Sums, products, and common functions

Sharon Goldwater
Institute for Language, Cognition and Computation
School of Informatics, University of Edinburgh

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Note: This document is unfinished, but I'm not sure when I'll get back to it, so I've made it available with sections 2-4 as placeholders only.

1 Sequences, sums and products

1.1 Definitions and notation

In computer science, we often need to work with sequences. A SEQUENCE is a collection of items (also called TERMS or ELEMENTS), where (unlike in a set) the equivalence of two sequences depends not only on what the elements are, but also on the order of the elements, and whether they are repeated. A sequence is basically just a list, whether finite or infinite. Here are some example sequences:

SEQUENCE
TERMS
ELEMENTS

a) 1, 3, -4, 6.2

b) 4, 2, 6, 8

c) 2, 2, 4, 6, 8

d) 2, 4, 6, 8

e) 2, 4, 6, 8, ...

f) `a`, `ab`, `abb`, `abbb`, ...

Example 1.1.1. Which of the sequences above are finite, and which are infinite? Which are equivalent to each other?

Solution: The first four are finite but none are equivalent: they all differ in terms of order or repetition of elements. The last two examples are infinite sequences (indicated by ...) so are also not equivalent to the others, nor are they equivalent to each other.

All of the sequences above are written *explicitly*; that is, each of the terms is written out, except in the case of the infinite sequences where the ... are used and the reader is left to extrapolate the remainder of the sequence. For example, sequence (1.0e)) contains all positive even integers in order. We could also use ... to help write out a finite but very long sequence, such as $2, 4, 6, \ldots, 100$, the sequence of positive integers between 2 and 100.

However, it is more common to write sequences using *formulas*. For example, we could refer to the sequence of even numbers between 2 and 100 as follows:

$$2i, \text{ for integers } 1 \leq i \leq 50$$

Notice what we did here. We defined an INDEX (plural INDICES), also called an INDEX VARIABLE i, which represents values from a very simple sequence: the sequence of integers

INDEX
INDICES
INDEX VARIABLE

from 1 to 50. We then defined the sequence we wanted (even integers between 2 and 100) in terms of that simpler sequence. The formula tells us that the ith element in our sequence is found by multiplying i by 2. That is, the explicit version of the sequence is:

$$2,4,6,\ldots,100$$

Example 1.1.2. Write a formula that expresses sequence 1.0f) from above. Use the notation $\texttt{b}^x$ to indicate x repetitions of the character b.

Solution: There are zero repetitions of b in the first element, one in the second, two in the third, and so on. So the number of repetitions for element i is $i-1$ and the formula is $\texttt{ab}^{i-1}$, for integers $i \geq 1$. (Note that italic a is a variable, while a is a particular character.)

Very often, we need to express the *sum* of a sequence of numbers. Again, we could do this explicitly (e.g., $2+4+6+\ldots+100$), but usually it's easier to write a formula using a SUMMATION symbol $\sum$. (The symbol is the Greek letter sigma, which sounds like s, as in "sum".) The beginning and ending values of the index are shown underneath and above the summation symbol, as follows (we write both the formula and the explicit version): SUMMATION

$$\sum_{i=1}^{50} 2i = 2\cdot 1+2\cdot 2+2\cdot 3+\ldots 2\cdot 50 \quad (1)$$

$$= 2+4+6+\ldots+100 \quad (2)$$

Example 1.1.3. Write down a formula that expresses the infinite sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$.

Solution: Let k be the index of each term. (You could just as well use i, it doesn't matter what we call the index variable.) The denominator of the kth term is k^2, so the sum is $\sum_{k=1}^{\infty}\frac{1}{k^2}$.

There is also a shorthand notation for the PRODUCT of a sequence of numbers, which uses the Greek letter pi, $\prod$. The rest of the notation is the same as for summation, with the index variable lower and upper limits written below and above the product symbol: PRODUCT

$$\prod_{i=1}^{50} 2i = (2\cdot 1)(2\cdot 2)(2\cdot 3)\ldots(2\cdot 50) \quad (3)$$

$$= 2\cdot 4\cdot 6\cdot\ldots\cdot 100 \quad (4)$$

1.2 Working with sums and products

Using Greek symbols does not change the basic algebraic properties of sums and products, so we can use those standard properties to rewrite formulas if needed. For example, sums and products are COMMUTATIVE: COMMUTATIVE

$$\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} y_i = \sum_{i=1}^{m}(x_i+y_i) \quad (5)$$

$$\left(\prod_{i=1}^{m} x_i\right)\cdot\left(\prod_{i=1}^{m} y_i\right) = \prod_{i=1}^{m}(x_i\cdot y_i) \quad (6)$$

where x_i and y_i just refer to the ith terms in each of the summations or products. Notice the use of parentheses in these equations to avoid ambiguity about the scope of the summation or product symbol (that is, which terms are in the sequence being summed). The scope of a summation symbol stops at the next $+$ or $-$, but (using standard order of operations) it could include a multiplied term. For example, we can write out the DISTRIBUTIVE property of DISTRIBUTIVE sums and products as follows:

$$\sum_{i=1}^{m}(c \cdot x_i) = \sum_{i=1}^{m} c \cdot x_i = c \cdot \sum_{i=1}^{m} x_i \tag{7}$$

Products are more ambiguous, which is why parentheses are important. Notice, for example, that $\prod_{i=1}^{m} x_i \cdot c$ could mean $x_1 \cdot x_2 \ldots \cdot x_n \cdot c$, or it could mean $x_1 \cdot c \cdot x_2 \cdot c \ldots \cdot x_n \cdot c$. So you should either use parentheses, or (if you really meant the latter) then put the *c closer* to the product symbol than the indexed term (here, x_i). That is, you should write $\prod_{i=1}^{m} cx_i$. In this case, the product must take scope over the x_i (otherwise the expression has no meaning), therefore since the c is closer to the product symbol, it must be included in each of the product terms as well. Notice that since c is a constant, we can choose to move the product of the c's outside the product symbol if we want:

$$\prod_{i=1}^{m} cx_i = c^m \prod_{i=1}^{m} x_i. \tag{8}$$

Example 1.2.1. Rewrite $\prod_{i=1}^{50} 2i$ as an expression where all constants are outside the product symbol.

Solution: Following the formula in equation 8 (or looking at the explicit product sequence in equation 3), we get:

$$\prod_{i=1}^{50} 2i = 2^{50} \prod_{i=1}^{50} i \tag{9}$$

$$= 2^{50} \cdot 50! \tag{10}$$

where the ! (FACTORIAL symbol) is just an alternative convenient notation: FACTORIAL

$$n! = 1 \cdot 2 \cdot \ldots \cdot n = \prod_{i=1}^{n} i \tag{11}$$

So far, the index variables have always started with 1. But this needn't be the case. For example, we could write $2+3+\ldots+21$ as $\sum_{k=2}^{21} k$. However, if we want to combine multiple summations or products into a single one, we need to make sure that the starting and ending values of the indices are the same.

Example 1.2.2. Simplify the following expression into a single summation:

$$\sum_{i=1}^{20} \frac{1}{i^2} + \sum_{i=2}^{21} i \tag{12}$$

Solution: First, we need to rewrite the second expression so that the starting and ending values of the indices match that of the first expression. To make things clear, I'll use a different index variable, k. We'd like an expression where

$$\sum_{i=2}^{21} i = \sum_{k=1}^{20} ?? \tag{13}$$

It's easy to solve for the ??: note that in the new indices, we substituted $k = i - 1$. Therefore $i = k + 1$, and we can simply substitute $k + 1$ inside the sum:

$$\sum_{i=2}^{21} i = \sum_{k=1}^{20} (k+1) \tag{14}$$

Now, we can go back and simplify the original expression:

$$\sum_{i=1}^{20} \frac{1}{i^2} + \sum_{i=2}^{21} i = \sum_{i=1}^{20} \frac{1}{i^2} + \sum_{k=1}^{20} (k+1) \tag{15}$$

$$= \sum_{k=1}^{20} (\frac{1}{k^2} + k + 1) \tag{16}$$

Notice that, although the minimum and maximum values of the indices need to match, the names of the index variables (i and k) do *not* need to match, because these are really just dummy variables. I could equivalently have written the final simplified expression using all i's instead of all k's.

Finally, it's worth pointing out that sometimes sums or products of sequences can be simplified to a single value with simple algebra.

Example 1.2.3. Compute the value of $\prod_{k=5}^{500} \frac{k}{k+1}$.

Solution: Begin by writing the product sequence explicitly: $\frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdot \ldots \frac{500}{501}$. This makes it clear that nearly all of the terms just cancel out, leaving the solution: $\frac{5}{501}$.

2 Logarithms and exponents

3 Logistic (sigmoid) function

4 max, argmax