Representations of Binding: Local Representations

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Outline

Naive Syntax Trees: Problems with binding

Local Representation

  Variable-Closed Sexprs
  Relations on terms
  Reasoning about Relations on Terms

Canonical Local Representations

  Locally Nameless Representation
  Sato Representation
  Adequacy of the Representation
  Example
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Canonical Local Representations
- Locally Nameless Representation
- Sato Representation
- Adequacy of the Representation
- Example
Pure Lambda terms: Raw Syntax

- Countable set $\mathbb{X}$ of *atoms* used for *variables*: $X, Y, Z$.
  - Only relation needed on $\mathbb{X}$ is decidable equality.
- Datatype of *terms* ranged over by $M, N, P, Q$:
  \[
  M ::= X \mid P \cdot Q \mid [X]M
  \]
- **Substitution**; naive definition as in “Software Foundations” and Greg’s Coq file for STLC:
  \[
  [M/Y]X ::= \text{if } Y = X \text{ then } M \text{ else } X \\
  [M/Y]N_1 \cdot N_2 ::= ([M/Y]N_1) \cdot [M/Y]N_2 \\
  [M/Y]([X]N) ::= [X]\text{(if } Y = X \text{ then } N \text{ else } [M/Y]N)
  \]
- This is a definition by structural recursion.
- Well known that naive substitution is wrong: allows capture
  \[
  [X/Y]([X]Y) = [X]X.
  \]
Substitution isn’t the only problem: Typing

- Let $A, B, \ldots$ be simple types (implicational propositions).
- Valid contexts $(\Gamma, \Delta)$ are lists of uniquely labelled assumptions:

\[
\Gamma ::= X_1, A_1, \ldots, X_n, A_n \quad \text{where the } X_i \text{ are pairwise distinct}
\]

- Consider the rules for Simply Typed Lambda Calculus (STLC):

\[
\begin{align*}
\Gamma \text{ valid} & \quad X:A \in \Gamma \\
\hline
\Gamma \vdash X : A \\
\end{align*}
\]

(ELIM) \[
\begin{align*}
\Gamma \vdash b : A \rightarrow B & \quad \Gamma \vdash a : A \\
\hline
\Gamma \vdash ba : B \\
\end{align*}
\]

(INTRO) \[
\begin{align*}
\Gamma, Y:A \vdash b : B \\
\hline
\Gamma \vdash [Y]b : A \rightarrow B \\
\end{align*}
\]

- Does this system accept the shadowing judgement

\[
\vdash [Y]([Y]Y) : A \rightarrow A \rightarrow A
\]
Substitution isn’t the only problem: Typing (2)

Greg’s Coq STLC does the rules differently to accept shadowing:

\[
\frac{\text{LIFO lookup } (X, \Gamma) = A}{\Gamma \vdash X : A}
\]

\[
\frac{\Gamma \vdash b : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash b \ a : B}
\]

\[
\frac{\Gamma, Y : A \vdash b : B}{\Gamma \vdash [Y] b : A \rightarrow B}
\]

- No validity requirement for contexts; LIFO lookup instead.
- Accepts the judgement \( \vdash [Y]([Y]Y) : A \rightarrow A \rightarrow A \).
- But this system doesn’t accept context permutation
  - which the previous system does.
- Worse, LIFO lookup cannot handle dependent types.
Shadowing in Dependent Types

- Consider a naive INTRO rule for dependent types

\[
\text{INTRO} \quad \frac{\Gamma, X:A \vdash M : B}{\Gamma \vdash \lambda X:A. M : \Pi X:A. B}
\]

- The following correct judgement with shadowing is not derivable:

\[
A:*, P:A \rightarrow * \vdash \lambda X:A. (\lambda X:P\cdot X \cdot X) : \Pi X:A. (\Pi Y:P\cdot X \cdot P\cdot X).
\]

- The stacking of dependencies in terms and types can differ.

- LIFO lookup doesn’t solve the shadowing problem for dependent types.
What is the problem?

- The raw syntax datatype doesn’t respect our idea of binding:
  - \([X]X \neq [Y]Y\) as elements of the datatype.
  - Structural induction doesn’t give the right induction hypothesis.
- Informal solution: quotient syntax by \(\alpha\)-equivalence
  - but \(\alpha\)-equivalence is usually defined in terms of substitution
    - Smallest congruence relation containing
      \[
      \frac{Y \neq X \quad Y \not\in \text{FV}(M)}{[X]M \equiv \alpha [Y]([Y/X]M)}
      \]
    - so first define substitution on raw syntax, then define \(\alpha\)-equivalence and show substitution respects it.

Not very pretty: our game is to avoid defining \(\alpha\)-equivalence at all.
Aside: $\alpha$ without substitution

- Let $(X \ Y) \cdot M$ be the operation that swaps $X$ and $Y$ in term $M$.
  - Viewing $M$ as raw syntax: no binding,
  - $\alpha$-equivalence is the smallest congruence containing
    \[
    Z \not\in (X, Y, M, N) \quad (X \ Z) \cdot M =_{\alpha} (Y \ Z) \cdot N
    \]
    \[
    [X]M =_{\alpha} [Y]N
    \]
  - where $Z \not\in M$ means $Z$ doesn’t appear in the raw syntax $M$.
  - The operations of swapping $\cdot$ and freshness $\not\in$ are definable on raw syntax by structural recursion.

Name permutation on raw syntax is the basic idea of nominal sets and nominal logic.

- This technique was already observed in [McKinna/Pollack 1993].
Back to the standard approach:
Carefully define substitution on raw syntax

- Commonly used in modern presentations since Church; Curry and Feys.

\[
[c/X]Y := \text{if } X = Y \text{ then } c \text{ else } Y \\
[c/X](b_1 \cdot b_2) := ([c/X]b_1) \cdot ([c/X]b_2) \\
[c/X]([Y]b) := [Z][c/X][Z/Y]b \quad \text{Z sufficiently fresh}
\]

- \([−/−]\) is defined by recursion on length of \(b\),
  - not on structure of \(b\), since \([Z/Y]b\) is not a subterm of \([Y]b\).
- Arbitrary choice of fresh \(Z\) could be made canonical . . .
  - e.g. “first name not occurring . . .”
  - this definition is deterministic given a choice function over names.
Aside: Simultaneous Substitution is Structural

- A simultaneous substitution, \( \rho \), is a finite partial function from \( X \) to terms.

\[
\begin{align*}
  x_{\rho} & := \rho x \\
  (a \cdot b)_{\rho} & := (a_{\rho}) \cdot (b_{\rho}) \\
  ([X]b)_{\rho} & := [Z](b(\rho, X=Z)) \quad \text{\( Z \) sufficiently fresh}
\end{align*}
\]

Primitive recursion with variable parameter.

- The choice of fresh \( Z \) can be canonical . . .
  - then applying any substitution alpha-normalizes (Stoughton)
  - so testing alpha-equivalence becomes a test for identity.

- Note: this operation has funny properties:

\[
[X/X]M \neq M
\]

Back to our quest to avoid \( \alpha \)-equivalence.
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Example
Distinct species of names

Why is it natural to identify bound names with free names?

- Example: →-intro rule for simple types

\[
\frac{\Gamma, X : A \vdash b : B}{\Gamma \vdash [X]b : A \rightarrow B}
\]

- ‘X’ really occurs in the premise (free, global).
- ‘X’ does not occur in the conclusion (locally bound).

This suggests:

- Syntactically separate local (bound) variables from global (free) variables.
- Not a new idea: Frege, Gentzen and Prawitz all informally used different species of names.
Syntax of locally named pre-terms for pure $\lambda$

As in McKinna/Pollack [TLCA 1993, JAR 1999].

- Countable set $V$ of atoms used for local *variables*: $x, y, z$.
- Countable set $X$ of atoms, used for global *parameters*: $X, Y, Z, p, q$.
- Only relation needed on $V, X$ is decidable equality.

Symbolic Expressions ($S$):

- Datatype of pre-terms ranged over by $M, N, P, Q$:

$$M ::= x \mid X \mid P \cdot Q \mid [x]M$$

  - No way to bind global names, $X$.
  - In general, may be other classes of variables, parameters and expressions
    - e.g. types and terms in System F.
Occurrences of Names

- Occurrences of global names (parameters)
  - $X \not\in A$ means “$X$ does not occur syntactically in $A$”.
    - Easily defined by structural recursion
    - Corresponds to nominal freshness (also written $\sharp$).
Substitution, Concretely

- Concretely defined by *structural* recursion:

  \[
  [M/X]x = x \\
  [M/X]Y = \text{if } X = Y \text{ then } M \text{ else } Y \\
  [M/X]N \cdot N = ([M/X]N) \cdot [M/X]N \\
  [M/X]([x]N) = [x][M/X]N
  \]

- Deterministic: no choosing arbitrary names.
  - Thus has natural properties; e.g.

  \[
  [X/X]M = M. \\
  X \not\in M \implies [P/X]M = M.
  \]

- Does not prevent capture, e.g. \([x/X][x]X = [x]x\).
  - Will only be used in safe ways.

- Substitution is a B-algebra homomorphism; see Pollack and Sato (J. Symb. Comp.).
Not Substitution: a purely technical operation

- Used to fill a “hole” (free variable) created by going under a binder.
- Defined by structural recursion:

\[
[M/y]x = \text{if } y = x \text{ then } M \text{ else } x
\]

\[
[M/y]X = X
\]

\[
[M/y]N_1 \cdot N_2 = ([M/y]N_1) \cdot [M/y]N_2
\]

\[
[M/y]([x]N) = [x](\text{if } y = x \text{ then } N \text{ else } [M/y]N)
\]

- Respects intended scope of binding.
- Does not prevent capture, e.g. \([x/y][x]y = [x]x\).
Overview: Symbolic expressions vs $\lambda$-terms

Sexprs do not faithfully represent $\lambda$-terms for two reasons.

1. Local variables may appear unbound in sexprs.
   - ‘$x$’ is an sexpr, not intended to represent any $\lambda$-term.
   - Remark: ‘$X$’ is an sexpr representing a $\lambda$-term with one (particular) global variable.
   - The fix: select the set of sexprs with no unbound local variables.
   - Call this subset $vclosed$ for variable closed.

2. Different sexprs may represent the same $\lambda$-term.
   - ‘$[x]x$’ and ‘$[y]y$’; not canonical.
   - Ignore this for the moment.
   - We show how to reason correctly with $vclosed$ expressions.
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Variable-Closed Sexprs

A predicate meaning “no free variables”.

\[ \text{vclosed} \quad \text{vclosed} \begin{array}{c} M \quad \text{vclosed} \quad N \quad \text{vclosed} \\
X \quad \text{vclosed} \quad M \cdot N \quad \text{vclosed} \\
\end{array} \begin{array}{c} [X/y]M \quad \text{vclosed} \\
\end{array} \]

- \text{vclosed} terms have no unbound local variables.
- An abstraction is \text{vclosed} when . . . .
- Every parameter is \text{vclosed} and no variable is \text{vclosed}.
- Use \text{vclosed} induction instead of sexpr structural induction . . .
- . . . no case for unbound variables.
Variable-Closed and Substitution

- Operations $[M/X]N$ and $[M/x]N$ are capture free on $\text{vclosed}$.
  - There are no free local names to get captured!
- $\text{vclosed}$ is trivially closed under substitution:
  $$\text{vclosed } M \land \text{vclosed } N \implies \text{vclosed } [M/X]N$$
- Think of $\text{vclosed}$ as a “weak typing judgement”.
  - $\text{vclosed}$ terms behave well for substitution, just as well-typed terms behave well for computation.
Aside: Alternative definitions of Variable-Closed

Only the rules for abstraction differ:

\[
\begin{align*}
X \not\in M & \quad \text{vclosed} [X/y]M \\
\vdash \text{vclosed} [y]M \\
\text{vclosed} [X/y]M & \quad \vdash \text{vclosed} [y]M \\
\text{vclosed} M & \quad \vdash \text{vclosed} [x][x/X]M
\end{align*}
\]

\[
\begin{align*}
\forall X. X \not\in M & \quad \Rightarrow \text{vclosed} [X/y]M \\
\forall X. \text{vclosed} [X/y]M & \quad \Rightarrow \text{vclosed} [y]M \\
\forall Y. (Y \not\in X \Rightarrow Y \not\in M) & \quad \Rightarrow \text{vclosed} [Y/X]M \\
\forall Y. \text{vclosed} [Y/X]M & \quad \Rightarrow \text{vclosed} [x][x/X]M
\end{align*}
\]

- These 6 relations are pairwise *extensionally* equivalent:
  - i.e. they derive the same judgments,
  - but their derivations are different,
  - they behave differently for induction.
- Each of the relations in the left hand column have infinitely many derivations of each derivable judgement.
- Each of the relations in the right hand column have at most one derivation of any judgement.
Simply Typed Lambda Calculus (STLC)

- Let *simple types* (A, B, ...) and *valid contexts* (Γ, Δ) be as above

\[
\Gamma \text{ valid } \quad p : A \in \Gamma \\
\Gamma \vdash p : A
\]

(ELIM) \quad \Gamma \vdash b : A \rightarrow B \quad \Gamma \vdash a : A

\[
\Gamma \vdash b a : B
\]

(INTRO) \quad \Gamma, p : A \vdash [p/y]b : B \quad p \not\equiv b

\[
\Gamma \vdash [y]b : A \rightarrow B
\]

- When going under a binder, substitute a *suitably fresh* parameter in the hole created.
  
  ▶ The choice of p is arbitrary; we will have more to say.

- Why no mention of vclosed?

  ▶ lemma: \( \Gamma \vdash b : B \implies vclosed b \).
Simply Typed Lambda Calculus (2)

\[ \Gamma \text{ valid} \quad p : A \in \Gamma \quad \frac{\Gamma \vdash p : A}{\Gamma \vdash p : A} \]

\[ (\text{ELIM}) \quad \frac{\Gamma \vdash b : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash b \ a : B} \]

\[ (\text{INTRO}) \quad \frac{\Gamma, p : A \vdash [p/y]b : B \quad p \not\in b}{\Gamma \vdash [y]b : A \rightarrow B} \]

- The side condition is needed in rule INTRO to prevent too many judgements being derivable:
  - if \( p \in b \) then \( p \in \lambda y . b \) in the conclusion, where \( p \) is not bound in the context \( \Gamma \).

- Validity side conditions are \textit{not} required in rules INTRO and ELIM because they follow from the premises.

- This definition of \( \vdash \) is easily formalised in Coq, Isabelle/HOL, ...
Beta Reduction

\[
(\beta) \quad \frac{\text{vclosed } [x]b \quad \text{vclosed } s}{([x]b) s > [s/x]b}
\]

\[
(\xi) \quad \frac{[p/x]s > [p/y]t \quad p \not=} (s, t)}{[x]s > [y]t}
\]

\[
\begin{align*}
\text{In rule } (\xi) \text{ we must let } x \text{ and } y \text{ possibly be different } & \ldots \\
& \text{or else some correct reduction judgements won’t be derivable.}
\end{align*}
\]

\[
\begin{align*}
\text{In } \beta, \text{ we must restrict to } (\text{vclosed } s) \text{ for safety.} \\
& \text{Otherwise free variables in } s \text{ might be captured in } [s/x]b.
\end{align*}
\]

\[
\begin{align*}
\text{The other } \text{vclosed } \text{ restrictions are for hygiene:} \\
& \text{lemma: } a > b \implies \text{vclosed } a \land \text{vclosed } b.
\end{align*}
\]
Dependent Types

- Now we see how to handle the problem of dependent types

\[
\Gamma, p: A \vdash \left[p/x\right] M : \left[p/y\right] B \quad p \not\in (M, B) \\
\Gamma \vdash \lambda x: A. M : \Pi y: A. B
\]

- Allowing different names to be bound in different parts of the judgement accounts for different dependencies in terms vs. types.
What can we do with these relations?

- Definitions and statements of lemmas are natural using names.
- All the expected judgements are derivable:
  - The set of derivable judgments is closed under $\alpha$-conversion and renaming.
- The standard metatheory can be developed:
  - Weakening, substitution lemma, subject reduction . . .
- We almost never need to define or reason about $\alpha$-conversion.
  - Church-Rosser for $\beta$-reduction does not hold on-the-nose for these $v$-closed terms: not canonical.
  - Church-Rosser does hold for Tait/Martin-Löf parallel reduction.
- We can use the Locally Nameless representation or the Sato representation to get canonicity.
Weakening for STLC

- Define subcontext:

  \[ \Gamma \subseteq \Delta \iff \forall p, A \cdot p : A \in \Gamma \implies p : A \in \Delta \]

  \(\Delta\) contains every assumption occurring in \(\Gamma\).

Lemma (Weakening)

\[ \Gamma \vdash a : A \implies (\forall \Delta \cdot \Gamma \subseteq \Delta \land \Delta \text{ valid} \implies \Delta \vdash a : A). \]

Remark

- We trivially have the lemma: \( \Gamma \vdash a : A \implies \Gamma \text{ valid} \).
- \( \ldots \) so the premise ‘\(\Delta \text{ valid}\)’ is needed in the statement of weakening.
Prove weakening

\[ \Gamma \vdash a : A \iff (\forall \Delta . \Gamma \sqsubseteq \Delta \land \Delta \text{ valid} \iff \Delta \vdash a : A) . \]

**Proof:** Attempt proof by induction on the derivation of \( \Gamma \vdash a : A \).

- Consider case for rule INTRO:
  \[ \Gamma, p : A \vdash [p/y]b : B \quad p \notin b \]
  \[ \Gamma \vdash [y]b : A \rightarrow B \]

- The goal is \( \Delta \vdash [y]b : A \rightarrow B \).
- The IH is:
  \[ \forall \Phi . (\Gamma, p_0 : A \sqsubseteq \Phi \land \Phi \text{ valid}) \iff \Phi \vdash [p_0/y]b : B \]
  for some particular \( p_0 \notin b \).

- By rule INTRO we need to show
  \[ \Delta, q : A \vdash [q/y]b : B \]
  for some \( q \notin b \).

- It seems we want to instantiate \( \Phi \) in IH with \( \Delta, p_0 : A \ldots \)

- \ldots but \( \Delta, p_0 : A \) may not be valid, as \( p_0 \) may occur in \( \Delta \).
Proof of weakening (contd)

- Since \( p_0 \) may not be fresh enough, we want to exchange it for a fresh parameter.
- Let \((p\ q)\bullet b\) mean permute all occurrences of \( p \) and \( q \) in \( b \).
- As a lemma (equivariance), show

\[
\Gamma \vdash a : A \implies \forall p\ q . \ (p\ q)\bullet \Gamma \vdash (p\ q)\bullet a : A . \quad (1)
\]

- This is easy to prove, but even better . . .
- nominal Isabelle defines polymorphic permutations and proves equivariance automatically.
- Now, pick \( q \not\in (\Delta, b, \Gamma) \). It suffices to show

\[
\Delta, q : A \vdash [q/y]b : B
\]

which, by (1) and IH is difficult but possible.
We have a proof; what’s the problem?

- We must use name swapping explicitly (as in the *weakening* proof above) to handle each example where eigenvariable problems appear.

Better: we can package this swapping reasoning for each relation (typing, reduction, . . . ) once and for all.

- This technique from McKinna/Pollack (1993).
A more uniform solution to eigenvariable problems

The following judgements are equivalent:

- Arbitrary choice of $p$ in INTRO: judgements may have infinitely many derivations:

  
  \[
  \frac{
    \Gamma \text{ valid} \quad p : A \in \Gamma
  }{
    \Gamma \vdash p : A
  }\]

  \[
  \frac{
    \Gamma \vdash b : A \rightarrow B \quad \Gamma \vdash a : A
  }{
    \Gamma \vdash ba : B
  }\]

  \[
  \frac{
    \Gamma, p : A \vdash [p/y]b : B \quad p \not\in \Gamma
  }{
    \Gamma \vdash [y]b : A \rightarrow B
  }\]

- No arbitrary choices: judgements have at most one derivation:

  \[
  \frac{
    \Gamma \text{ valid} \quad p : A \in \Gamma
  }{
    \Gamma \not\vdash \vdash p : A
  }\]

  \[
  \frac{
    \Gamma \not\vdash b : A \rightarrow B \quad \Gamma \not\vdash a : A
  }{
    \Gamma \not\vdash \vdash ba : B
  }\]

  \[
  \frac{
    \forall p. \, p \not\in \Gamma \quad \Rightarrow\quad \Gamma, p : A \not\vdash [p/y]b : B
  }{
    \Gamma \not\vdash \vdash [y]b : A \rightarrow B
  }\]
A more uniform solution to eigenvariable problems

- $\vdash$ is the “official” relation, $\models$ is an auxiliary notion.
- $\vdash$ is ordinary syntax: formalizable in Primitive Recursive Arithmetic.
  - $\vdash$ derivations are well-founded, finitely branching trees.
- $\models$, defined by generalized inductive definition, is not formalizable in PRA.
  - $\models$ derivations are well-founded but infinitely branching trees.
- By induction on the derivation of $\Gamma \models a : A$, it is trivial that

  $$\Gamma \models a : A \iff \Gamma \vdash a : A.$$ 

The other direction takes some work.
Why are we interested in this equivalence?

- With this equivalence we can prove weakening by rule induction without name swapping.
- The equivalence of $\vdash$ and $\Vdash$ “packages” the eigenvariable reasoning that we need for all examples.
  - Introduction of $\vdash$ is easy: only need property for one fresh variable.
  - Elimination of $\Vdash$ is powerful: get the IH for all sufficiently fresh variables.
  - We use $\Vdash$ as a “derived induction principle” for $\vdash$
- Instead of using swapping arguments for every rule induction on every relation (typing, reduction, . . . )
- . . . we use it once for each relation, to prove strengthened induction principle.
- There is much more to say about this: cofinite quantification, nominal freshness contexts, . . .
Applying the equivalence to prove weakening

- It is easy to prove weakening:
  \[ \Gamma \vdash a : A \iff (\Gamma \sqsubseteq \Delta \land \Delta \text{ valid} \implies \Delta \vdash a : A). \]
  hence, equivalently, weakening for \( \vdash \).

- **Proof:** by induction on the derivation of \( \Gamma \vdash a : A \).
- Consider case for **INTRO:**
  \[ \forall p. p \not\vdash \Gamma \implies \Gamma, p:A \vdash [p/y]b : B \]
  \[ \Gamma \vdash [y]b : A \to B \]
- By rule **INTRO** (for \( \vdash \)) we need to show
  \[ \Delta, p:A \vdash [p/y]b : B \quad \text{for some } p \not\vdash b. \]
  using IH:
  \[ \forall \Phi \forall p. (p \not\vdash \Gamma \land \Gamma, p:A \sqsubseteq \Phi \land \Phi \text{ valid}) \implies \Phi \vdash [p/y]b : B. \]
- Select \( p_0 \not\vdash (b, \Gamma, \Phi) \) and instantiate \( \Phi \) in IH with \( \Delta, p_0:A \). \square
Proof of the equivalence of ⊢ and ⊩

**Lemma** \( \Gamma \vdash a : A \iff \Gamma \vDash a : A \).

- Proof by induction on the derivation of \( \Gamma \vdash a : A \).
- Consider the case of rule **INTRO**.
- Any derivation of \( \vdash \) will use a particular parameter, say \( p_0 \).
- The IH for this case is

\[
\Gamma, p_0 : A \vdash [p_0/y]b : B \quad (p_0 \not\vDash b) \quad (\text{also } p_0 \not\vDash \Gamma)
\]

but to use the **INTRO** rule for \( \vDash \) we need the premise

\[
\forall p . \ p \not\vDash \Gamma \iff \Gamma, p : A \vdash [p/y]b : B
\]

- How to reason from a particular parameter to all parameters?
Proof continued: \( \Gamma \vdash a : A \iff \Gamma \not\vdash a : A \)

We solve the problem using swapping, as in the *weakening* proof.

- As a lemma, have *equivariance* of \( \vdash \):
  \[ \Gamma \vdash a : A \iff \forall p \ q. (p \ q) \Gamma \vdash (p \ q) \bullet a : A. \tag{2} \]

  Nominal Isabelle can prove this automatically, and provides the lemmas about \# and \# that we need.

- We are trying to prove
  \[ \forall p \bullet p \not\# \Gamma \iff \Gamma, p:A \vdash [p/y]b : B. \]

  So pick \( p \not\# \Gamma \). (Hence \( p \not\# b \)).

- From IH and (2) have
  \[ (p \ p_0) \bullet (\Gamma, p_0:A) \vdash (p \ p_0) \bullet ([p_0/y]b) : B \]

  i.e. \( \Gamma, p:A \vdash [p/y]b : B \) as required.
Aside: Stronger inversion principles

- Rule **INTRO**
  \[
  \Gamma, p:A \vdash [p/y]b : B \quad p \not\in b \\
  \Gamma \vdash [y]b : A \to B
  \]
  gives rise (by induction) to an *inversion principle*:

  \[
  \Gamma \vdash [y]b : T \implies \\
  \exists A, B, p. \, \Gamma, p:A \vdash [p/y]b : B \land p \not\in b \land T = A \to B.
  \]

- Rule **INTRO**
  \[
  \forall p. \, p \not\in \Gamma \implies \Gamma, p:A \vdash [p/y]b : B \\
  \Gamma \vdash [y]b : A \to B
  \]
  gives a stronger inversion principle (using \( \vdash \iff \models \)):

  \[
  \Gamma \vdash [y]b : T \implies \\
  \exists A, B. \, \forall p \not\in \Gamma. \, \Gamma, p:A \vdash [p/y]b : B \land p \not\in b \land T = A \to B.
  \]

- Some proofs need stronger inversion.
Outline

Naive Syntax Trees: Problems with binding

Local Representation
  Variable-Closed Sexprs
  Relations on terms
  Reasoning about Relations on Terms

Canonical Local Representations
  Locally Nameless Representation
  Sato Representation
  Adequacy of the Representation
  Example
Limitations of $vclosed$ representation

- Feasible way to work with representatives of $\alpha$ classes . . .
  - Never needed to define $\alpha$-conversion in [McKinna/Pollack].
  - Can prove parallel-$\beta$-conversion is confluent.
- . . . but Church–Rosser for $\beta$-reduction fails on-the-nose for $vclosed$.
  - If we wanted to reason about $\beta$-reduction we would need to reason about $\alpha$-conversion.
- $vclosed$ representation not canonical: $[x]x \neq [y]y$.
- We can get canonical representations:
  - using de Bruijn indexes (locally nameless)
  - or by choosing bound variable names canonically (Sato representation).
- In both these approaches, use strengthened induction principles as above.
Locally Nameless Representation

- This is described in detail [POPL 2008] and in a JAR paper by Arthur Charguéraud.
- Charguéraud’s paper comes with a Coq library and many examples.
  - Arthur’s scripts work in Coq 8.3pl2 (26 Oct 2011), but there are some assumptions waiting to be filled in with proofs.
- Use de Bruijn indexes (numbers, \(n\)) for local (bound) variables.
- Use \(X, Y, \ldots\) for global parameters, as above.
- Datatypref of raw terms: \(M ::= X \mid n \mid P \cdot Q \mid \[] M\)
Locally Nameless

- Substitution, \([M/p]N\) is defined as before.
  - No need to lift the free indexes in \(M\) because \(v\text{closed } M\) have no free indexes.

- "Hole filling", \([M/n]N\), is defined by structural recursion on \(N\)

\[
\begin{align*}
[M/k]i &= \text{ if } k = i \text{ then } M \text{ else } i \\
[M/k]X &= X \\
[M/k]N_1 \cdot N_2 &= ([M/k]N_1) \cdot [M/k]N_2 \\
[M/k][]N &= []([M/k+1]N)
\end{align*}
\]

- This index arithmetic does make reasoning more difficult.

\[
\begin{align*}
v\text{closed } X &\quad v\text{closed } M &\quad v\text{closed } N &\quad v\text{closed } [X/0]M \\
v\text{closed } M &\quad v\text{closed } M \cdot N &\quad v\text{closed } []M
\end{align*}
\]

- I leave you to look up the details.
A Canonical Locally Named Representation

- Consider again the $\textit{vclosed}$ rules:

$$
\begin{align*}
\text{vclosed } X & \quad \text{vclosed } M \quad \text{vclosed } N \\
\hline
\text{vclosed } X & \quad \text{vclosed } M \cdot N \\
\hline
\text{vclosed } [x][x/X]M & \quad \text{vclosed } \text{M}
\end{align*}
$$

Local variable ‘$x$’ not determined in the rule for abstraction.

- To define a canonical subset $\mathbb{L}_F$, choose ‘$x$’ deterministically:

$$
\begin{align*}
X : \mathbb{L}_F & \quad M : \mathbb{L}_F \quad N : \mathbb{L}_F \\
\hline
M \cdot N : \mathbb{L}_F & \quad M : \mathbb{L}_F \quad x = F_x(M) \\
\hline
\text{[x][x/X]M} : \mathbb{L}_F
\end{align*}
$$

parameterized by a height function $F : X \times S \rightarrow V$.

- Clearly $M : \mathbb{L}_F \Rightarrow \textit{vclosed } M$, so substitution is capture free.

- Not obvious that $\mathbb{L}_F$ is closed under substitution.

- Still to do: specify $F$ such that $\mathbb{L}_F$ well behaved.
**Improve Notation**

- Everything is parameterised by a height function $F$; drop explicit subscript.
- Considering rule:

\[
M : \mathbb{L} \quad x = F_X(M) \\
\frac{}{[x][x/X]M : \mathbb{L}}
\]

define “abstraction”:

\[
\text{abs}_x M \triangleq [F_X(M)][F_X(M)/X]M.
\]

- Abstraction rule can now be written more abstractly.

\[
\begin{align*}
X : \mathbb{L} & \quad M : \mathbb{L} \quad N : \mathbb{L} \\
M \cdot N : \mathbb{L} & \quad \text{abs}_x M : \mathbb{L}
\end{align*}
\]

- $\text{abs}_x M$ will behave like informal ‘$\lambda X. M$’.
- $X$ does not occur in $\text{abs}_x M$; $\text{abs}_x X \equiv \text{abs}_Y Y$. 
A Datatype of Lambda Terms?

Assume good properties for $F$ (to be discussed below).

- $\mathbb{L}$ can be formalized as a type
  - $\mathbb{L}$ is a decidable predicate on $S$;
    - can be a proof irrelevant $\Sigma$-type in Type Theory.
  - $\mathbb{L}$ is a non-empty predicate on $S$;
    - can be a defined type in HOL.

- But no amount of clever indexing . . ., inductive-recursive . . . can make $\mathbb{L}$ into a datatype . . .

- . . .because ‘abs’ isn’t injective.

Another topic for another talk.
Three Good Properties of $F : \mathcal{X} \times \mathcal{S} \to \mathcal{V}$

(HE) $F$ is equivariant: Let $\pi$ be a permutation over $\mathcal{X}$, then

$$M : \mathcal{L} \implies F_X(M) = F_{\pi \cdot X}(\pi \cdot M).$$

(HP) $F$ is preserved by substitution:

$$M : \mathcal{L} \land Q : \mathcal{L} \land X \neq Y \land X \nmid Q \implies F_X(M) = F_X([Q/Y]M).$$

(HF) $F_X(M)$ does not occur in binding position on any path from the root of $M$ to any occurrence of $X$ in $M$.

$$M : \mathcal{L} \implies F_X(M) \notin E_X(M)$$

where $E_X(M) : \mathcal{X} \times \mathcal{S} \to (\forall \text{ set})$ is defined:

$$E_X(\alpha) \triangleq \{\} \quad \text{if } \alpha \text{ is atomic}$$

$$E_X(M \cdot N) \triangleq E_X(M) \cup E_X(N)$$

$$E_X([x]M) \triangleq \begin{cases} \{\} & \text{if } X \nmid M \\ \{x\} \cup E_X(M) & \text{otherwise} \end{cases}$$
Consistency and independence of goodness

- **Consistency**: A good height function exists.
  - Interpret \( V \) as natural numbers, \( \mathbb{N} \).
  - \( H : X \times S \rightarrow \mathbb{N} \) defined by structural recursion:

\[
H_X(Y) \triangleq \begin{cases} 
1 & \text{if } X = Y \\
0 & \text{if } X \neq Y 
\end{cases}
\]

\[
H_X(x) \triangleq 0
\]

\[
H_X(M \cdot N) \triangleq \max(H_X(M), H_X(N))
\]

\[
H_X([x]M) \triangleq \begin{cases} 
H_X(M) & \text{if } H_X(M) = 0 \text{ or } H_X(M) > x \\
x + 1 & \text{otherwise}
\end{cases}
\]

- **Independence**: No two of (HE), (HP) and (HF) imply the third.

**Proof** by examples; see JAR paper.
abs_{X} M \text{ behaves like abstraction should}

We develop a theory of good F. For example:

- From (HE)

  \[
  \lambda X . M \overset{\alpha}{\sim} \lambda Y . N'' \quad \Longrightarrow \quad \text{abs}_{X} M = \text{abs}_{Y} N
  \]

- From (HF)

  \[
  \text{abs}_{X} M = \text{abs}_{Y} N \quad \Longrightarrow \quad \lambda X . M \overset{\alpha}{\sim} \lambda Y . N''
  \]

- From (HP), substitution “under a binder”

  \[
  X \neq Y \land X \not\in Q \quad \Longrightarrow \quad [Q/Y] \text{abs}_{X} M = \text{abs}_{X}[Q/Y] M
  \]

Together, (HE), (HF) and (HP) show ‘abs’ behaves correctly for \(\alpha\)-conversion and substitution.
Is the representation “adequate”?

**Formal vs informal** Relationship between a formal thing and an informal thing is not formalizable.

**Formal vs formal** Adequacy of representation of one formal thing by another formal thing depends on which properties we intend to preserve.

We show formally that $L$ is an adequate representation of pure $\lambda$-terms in Nominal Isabelle.

- Let $A, B, C$ range over nominal terms

$$A ::= X \ | \ B \cdot C \ | \ [X]A$$

- One more definition: ‘instantiation’ (nominal and Sato terms).

$$([x]M)\triangleright N \triangleq [N/x]M \quad (X)A\triangleright B \triangleq A[X:::B]$$
Isomorphism with Nominal Lambda Terms

▶ Define a *representation function* by “structural recursion”:

\[
!X \triangleq X \\
!(A \cdot B) \triangleq !A \cdot !B \\
![X]A \triangleq \text{abs}_X!A
\]

▶ Need (HE) (F equivariant), to show ! is a function.

▶ Assuming F is good, ! is a bijection that preserves substitution and instantiation:

\[
M : \mathbb{L} \implies \exists A. !A = M \\
!A = !B \implies A = B \\
!(A[X::=B]) = ![B/X]!A \\
!(([X]A) \triangle V) = (\text{abs}_X!A) \triangle V
\]

! is surjective, ! is injective, ! respects substitution, ! respects instantiation by parameters.
A Converse

- Assume ! is a bijection that preserves instantiation. Then (HE), (HP) and (HF) hold.

- (Thus preservation of substitution is not independent of the other properties. There is still something to figure out here.)
Example: $\beta$-reduction

\[
\frac{P : \mathbb{L} \quad N : \mathbb{L}}{(\text{abs}_x P) \cdot N \rightarrow (\text{abs}_x P) \succeq N} \quad (\beta)
\]

\[
\frac{M_1 \rightarrow M_2 \quad N : \mathbb{L}}{M_1 \cdot N \rightarrow M_2 \cdot N} \quad \frac{M : \mathbb{L} \quad N_1 \rightarrow N_2}{M \cdot N_1 \rightarrow M \cdot N_2}
\]

\[
\frac{M \rightarrow N}{\text{abs}_x M \rightarrow \text{abs}_x N} \quad (\xi)
\]

\[\rightarrow\] is well behaved, e.g.

\[\rightarrow\] is equivariant.

Reduction and well-formedness: $M \rightarrow N \iff M : \mathbb{L} \land N : \mathbb{L}$.

Reduction respects representation: $A \rightarrow B \Leftrightarrow !A \rightarrow !B$