\[ \text{We assume it is well known that, for } \beta \text{ reduction of pure } \lambda \text{ terms, the } \xi \text{ rule is invertible:} \\
\lambda x.s \xrightarrow{\beta} \lambda x.t \implies s \xrightarrow{\beta} t \]

With this observation we give a de Bruijn-like representation of pure \( \lambda \) terms, and rules for \( \beta \) reduction in this representation that need no rule \( \xi \) because rule \( \xi \) is admissible. This work has been formalized in Isabelle/HOL and proved adequate w.r.t. nominal Isabelle.

Fix a countable set of names, ranged over by \( x, y \). Let \( i, j, m, n \) range over natural numbers. The raw syntax of preterms is

\[
\text{pt ::= } X_n x \mid J_n j \mid (M N)_n
\]

Preterms are ranged over by \( M, N, P, Q \), and indexed by their \textit{height}, \( n \). We write \( hgt M = n \).

There is a notion of \textit{well formedness} of preterms, \( W \), defined inductively by

\[
\begin{align*}
W X_n x & \quad i < n \\
W J_n i & \\
W (M N)_n & \quad n \leq hgt M \quad n \leq hgt N
\end{align*}
\]

If \( WM \) we call \( M \) a \textit{term}, and write \( W_n M \) to mean \( WM \) and \( n \leq hgt M \). The height of a term shows how many bindings it implicitly sits under.

We can define \textit{abstraction} as a function on preterms:

\[
\begin{align*}
\text{lam}_x (X_n y) & := \text{if } x = y \text{ then } J_{n+1} 0 \text{ else } X_{n+1} y \\
\text{lam}_x (J_n j) & := J_{n+1} (i+1) \\
\text{lam}_x ((M N)_n) & := (\text{lam}_x (M) \text{ lam}_x (N))_{n+1}
\end{align*}
\]

Abstraction preserves well formedness and raises height by one.

\[
W_n M \implies W_{n+1} \text{lam}_x (M)
\]

Conversely, every term with height a successor is an abstraction. We use \( A, B \) as metavariables over abstractions.

The intended interpretation of preterms is given by the relation

\[
x \sim X_0 x \quad t_1 \sim M_1 \quad t_2 \sim M_2 \quad t \sim M \quad \lambda x.t \sim \text{lam}_x (M)
\]

which is an isomorphism between conventional \( \lambda \) terms (e.g. nominal terms) and terms of our formal language.

To define instantiation we first introduce a lifting function

\[
(X_n y)^\uparrow := X_{n+1} y \quad (J_n j)^\uparrow := J_{n+1} (i+1) \quad ((M N)_n)^\uparrow := ((M)^\uparrow (N)^\uparrow)_{n+1}
\]

which we iterate as: \((M)^\uparrow^0 := M\) and \((M)^\uparrow^{m+1} := ((M)^\uparrow^m)^\uparrow\).

\textit{Instantiation} is a binary function, \( M[N] \). If \( hgt M = 0 \) (\( M \) is under no binders), \( M[N] = M \).

Otherwise \( M[N] \) fills any holes \( J_{n+1} 0 \) in \( M \) and adjusts the rest of the term:

\[
\begin{align*}
X_{n+1} y[N] & := X_n y \\
J_{n+1} 0[N] & := (N)^\uparrow^n \\
J_{n+1} (j+1)[N] & := J_n j
\end{align*}
\]
Instantiation preserves well formedness and lowers height by one:

\[ \mathcal{W}_{n+1} M \land \mathcal{W} N \implies \mathcal{W}_n M[N] \]

Using abstraction we have a natural definition of \( \beta \) reduction:

\[
\begin{align*}
\mathcal{W} M & \cdot \mathcal{W} N \\
\frac{M \xrightarrow{\beta} M' \quad \mathcal{W} M \cdot \mathcal{W} N}{(\text{lam}_x(M) \cdot N)_0 \xrightarrow{\beta} (\text{lam}_x(M))[N]}
\end{align*}
\]

Any preterm that participates in this relation is well-formed. This relation is correct \( \beta \) reduction w.r.t. the meaning of preterms given above, but still contains an invertible \( \xi \) rule. To define an equivalent relation with no \( \xi \) rule we need to define generalized lifting, \( (M)^{\text{\#}} \):

\[
(X_n y)^{\text{\#}} := X_{n+1} y \quad (J_n j)^{\text{\#}} := \begin{cases} 
J_{n+1} j & (j < i) \\
J_{n+1} j+1 & (j \geq i)
\end{cases}
\]

which we iterate as \( (M)^{\text{\#}0} := M \) and \( (M)^{\text{\#}n+1} := ((M)^{\text{\#}n})^{\#\text{\#}} \). As with instantiation, generalized instantiation, \( (M)[N]^i \), leaves terms \( M \) of height 0 unchanged, and updates abstractions:

\[
(X_{n+1} y)[M]^i := X_n y \quad (J_{n+1} j)[M]^i := (M)^{\text{\#}n-i} \quad ((P Q)_{n+1})[M]^i := ((P)[M]^i (Q)[M]^i)_n
\]

Claim the relation \( \bullet \succ \bullet \) defined without a \( \xi \) rule:

\[
\frac{\mathcal{W}_{n+1} A \quad \mathcal{W}_n N}{(A N)_n \succ (A)[N]^n} \quad \frac{M > M'}{\mathcal{W}_n M} \quad \frac{M > M'}{\mathcal{W}_n N} \quad \frac{N > N'}{\mathcal{W}_n M} \quad \frac{N > N'}{\mathcal{W}_n N}
\]

is equivalent to the relation \( \bullet \xrightarrow{\beta} \bullet \) given above (and thus to the usual notion of \( \beta \) reduction).

Proof that \( M > N \implies M \xrightarrow{\beta} N \) goes by induction on the relation \( M > N \). Both congruence rule cases use invertibility of rule \( \xi \) for the relation \( \bullet \xrightarrow{\beta} \bullet \). The converse direction is straightforward. \( \square \)

Here is Tait–Martin-Löf parallel reduction without a \( \xi \) rule:

\[
X_n y \succ X_n y \quad \frac{n \leq \text{hgt} M \quad M \succ M'}{\mathcal{W}_n M} \quad \frac{n \leq \text{hgt} N \quad N \succ N'}{\mathcal{W}_n N} \quad \frac{j < n}{J_n j \succ J_n j}
\]

\[
\frac{n < \text{hgt} A \quad A \succ B}{\mathcal{W}_n A} \quad \frac{n \leq \text{hgt} M \quad M \succ N}{\mathcal{W}_n M} \quad \frac{(A M)_n \succ (B)[N]^n}{(A M)_n \succ (B)[N]^n}
\]

This (nondeterministic) parallel reduction can be made into (deterministic) complete development by replacing the application congruence rule with

\[
\frac{n = \text{hgt} M \quad M \succ M'}{\mathcal{W}_n M} \quad \frac{n \leq \text{hgt} N \quad N \succ N'}{\mathcal{W}_n N}
\]

which removes overlap with the \( \beta \) rule.

Unfortunately this approach doesn’t seem to extend to \( \beta\eta \) reduction, as rule \( \xi \) is not invertible in that case. On this point it is interesting to note that none of the reduction relations in this note can reduce the height of a term, but \( \eta \) reduction can do that.