p_{disc}, which switches the net back to normal discrete mode. Note that after a time-passing phase the only tokens on places $p(k)$ are those which

came from $p(k-)$, because all tokens on $p(k)$ were first transferred to $p(k+)$ by the transfer transition.

Definition 10. *Let N be a TPN and N' the corresponding SD-TN, defined as in Def. 9. We say that a marking M' of N' is a* standard marking *if $M'(p_{disc}) = 1$ and $M'(p_{time}) = 0$. We denote by INF the set of all markings of N' from which infinite computations start. Since INF is upward-closed it can be characterized by its finitely many minimal elements. Let INF_{min} be the finite set of these minimal elements. Let INF' and INF'_{min} be the restriction to standard markings of INF and INF_{min}, respectively.*

The following definitions establish the connection between the markings of the timed Petri net N and the markings of the SD-TN N'.

Definition 11. *For every δ with $0 < \delta < 1$ we define a function $int_\delta : (P \times \mathcal{R}^{\geq 0})^{\odot} \to (P' \to \mathcal{N})$ that maps a marking M of N to its corresponding marking M' in N'. $M' := int_\delta(M)$ is defined as follows. For any k we have*

$$\begin{array}{lll} M'(p(k)) & := M((p,k)) & M'(p_{disc}) := 1 \\ M'(p(k+)) & := \sum_{k<x\leq k+\delta} M((p,x)) & M'(p_{time}) := 0 \\ M'(p((k+1)-)) & := \sum_{k+\delta<x<k+1} M((p,x)) & \end{array}$$

Note that $M' = int_\delta(M)$ is a standard marking by Def. 10.

The intuition is as follows. In an infinite computation π starting at M with $\Delta(\pi) < 1-\delta$, no token (p,x) with $k < x \leq k+\delta$ can reach age $k+1$ by aging. This is reflected in N' by the fact that $p(k+)$ tokens stay $p(k+)$ tokens during the symbolic time-passing phase. On the other hand, tokens (p,x) with $k+\delta < x < k+1$ can reach an age $\geq k+1$ by aging. This is reflected in N' by the fact that $p((k+1)-)$ tokens can become $p(k+1)$ or $p((k+1)+)$ tokens during the symbolic time-passing phase.

Lemma 12. *Given a TPN N with marking M_0, the corresponding SD-TN N' from Def. 9 and $0 < \delta < 1$. If there exists an infinite M_0-computation π s.t. $\Delta(\pi) < 1 - \delta$ then in N' there exists an infinite $int_\delta(M_0)$-computation π', i.e., $int_\delta(M_0) \in INF'$.*

The reverse implication of Lemma 12 does not generally hold. The fact that $int_\delta(M) \in INF'$ for some marking M of a TPN N does not imply that $M \in ZENO$. The infinite $int_\delta(M)$-run in N' depends on the fact that the $p(k-)$ tokens do (or don't) become $p(k)$ or $p(k+)$ tokens at the right step in the computation.
To establish a reverse correspondence between markings of N' and markings of N we need the following definitions.

Definition 13. *Let N' be a SD-TN with places $P' = \{p(s) \mid p \in P, s \in Sym\} \cup \{p_{disc}, p_{time}\}$ and a standard marking $M' : P' \to \mathcal{N}$. Let $M'^{-} := M'_{|\{p(k-) \mid k-\in Sym\}}$ and $M'^{+} := M'_{|\{p(k+) \mid k+\in Sym\}}$ be the sub-markings of M, restricted to $p(k-)$ and $p(k+)$ tokens, respectively. We define $perm(M'^{-})$ as the set of all words $w = b_1 \ldots b_n \in \left((P \times \{0,\ldots,max-1\})^{\odot}\right)^*$ s.t. for all p and $k < max$ we have that $\left(\bigcup_{1\leq i\leq n} b_i\right)((p,k)) = M'^{-}(p((k+1)-))$. Similarly, let $perm(M'^{+})$ be the set of all words $w = b_1 \ldots b_n \in \left((P \times \{0,\ldots,max-1\})^{\odot}\right)^*$ s.t. for all p and $k < max$ we have $\left(\bigcup_{1\leq i\leq n} b_i\right)((p,k)) = M'^{+}(p(k+))$.*

Intuitively, $perm(M'^{-})$ describes all possible permutations of the fractional parts of tokens in a TPN marking M which are symbolically encoded as $p(k-)$ tokens in the corresponding SD-TN standard marking M'. Every symbolic marking M' of the SD-TN defines a set of TPN markings, depending on which permutation of the $p(k-)$ tokens and $p(k+)$ tokens is chosen.

Definition 14. *Let N' be a SD-TN. Every standard marking $M' : P' \to \mathcal{N}$ induces a set of regions $R(M', x, y)$ given by $R(M', x, y) = (b_0, xy, b_{max})$, where $b_0((p, k)) = M'(p(k))$ for all p and all $k \leq max$, $x \in perm(M'^{+})$, $y \in perm(M'^{-})$ and $b_{max}(p) = M'(p(max+))$ for all p.*

Lemma 15. *Let N be a TPN with corresponding SD-TN N' and $M' \in INF'$.*

$$\exists y \in perm(M'^{-}).\, \forall x \in perm(M'^{+}).\, [\![R(M', x, y)]\!]\uparrow \subseteq ZENO$$

Definition 16. *Let N be a TPN with corresponding SD-TN N'.*

$$Z := \bigcup_{M' \in INF'_{min}} \bigcup_{x \in perm(M'^{+})} \bigcap_{y \in perm(M'^{-})} Pre^*([\![R(M', x, y)]\!]\uparrow)$$

Now we show that the set $ZENO$ is effectively constructible.

Lemma 17. $ZENO = [\![Z]\!]\uparrow$.

Now we show that, for any SD-TN, the set INF'_{min} is effectively constructible. For this we use a result by Valk and Jantzen.

Theorem 18. *[VJ85] A finite basis of an upward-closed set $K \subseteq \mathcal{N}^k$ is effectively computable iff for any vector $u \in \mathcal{N}^k_\omega$, the predicate $u\downarrow \cap K \neq \emptyset$ is decidable.*

Lemma 19. *For any SD-TN N with initial marking M_0 the coverability graph can be effectively constructed.*

Note that Lemma 19 implies that place-boundedness is decidable for simultaneous-disjoint transfer nets, while it is undecidable for general transfer nets [DJS99, May03].

Lemma 20. *Given an SD-TN N with k places and an ω-marking $M_0 \in \mathcal{N}^k_\omega$, it is decidable if $M_0\downarrow \cap INF \neq \emptyset$.*

Lemma 21. *For any SD-TN N' the set INF'_{min} can be effectively constructed.*

Proof. Since INF is upward-closed, we can, by Lemma 20 and Theorem 18, construct a finite basis of INF, i.e., the set INF_{min}. We obtain INF'_{min} by the restriction of INF_{min} to standard markings. □

Theorem 22. *Let N be a TPN. The set $ZENO$ is effectively constructible.*

Proof. We first construct the SD-TN N' corresponding to N, according to Def. 9. Then we consider the set Z from Def. 16. We have $ZENO = [\![Z]\!]\uparrow$ by Lemma 17. The set Z is effectively constructible by Lemma 21, Definition 16 and Lemmas 6 and 4. □

Remark 23. To compute *ZENO* for discrete-time Petri nets (left open in [dFERA00]), one can simply remove the transfer-transition from the net N' in Def. 9. This modified construction would yield $ZENO$ for the discrete-time case, because (unlike in the dense-time case) every infinite zeno-run in a discrete-time net has an infinite suffix taking no time at all.

4 Token Liveness

First, we define the *liveness* of a token in a marking. Let M be a marking in a TPN $N = (P, T, In, Out)$. A token in M is called *syntactically k-dead* if its age is $\geq k$. It is trivial to decide whether a token is k-dead from a marking. A token is called *semantically live* from a marking M, if we can fire a sequence of transitions starting from M which eventually consumes the token. Formally, given a token (p, x) and a marking M, we say that (p, x) can be *consumed* in M if there is a transition t satisfying the following properties: (a) t is enabled in M, (b) $In(t, p)$ is defined and $x \in In(t, p)$.

Definition 24. *A token (p, x) in M is semantically live if there is a finite M-computation $\pi = M M_1 \cdots M_r$ such that $(p, x + \Delta(\pi))$ can be consumed in M_r.*

For a marking M, we let $L(M)$ be the multiset of live tokens in M.

SEMANTIC LIVENESS OF TOKENS IN TPN

Instance: A timed Petri net N with marking M and a token $(p, x) \in M$.
Question: Is (p, x) live, i.e., $(p, x) \in L(M)$?
We show decidability of the semantic token liveness problem by reducing it to the *coverability problem* for TPNs.

COVERABILITY PROBLEM

Instance: A TPN N, a finite set of initial markings M_{init} of N, and an upward closed set $\mathsf{M}_{fin}\uparrow$ of markings of N, where M_{fin} is finite.
Question: $\mathsf{M}_{init} \xrightarrow{*} \mathsf{M}_{fin}\uparrow$?

Theorem 25. *The coverability problem is decidable for TPN [AN01].*

Suppose that we are given a TPN $N = (P, T, In, Out)$ with marking M and a token $(p, x) \in M$. We shall translate the question of whether $(p, x) \in L(M)$ into (several instances of) the coverability problem. To do that, we construct a new TPN N' by adding a new place p^* to the set P. The new place is not input or output of any transition. Now there are 2 cases. Either there is no transition in N which has p as its input place. Then it is trivial that $(p, x) \notin L(M)$. Otherwise, we consider all instances of the coverability problem defined on N' such that

- M_{init} contains a single marking $M - (p, x) + (p^*, x)$.
- M_{fin} is the set of markings of the form $[(p_1, x_1), \ldots, (p_n, x_n), (p^*, x')]$ such that there is a transition t and
  - the set of input places of t is given by $\{p, p_1, \ldots, p_n\}$.
  - $x' \in In(t, p)$ and $x_i \in In(t, p_i)$ for each $i : 1 \leq i \leq n$.

In the construction above, we replace a token (p, x) in the initial marking by a token (p^*, x); we also replace a token (p, x') in the final marking with $x' \in In(t, p)$, by a token (p^*, x'). The fact that the token in the question is not consumed in any predecessor of a marking in M_{fin}, is simulated by moving the token into the place p^* (in both the initial and final markings), since $p^* \notin P$ and not an input or output place in N'. Therefore, the token is live in M of N iff the answer to the coverability problem is 'yes'. From Theorem 25, we get the following.

Theorem 26. *The token liveness problem is decidable.*

5 Boundedness

SYNTACTIC BOUNDEDNESS OF TPN

Instance: A timed Petri net N with initial marking M_0.
Question: Is $|Reach(M_0)|$ bounded ?

We give an algorithm similar to the Karp-Miller algorithm [KM69] for solving the syntactic boundedness problem for TPNs. The algorithm builds a tree, where each node is labeled with a region. We build the tree successively, starting from the root, which is labeled with R_{M_0}: the unique region satisfied by M_0 (it is easy to compute this region). At each step we pick a leaf with label R and perform one of the following operations:

1. If $post(R)$ is empty we declare the current node *unsuccessful* and close the node.
2. If there is another (already generated) node which is labeled with R then declare the current node *duplicate* and close the node.
3. If there is a predecessor of the current node labeled with $R' \prec R$ then declare $|Reach(M_0)|$ unbounded (the TPN is unbounded), and terminate the procedure.
4. Otherwise, declare the current node as an *interior* node, add a set of successors to it, each labeled with an element in *Post*(R). This step is possible due to Lemma 5.

If the condition of step 3 is never satisfied during the construction of the tree, declare $|Reach(M_0)|$ bounded (the TPN is bounded).

The proof of correctness of the above algorithm is similar to that of original Karp-Miller construction [KM69]. The termination of the algorithm is guaranteed due to the fact that the ordering $\preceq$ on the set of regions is a well-quasi-ordering (follows from [Hig52]).

Theorem 27. *Syntactic boundedness is decidable for TPNs.*

It follows that termination is also decidable for TPNs. Since dead tokens cannot influence the behavior of a TPN, one would like to abstract from them. Let N be a TPN with marking M. Then we define the live part of TPN markings as $Reach^l(M) := \{L(M') \mid M' \xrightarrow{*} M'\}$, i.e, the set of reachable markings without dead tokens.

SEMANTIC BOUNDEDNESS OF TPN

Instance: A timed Petri net N with initial marking M_0.
Question: Is $|Reach^l(M_0)|$ bounded ?

Using slightly modified constructions of [RGdFE99] or [AN02], we can easily derive the undecidability of semantic boundedness even for dense-timed Petri nets.

6 Conclusions

We have shown decidability of zenoness, token-liveness and syntactic boundedness for dense-time *timed Petri nets (TPNs)* in which each token has an age represented by a real number, and where the transitions are constrained by the ages of the tokens. This class is closely related to the class of parameterized systems of timed processes where each process is restricted to a single clock [AJ03]. We have considered TPNs with just one real-valued clock per token. For all the problems studied so far, the decidability results coincide for dense-time and discrete time (although the proofs for dense-time are harder). However, if we consider TPNs with *two* clocks per token, there is a decidability gap between the dense-time and the discrete time domain. The coverability

problem becomes undecidable for dense-time TPNs with only *two* clocks per token, while it remains decidable for discrete TPNs with any finite number of clocks per token [ADM04]. The class of TPNs with multiple clocks per token is related to parameterized systems of timed processes, with multiple clocks per process [ADM04]. It is therefore worth investigating whether this more general class induces a similar gap for the problems we have considered in this paper.

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