Modal Effect Types

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Effect handlers are a powerful abstraction for defining, customising, and composing computational effects. Statically ensuring that all effect operations are handled requires some form of effect system, but using a traditional effect system would require adding extensive effect annotations to the millions of lines of existing code in these languages. Recent proposals seek to address this problem by removing the need for explicit effect polymorphism. However, they typically rely on fragile syntactic mechanisms or on introducing a separate notion of second-class function. We introduce a novel approach based on modal effect types.

1 Introduction

Effect handlers [44] allow programmers to define, customise, and compose a range of computational effects including concurrency, exceptions, state, backtracking, and probability, in direct-style inside the programming language. Following their pioneering use in languages such as Eff [4], Effekt [8, 9], Frank [13, 34], Koka [32], and Links [22], they are now increasingly being adopted in production languages and systems such as OCaml [50], Scala [7], and WebAssembly [42].

In a statically typed programming language with effect handlers some form of effect system to track effectful operations is necessary in order to ensure that a given program handles all of its effects. However, traditional effect systems require extensive effect annotations even for code that does not use effects. Consider the standard `map` function:

```
map : ∀ a b . (a → b) → List a → List b
```

This type is a statement about the values that map accepts and returns, but is silent about which effects may occur during its evaluation. In the effect system of Koka, for instance, this `map` function is thus presumed to be a total function that takes a function which cannot perform any effects and itself does not perform any effects.

However, this would prevent programmers from passing any effectful function to `map`. To use `map` in effectful code, in Koka we must give it a more permissive type such as:

```
map' : ∀ a b e . (a -e-> b) -e-> List a -e-> List b
```

This type uses *effect polymorphism*, quantifying over an *effect variable* `e`, which occurs on every arrow. Such effect annotations pollute the type signature of `map` to convey the obvious: `map'` is polymorphic in its effects, that is, the effects of `map' f xs` depend on the effects of the function argument `f`. Effect annotations impose a mild burden to authors of new code, but pose a significant problem when extending an existing language with effectful features.

Signatures of existing library code must be rewritten to support effect polymorphism [7, 39], even in legacy libraries that do not use effects, making it challenging to retrofit such an effect system onto an existing language in a backwards-compatible way without causing friction for existing codebases. However, if we can eliminate the need to annotate effect polymorphism, then

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retrofitting an effect system ought to become a tractable problem. Our goal is to design a principled effect system, where effect polymorphism silence is a virtue.

An important step towards that goal was taken by the language Frank [13, 34]. Frank gives `map` its original unannotated type, whilst still allowing it to be passed effectful functions. The key idea is that expressions are typed assuming an unknown set of possible effects—the *ambient effects*—will be provided by the context in which the expression occurs. Rather than assuming unannotated function types perform no effects, they are assumed to perform the ambient effects.

Frank still uses effect variables behind the scenes, implicitly inserting effect variables for passing the ambient effects around. For instance, Frank simply treats the type signature of `map` above as syntactic sugar for `map'`. (Frank has certain other syntactic idiosyncrasies, so in order to ease readability, we render Frank code using the syntax of the rest of the paper.) This syntactic mechanism is fragile. For instance, effect variables can appear in error messages as in the following example in which we use a `yield` effect to write a function that yields all values in a list.

```
gen : List Int ⟶[yield] 1
gen xs = map (fun x ⟶ do yield x) xs; ()
```

If the user forgets the `yield` annotation, Frank complains:

```
cannot unify effects e and yield, £
```

Here `£` and `e` are the underlying effect variables inserted by the Frank compiler. They do not appear in the source program and in larger programs it can be unclear how to fix such errors.

Effekt [9] and Scala [7] also make use of ambient effects to avoid effect polymorphism by tracking effects as capabilities. However, they either restrict functions to be second-class or require having capability variables in types for certain use cases, as we discuss further in Section 8.2.

We build on the insight that ambient effect contexts can substantially reduce the annotation burden. Instead of relying on desugaring to traditional effect polymorphism like Frank, we develop Met (Modal Effect Types), a novel effect system with a theoretical foundation based on modal types. We follow multimodal type theory (MTT) [19, 20] in tracking *modes* for types and terms and consider *modalities* as the transitions between such modes. We treat each possible ambient effect context as a mode, and each possible transition between effect contexts as a *modality*. We support *absolute* modalities, which override the ambient effect context. We also support *relative* modalities, which describe a local change to the ambient effect context, as exemplified by effect handlers which handles certain effects and let all others pass through unchanged.

Met precludes hidden effect variables in error messages as there are no hidden effect variables. Moreover, Met works smoothly with pure first-class higher-order functions, which require neither hidden effect variables nor extra annotations, and can be applied to effectful arguments. Both Frank and Met strive to capture the essence of modular programming with effects. Frank relies on a fragile syntactic characterisation based on polymorphic types. In contrast, Met provides a more robust characterisation based on simple types.

The main contributions of this paper are as follows.

- We give a high-level overview of the key ideas of modal effect types: effect contexts, absolute and relative modalities. We provide a series of practical examples to show how modal effect types enable us to write modular effectful programs without effect polymorphism (Section 2).
- We briefly recall the design of multimodal type theory (MTT), the basis of modal effect types, and outline why MTT works well for designing an effect system (Section 3).
- We introduce Met, a simply-typed core calculus with effect handlers and modal effect types (Section 4). We prove its type soundness and effect safety.

- Intuitively, Met can type check all functions that can be written in traditional effect systems using a single effect variable, which is the most common case in practice. We formally prove this intuition by presenting a calculus for row-based effect systems with a single effect variable and encoding it in Met (Section 5).
- We extend Met with data types and polymorphism for value types. To recover the full power of traditional effect systems, we also extend Met with effect polymorphism which can be seamlessly used alongside modal effect types to express effectful programs that use higher-order effects modularly (Section 6).
- We outline and prototype a surface language MetL which uses bidirectional type checking to infer the introduction and elimination of modalities (Section 7).
- We present a direct comparison of type signatures in MetL, Koka, and Effekt in order to highlight the practical appeal of modal effect types (Section 8.3).

Section 8 also discusses related and future work. The full specifications, proofs, and appendices can be found in the supplementary material.

2 Programming with Modal Effect Types

In this section we illustrate the main ideas of modal effect types through a series of examples. We demonstrate how modal effect types support modular composition of higher-order functions and effect handlers without effect polymorphism. The examples are written in MetL, which translates to the core calculus Met via a simple type-directed elaboration. In order to elucidate the core idea that modal effect types support modular effectful programming without polymorphism we begin with examples in the simply-typed fragment of MetL.

2.1 From Function Arrows to Effect Contexts

Traditional effect systems annotate a function type with the effects that the function may perform when invoked. For instance, consider the following typing judgement for the `app` function specialised to take a pair of a function from integers to the unit type and an integer.

```
⊢ fun (f, x) → f x : (Int -E-> 1, Int) -E-> 1
```

The effect annotation `E` is a row of typed operations that `f` may perform. For instance, if `E` is `get : 1 ↠ Int, put : Int ↠ 1` then `f` may perform a `get` operation which takes a unit value and returns an integer and a `put` operation which takes an integer and returns a unit value. As in Frank [13] and Koka [32], rows are scoped [31] meaning that they allow duplicate operations with the same name (but possibly different signatures). The order of duplicates matters, but the relative order of distinct operations does not.

Since `app` invokes its argument, `E` also denotes the operations that invoking `app` might perform. As we saw in the introduction, the standard way to support modularity is to be polymorphic in `E`. But this introduces an annotation burden for all higher-order functions, including those (like `app`) which do not themselves perform effects.

In the spirit of Frank, Met decouples effects from function types and tracks *effect contexts* in typing judgements. All components of the term and type share the same effect context (unless manipulated by modalities as we will see in Sections 2.2 and 2.3). For instance, we have the following typing judgement for the same `app` function as above.

```
⊢ fun ( f , x) → f x : (Int → 1, Int) → 1 @ E
        @E     @E      @E          @E
```

As a visual aid, we use braces to explicitly annotate the effect contexts for the argument `f` and the whole function in the term and type. The `@ E` annotation belongs to the judgement and indicates

the effect context `E`. This is the *ambient effect context* for the whole term and type of this typing judgement. An effect context specifies which operations may be performed. In this example, the effect contexts are all the same as the ambient effect context. We know that `app` can perform the same effects as its argument `f` as they share the same effect context.

2.2 Overriding the Ambient Effect Context with Absolute Modalities

An *absolute modality* `[E]` defines a new effect context `E` that overrides the ambient effect context. For instance, the following function invokes the operation `yield` via the **`do`** keyword. The `yield` operation takes an integer and returns a unit value.

```
⊢ fun x → do yield x : [yield : Int ↠ 1](Int → 1) @ .
                @ yield : Int ↠ 1          @ yield : Int ↠ 1
```

The absolute modality `[yield : Int ↠ 1]` specifies a singleton effect context in which the `yield` operation with signature `Int ↠ 1` may be performed. Here, it overrides the empty ambient effect context (`.`), allowing `yield` to be performed in the function body.

Effect contexts percolate through the structure of a type. For example, a function of type `[E](A → B)` may perform effects `E` when invoked, and a list of type `[E](List (A → B))` may perform effects `E` when its components are invoked. For brevity, we define an effect context abbreviation.

```
eff Gen a = yield : a ↠ 1
```

Such abbreviations are merely macros, such that, for instance, `[Gen Int]` denotes the modality `[yield : Int ↠ 1]` and `[Gen Int, E]` denotes the modality `[yield : Int ↠ 1, E]`.

For higher-order functions like `map` and `app` which do not directly perform any effects, we use the empty absolute modality `[]`. For instance, in Metl, the curried first-class higher-order `iter` function, specialised to iterate over a list of integers, is defined as follows.

```
iter : []((Int → 1) → List Int → 1)
iter f nil         = ()
iter f (cons x xs) = f x; iter f xs
```

The empty absolute modality `[]` specifies an empty effect context in which the function is defined. However, due to subeffecting, `iter` is not limited to only the empty effect context. For instance, we can apply `iter` to the previous function which uses `yield`.

```
⊢ iter (fun x → do yield x) : List Int → 1 @ Gen Int
```

Metl allows us to use `iter` here directly even though its type contains an absolute modality. This is allowed since an elaboration step implicitly unboxes `iter` before it is applied to the function that invokes `yield`. Though the modality `[]` requires us to use `iter` under the empty effect context, subeffecting then upcasts the empty effect context to the singleton effect context `Gen Int`.

To achieve the same flexibility of applying `iter` to any effectful arguments in a traditional row-based effect system, we would need effect polymorphism:

```
iter : ∀ e . (Int -e-> 1) -e-> List Int -e-> 1
```

2.3 Transforming the Ambient Effect Context with Relative Modalities

So far we have only seen examples that are either pure or just perform effects. Absolute modalities suffice for modular programming with such examples without requiring any use of effect polymorphism. However, the situation becomes more interesting when we introduce constructs that manipulate effect contexts non-trivially, such as effect handlers. Effect handlers provide a way of interpreting such effects inside the object language. For instance, we can use an effect handler to interpret `Gen Int` thunks by simply generating a list of integers.

```
asList f = handle f () with
  return ()              ↦ nil
  (yield : Int ↠ 1) x r ↦ cons x (r ())
```

The body of `asList` invokes the function `f` inside a handler. The handler have two clauses that account for two cases: 1) what happens when `f` returns; and 2) what happens when `f` performs `yield`. In the first case, it directly returns the empty list `nil`. In the second case, it conses the integer `x` onto the head of the list returned by the application of `r`. Here `r` is bound to the continuation of performing `yield` inside `f`. The argument type of `r` is determined by the return type of the operation being handled (unit in the case of `yield`) and its return type is determined by the return type of the handler. Thus `r : 1 → List Int`. The continuation `r` reinstalls the handler around its body such that if `yield` is performed again then it will be handled by the same handler. (This kind of handler is known as *deep* in the literature [27].) We write `H` for the handler clauses in `asList`.

What type should `asList` have? Naively, we might simply expect to ignore the handler.

```
asList : []((1 → 1) → List Int)
```

This would be unsound as it would allow us to write:

```
crash : [Gen String](String → List Int)
crash s = asList (fun () → do yield s)
```

The function passed to `asList` yields a string. This is then accidentally handled by the handler in `asList`, which expects an integer.

One fix is to box the argument of `asList` with an absolute modality `[Gen Int]`.

```
asList : []([Gen Int](1 → 1) → List Int)
```

To see what happens here, consider the following typing judgement for the inlined function body of `asList` under some effect context `E`.

```
⊢ fun f → handle f () with H : [Gen Int](1 → 1) → List Int @ E
      @ Gen Int   @ Gen Int, E        @ Gen Int
```

The effect handler extends the ambient effect context `E` with a `yield` operation to give an effect context of `Gen Int, E`. Meanwhile, the argument `f` has the effect context `Gen Int` specified by the absolute modality `[Gen Int]`. This is sound, because it is safe to invoke a function which can only use `Gen Int` under the effect context `Gen Int, E`.

However, the restriction that the argument can only use `Gen Int` severely hinders reusability. We would like to apply `asList` to arguments that may perform other operations in addition to `yield`. To this end, we introduce *relative modalities* which enable us to describe the relative change that a handler makes to the effect context. For instance, consider:

```
asList : [](<Gen Int>(1 → 1) → List Int)
```

The relative modality `<Gen Int>` is part of the argument type and extends the ambient effect context with `Gen Int` for the inner function `1 → 1`. The typing judgement becomes:

```
⊢ fun f → handle f () with H : <Gen Int>( 1 → 1 ) → List Int @ E
      @ Gen Int, E   @ Gen Int, E          @ Gen Int, E
```

Now, the effect context for the function of argument `f` is also `Gen Int, E`, matching the effect context at its invocation. This allows the argument `f` to perform other effects from the ambient effect context `E` (which will be forwarded to outer handlers).

In practice relative modalities often appear in an argument position and specify which effects of an argument will be handled in the function body. A function that handles effects `D` of its argument typically has a type of the form `<D>(A → B) → C`.

In a traditional row-based effect system, in order to be able to use `asList` across different effect contexts, we would typically require effect polymorphism.

```
asList : ∀ e . (1 -[Gen Int, e]-> 1) -[e]-> List Int
```

2.4 Coercions Between Modalities

The implicit unboxing and boxing performed by Metl allows values to be coerced between different modalities. For instance, we can extend an absolute modality.

```
⊢ fun f → f : [Gen Int](1 → 1) → [Gen Int, Gen String](1 → 1) @ E
                  @ Gen Int                  @ Gen Int, Gen String
```

In contrast, a relative modality cannot be similarly extended

```
⊬ fun f → f : <>(1 → 1) → <Gen Int>(1 → 1) @ E  # Ill-typed
                 @ E            @ Gen Int, E
```

as doing so would insert a fresh `yield : Int ↠ 1` operation which may shadow other `yield` operations in `E`, consequently permitting bad programs like `crash` in Section 2.3.

An absolute modality can be coerced into the corresponding relative modality.

```
⊢ fun f → f : [Gen Int](1 → 1) → <Gen Int>(1 → 1) @ E
                  @ Gen Int          @ Gen Int, E
```

But the converse is not permitted

```
⊬ fun f → f : <Gen Int>(1 → 1) → [Gen Int](1 → 1) @ E  # Ill-typed
                  @ Gen Int, E          @ Gen Int
```

because the argument may also use effects from the ambient effect context `E`.

Similarly, the following typing judgement is invalid

```
⊬ fun f → f () : <Gen Int>(1 → 1) → 1 @ E  # Ill-typed
       @ Gen Int, E  @ E      @ Gen Int, E
```

because the argument may use `Gen Int` in addition to the ambient effect context `E`.

2.5 Composing Handlers

We can compose handlers modularly. For example, consider state operations `get` and `put`.

```
eff State s = get : 1 ↠ s, put : s ↠ 1
```

We can implement a standard state handler, specialised to integer state, by interpreting a computation over state operations as a state-passing function.

```
state : [](<State Int>(1 → 1) → Int → 1)
state m = handle m () with
  return x              ↦ fun s → x
  (get : 1 ↠ Int) () r  ↦ fun s → r s s
  (put : Int ↠ 1) s' r  ↦ fun s → r () s'
```

Using integer state we can write a generator which yields the prefix sum of a list.

```
prefixSum : [Gen Int, State Int](List Int → 1)
prefixSum xs = iter (fun x → do put (do get () + x); do yield (do get ())) xs
```

The absolute modality `[Gen Int, State Int]` aggregates all effects performed in `prefixSum`.

We can now handle `prefixSum` by composing two handlers in sequence.

```
> asList (fun () → state (fun () → prefixSum [3,1,4,1,5,9]) 0)
# [3,4,8,9,14,23] : List Int
```

The type signature of `state` mentions only `State Int` even though it is applied to a computation which invokes `prefixSum`, which also uses `Gen Int`. In contrast, to achieve the same modularity, conventional row-based effect systems would ascribe the following type to `state`.

```
state : ∀ e . (1 -{State Int, e}-> 1) -{e}-> Int -{e}-> 1
```

2.6 Storing Effectful Functions in Data Types

We show how modal effect types allow us to smoothly store effectful functions into data types. We consider a richer effect handler example that implements cooperative concurrency using a UNIX-style fork operation [25, 47]. A `Coop` effect context includes two operations.

```
eff Coop = ufork : 1 ↠ Bool, suspend : 1 ↠ 1
```

The `ufork` operation returns a boolean. As we shall see, concurrency can be implemented by a handler that invokes the continuation twice. The idea is that passing true to the continuation defines the behaviour of the parent, whereas passing false defines the behaviour of the child. The `suspend` operation suspends the current process allowing another process to run.

We model a process as a data type that embeds a continuation function which takes a list of suspended processes and returns unit. In addition, we define auxiliary functions `push` to append a process onto the end of the list and `next` to remove and then run the process at the head of the list.

```
data Proc = proc (List Proc → 1)

push : [](Proc → List Proc → List Proc)
push x xs = xs ++ cons x nil
```

```
next : [](List Proc → 1)
next q = case q of
  nil              → ()
  cons (proc p) ps → p ps
```

The following handler implements a scheduler parameterised by a list of suspended processes.

```
schedule : [](<Coop>(1 → 1) → List Proc → 1)
schedule m = handle m () with
  return ()                 ↦ fun q → next q
  (suspend : 1 ↠ 1)  () r ↦ fun q → next (push (proc (r ())) q)
  (ufork : 1 ↠ Bool) () r ↦ fun q → r true (push (proc (r false)) q)
```

The **return**-case is triggered when a process finishes, and runs the next available process. The `suspend`-case pushes the continuation onto the end of the list, before running the next available process. The `ufork`-case implements the process duplication behaviour of UNIX fork by first pushing one application of the continuation onto the end of the list, and then immediately applying the other. Observe that the above code seamlessly stores continuation functions in `Proc` and then puts `Proc` in `List` without even mentioning any effects. These functions are not restricted to be pure; they may use any effects from the ambient effect context.

The `schedule` function allows processes to use any other effects. To achieve this flexibility, a traditional row-based effect system requires effect polymorphism and a parameterised data type.

```
data Proc e = proc (List Proc -{e}-> 1)
schedule : ∀ e . (1 -{Coop, e}-> 1) -{e}-> List (Proc e) -{e}-> 1
```

2.7 Masking

Whereas handlers extend the effect context, masking restricts the effect context [5]. Masking is a useful device to conceal private implementation details [36]. We illustrate masking by using a generator to implement a function to find an integer satisfying a predicate.

```
findWrong : []((Int → Bool) → List Int → Maybe Int)  # ill-typed
```

```
findWrong p xs = handle (iter (fun x → if p x then do yield x) xs) with
  return _               ↦ nothing
  (yield : Int ↠ 1) x _ ↦ just x
```

The `findWrong` program is ill-typed because it is unsound to invoke predicate `p` inside the handler, as this would accidentally handle any `yield` operations performed by `p`.

```
⊢ ... handle (iter (fun x → if (p x) then do yield x) xs) with ... : _ @ E
                               @ Gen Int, E
```

Changing the type of `p` from `Int → Bool` to `<Gen Int>(Int → Bool)` would fix the type error but leak the implementation detail that `findWrong` uses `yield`. A better solution is to mask `yield` for the argument `p` and rewrite the handled expression as follows.

```
⊢ ... handle (iter (fun x → if mask<yield>(p x) ... ) with ... : _ @ E
                               @ E
```

The term `mask<yield>(M)` masks the effect `yield` from the ambient effect context for `M`. Now the effect context for `p` is equivalent to the ambient one, since the transformations of extending with `yield` (performed by the handler) followed by masking with `yield` (performed by the mask) cancel each other out. Like a handler, a mask wraps its return value in a relative modality. The term `mask<yield>(p x)` initially returns a value of type `<yield|>Bool` instead of `Bool`, where `<yield|>` is a relative modality masking `yield` from the ambient effect context. In this case Metl automatically unboxes the relative modality, as booleans are pure and do not rely on effect contexts.

In general, relative modalities have the form `<L|D>` which specifies a local transformation on the effect context: `L` is a row of effect labels that are removed from the effect context and `D` is a row of effects that are added to the effect context. We write `<D>` as a shorthand for `<|D>`.

2.8 Kinds

A handler extends the effect context with those effects it handles. When a value leaves the scope of a handler, its effect context changes, and we must keep track of this change.

Let us now consider `state'`, a variation of the `state` function defined in Section 2.5 in which the return type of the handled computation is changed from `1` to `1 → 1`. The body of `state'` is exactly the same as that of `state`. We might naively expect its type signature to be the following.

```
state' : [](<State Int>(1 → (1 → 1)) → Int → (1 → 1))
```

However, this type is unsound. Suppose we apply `state'` as follows.

```
state' (fun () → fun () → do put (do get () + 42)) 0 : 1 → 1
```

The function `fun () → do put (do get () + 42)` is returned by the return clause of `state'`, escaping the scope of their handler. To guarantee effect safety, we must capture the fact that the returned function might perform `get` and `put` when invoked. The following type signature is sound.

```
state' : [](<State Int>(1 → (1 → 1)) → Int → <State Int>(1 → 1))
```

Let us contrast the types of `state` and `state'`:

```
state  : [](<State Int>(1 → 1)        → Int → 1)
state' : [](<State Int>(1 → (1 → 1)) → Int → <State Int>(1 → 1))
```

The crucial difference is that the former cannot leak the state effect as the handled computation has unit type, whereas the latter can as the handled computation is a function.

In practice, it is useful to allow a value of base type or an algebraic data type that contains only base types or a type boxed with absolute modalities to appear anywhere, including escaping the scope of a handler. Such values can never depend on the effect context in which they are used. We

introduce a kind system in which the `Abs` kind classifies such *absolute types*, whereas the `Any` kind classifies unrestricted types. Subkinding allows absolute types to be treated as unrestricted.

2.9 Polymorphism for Value Types

Now that we have explored the simply-typed fragment of modal effect types, we briefly outline its extension with polymorphism for value types. For simplicity, Metl requires explicit type abstraction and type application. We write explicit type abstractions and applications using braces. For instance, we can define the polymorphic iterate function as follows.

```
iter : ∀ a . [](a → 1) → List a → 1
iter {a} f nil         = ()
iter {a} f (cons x xs) = f x; iter {a} f xs
```

The extension is mostly routine, however, we must respect kinds. The `state` and `state'` examples in Section 2.8 illustrate a non-uniformity that we must account for. We may generalise them such that the former allows any absolute return type and the latter allows any return type at all.

```
state  : ∀ [a] . [](<State Int>(1 → a) → Int → a)
state' : ∀  a  . [](<State Int>(1 → a) → Int → <State Int>a)
```

The syntax `∀ [a]` ascribes kind `Abs` to `a`, allowing values of type `a` to escape the handler. The syntax `∀ a` ascribes kind `Any` to `a`, not allowing values of type `a` to escape the handler. Though in practice it is usually desirable for return types of computations inside handler scopes to be absolute, the latter type signature is the more general in that simply by η-expanding we can coerce it to the former.

```
⊢ fun {a} m s → state' {a} m s : ∀ [a] . [](<State Int>(1 → a) → Int → a) @ .
```

2.10 Effect Polymorphsim

Though modal effect types alone suffice for writing a remarkably rich class of modular effectful programs, occasionally effect variables are still useful. In particular, they are required for the implementation of higher-order operations [53, 54, 57], which take closures as arguments.

Modal effect types restrict operation arguments and results to be absolute. This is because effect handlers provide non-trivial manipulation of control-flow, which allows operation arguments and results to jump between different effect contexts. For example, if we were to allow an operation `leak : (1 → 1) ↠ 1`, then we could write the following unsafe program.

```
handle asList (fun () → do leak (fun () → do yield 42)) with
  return                _   ↦ fun () → 37
  (leak : (1 → 1) ↠ 1) p _ ↦ p
```

The `asList` handler extends the ambient effect context with `yield`. However, the `leak` handler binds the closure `(fun () → do yield 42)` to `p` and returns this closure, leaking the `yield` operation.

Consider a higher-order fork operation which takes a thunk as an argument. (One reason to prefer such an operation over the UNIX fork operation of Section 2.6 is that it can be implemented using an affine handler that invokes each captured continuation at most once.) We may define a recursive effect context for cooperative processes as follows.

```
eff Coop = fork : [Coop](1 → 1) ↠ 1, suspend : 1 ↠ 1
```

In order to allow processes to use additional operations as well as `fork` and `suspend` we must extend our modal type system with effect variables. With an effect variable `e`, we can define the following higher-order `Coop` parameterised over effect context `e`.

```
eff Coop e = fork : [Coop e, e](1 → 1) ↠ 1, suspend : 1 ↠ 1
```

As we show in Section 6.2, modal effect types are compatible with effect variables. Nonetheless, effect variables are only necessary for use-cases such as higher-order effects in which a computation needs to be stored for use in an effect context different from the ambient one.

3 A Tale of Locks and Keys: Elaborating MetL into Met

So far, we have presented a series of examples in MetL illustrating the core ideas of modal effect types. While MetL is an easy-to-use surface language, it contains too many implicit coercions to be a suitable basis for a core calculus. Instead, our core calculus Met is a more explicit language based on (simply-typed) multimodal type theory (MTT) [19, 20]. In this section, we motivate the design of Met and introduce core concepts of MTT. For a more detailed account of the simply-typed fragment of MTT, we refer the reader to the work of Kavvos and Gratzer [30].

MTT extends type theory with the notions of *modes* and *modalities*. A typing judgement has the form $\Gamma \vdash M : A \mathbin{@} E$, which means that term M has type A under context Γ at mode E. In our work, we use modes to represent effect contexts and modalities to represent absolute modalities `[E]` and relative modalities `<L|D>`. In MTT, most type constructors are mode-local: the components have the same mode as the whole type. For example, if a function type $A \to B$ is at mode E, then both A and B are at mode E. This is exactly the behaviour needed for effect contexts in Met (Section 2.1), and is a primary motivation for treating effect contexts as modes in Met.

In MTT, modes can be transformed by modalities. A modality $\mu : E \to F$ transforms types and terms from mode E to mode F. While this is implicit in MetL, Met requires explicit terms for modality introduction and elimination. We write $\mathbf{mod}_\mu$ for the introducing the modality μ. For example, the function `fun x → do yield x` of type `[yield : Int ↠ 1](Int → 1)` from Section 2.2 is elaborated to the following term in Met with an explicit modality introduction.

$$\mathbf{mod}_{[\texttt{yield:Int} \twoheadrightarrow 1]}\ (\lambda x^{\texttt{Int}}.\mathbf{do}\ \texttt{yield}\ x)$$

In order to invoke this function, we need to eliminate its modality first. Following the literature on modal types, we refer to introduction of modalities as *boxing* and elimination as *unboxing*. In MetL, we need not manually unbox variables; elaboration does so for us. Met adopts let-style unboxing, which requires binding a term in order to unbox it. For example, consider that we bind the above function to the variable `gen` and then apply it to the integer 42. The binding and application are elaborated to the following Met term.

$$\mathbf{let}\ \mathbf{mod}_{[\texttt{yield:Int} \twoheadrightarrow 1]}\ \texttt{gen} = \mathbf{mod}_{[\texttt{yield:Int} \twoheadrightarrow 1]}\ (\lambda x^{\texttt{Int}}.\mathbf{do}\ \texttt{yield}\ x)\ \mathbf{in}\ \texttt{gen}\ 42$$

The let-binding eliminates the absolute modality. Whenever `gen` is used, the type system ensures that the effect context contains at least the operation `yield : Int ↠ 1`. Both boxing and unboxing interact with the typing context non-trivially. Their typing rules in MTT are as follows[1].

$$\frac{\mu : E \to F \qquad \Gamma, \mathbin{🔒_\mu} \vdash M : A \mathbin{@} E}{\Gamma \vdash \mathbf{mod}_\mu\ M : \mu A \mathbin{@} F} \qquad \frac{\Gamma \vdash M : \mu A \mathbin{@} E \qquad \Gamma, x :_\mu A \vdash N : B \mathbin{@} E}{\Gamma \vdash \mathbf{let}\ \mathbf{mod}_\mu\ x = M\ \mathbf{in}\ N : B \mathbin{@} E}$$

Contexts Γ are ordered. When we box a term with modality μ, we put a lock with index μ in the context. When we unbox a term with modality μ, we annotate the variable binding with index μ in the context. These locks and annotations are crucial for controlling variable access. For instance, consider the following typing judgement

$$x : \mu A \vdash \mathbf{let}\ \mathbf{mod}_\mu\ x' = x\ \mathbf{in}\ \mathbf{mod}_\nu\ x' : \nu A \mathbin{@} F$$

[1] As we will see in Section 4, in Met modalities on bindings and locks have indexes and the $\mathbf{let}\ \mathbf{mod}_\mu$ syntax also has an additional annotation. We opt for a simplified version here to convey the core intuition.

which unboxes a variable of type μA and re-boxes it with modality ν. In the derivation tree, we obtain the following judgement for x':

$$x : \mu A, x' :_{\mu} A, \blacksquare_{\nu} \vdash x' : A \mathbin{@} E \quad \text{where } \nu : E \to F$$

Whether this usage of x' is valid or not depends on the mode theory. Beyond modes and modalities, a mode theory also specifies a set of modality transformations $\alpha : \mu \Rightarrow \nu$. We can use a variable $x' :_{\mu} A$ across a lock $\blacksquare_{\nu}$ if there exists a transformation $\alpha : \mu \Rightarrow \nu$.

In our work, modality transformations describe possible coercions between modalities. In the second example of Section 2.4, we saw that MetL disallows coercing a function of type `<>(1 → 1)` into a function of type `<Gen Int>(1 → 1)`. This example is elaborated into Met as follows:

$$\lambda f^{\langle|\rangle(1\to 1)}.\mathbf{let\ mod}_{\langle|\rangle}\ \hat{f} = f\ \mathbf{in\ mod}_{\langle\!|\texttt{Gen Int}\rangle}\ \hat{f} : \langle\!|\texttt{Gen Int}\rangle(1 \to 1) \mathbin{@} E$$

First, f is unboxed to obtain $\hat{f} :_{\langle|\rangle} (1 \to 1)$ in the context. Then, it is boxed again with the $\langle\!|\texttt{Gen Int}\rangle$ modality. However, just as in MetL, this example does not check in Met. The reason for this is that the boxing term $\mathbf{mod}_{\langle\!|\texttt{Gen Int}\rangle}$ introduces a lock $\blacksquare_{\langle\!|\texttt{Gen Int}\rangle}$ in the context. The variable $\hat{f}$ can only be used under the lock if there is a modality transformation of form $\langle|\rangle \Rightarrow \langle\!|\texttt{Gen Int}\rangle$. But as explained in Section 2.4, such a transformation would break effect safety and is not thus permitted.

Locks are introduced in the context whenever a typing rule changes the effect context. This happens not only during boxing but also in the rules for handlers and masks. For example, the definition of `asList : [](<Gen Int>(1 → 1) → List Int)` from Section 2.3 is elaborated to

$$\mathbf{mod}_{[]}\ (\lambda f^{\langle\!|\texttt{Gen Int}\rangle(1\to 1)}.\mathbf{let\ mod}_{\langle\!|\texttt{Gen Int}\rangle}\ \hat{f} = f\ \mathbf{in\ handle}\ \hat{f}\ ()\ \mathbf{with}\ \hat{H})$$

where $\hat{H}$ is the elaboration of handler clauses H. The handler for the `Gen` effect introduces a lock $\blacksquare_{\langle\!|\texttt{Gen Int}\rangle}$ in the context. We can use the variable $\hat{f}$ under the lock if there is a modality transformation $\alpha : \langle\!|\texttt{Gen Int}\rangle \Rightarrow \langle\!|\texttt{Gen Int}\rangle$. This identity transformation always exists. (For those readers familiar with category theory, the structure of modes E, modalities $\mu : E \to F$, and transformations $\alpha : \mu \Rightarrow \nu$ forms a 2-category.)

4 A Multimodal Core Calculus with Effect Handlers

In this section we introduce Met, a simply-typed call-by-value calculus with effect handlers and modal effect types. We present its static and dynamic semantics as well as its meta theory. We aim at a minimal core calculus here and defer extensions such as data types, alternative forms of handlers, and polymorphism (including both for values and effects) to Section 6.

4.1 Syntax

The syntax of Met is as follows.

Types	$A, B ::= 1 \mid A \to B \mid \mu A$	Contexts	$\Gamma ::= \cdot \mid \Gamma, x :_{\mu_F} A \mid \Gamma, \blacksquare_{\mu_F}$
Masks	$L ::= \cdot \mid \ell, L$	Terms	$M, N ::= () \mid x \mid \lambda x^A.M \mid M\,N \mid \mathbf{mod}_{\mu}\ V$
Extensions	$D ::= \cdot \mid \ell : P, D$		$\mid \mathbf{let}_{\nu}\ \mathbf{mod}_{\mu}\ x = V\ \mathbf{in}\ M$
Effect Contexts	$E, F ::= \cdot \mid \ell : P, E$		$\mid \mathbf{do}\ \ell\ M \mid \mathbf{mask}_{L}\ M$
Signatures	$P ::= A \twoheadrightarrow B \mid -$		$\mid \mathbf{handle}\ M\ \mathbf{with}\ H$
Modalities	$\mu, \nu ::= [E] \mid \langle L \mid D \rangle$	Values	$V, W ::= () \mid x \mid \lambda x^A.M \mid \mathbf{mod}_{\mu}\ V$
Kinds	$K ::= \mathsf{Abs} \mid \mathsf{Any}$	Handlers	$H ::= \{\mathbf{return}\ x \mapsto M\} \mid \{\ell\ p\ r \mapsto M\} \uplus H$

Met extends a simply-typed λ-calculus with standard constructs for effects and handlers as well as the main novelty of this work: modal effect types. We highlight the novel parts in grey.

We have provided a brief introduction to MTT in Section 3. We present Met without assuming deep familiarity with MTT and discuss further in Section 8.5. MTT provides us with the flexibility to define our own mode theory. In the following, we first illustrate the structures of modes, modalities, and modality transformations for Met before presenting the typing rules.

4.2 Effect Contexts as Modes

The *modes* of Met are effect contexts E. Each type and term is at some effect context E, specifying the available effects from the context.

Effect contexts E are defined as scoped rows of effect labels [31]. Each label denotes an effectful operation. An effect context may contain the same label multiple times. Each label has an operation signature. An operation signature can be an arrow of the form $A \twoheadrightarrow B$, which indicates that the operation takes an argument of type A and returns a value of type B, or absent $-$ (similar to presence types [46]), which indicates that the operation of this label cannot be invoked.

Following Rémy [46] and Leijen [31], we identify effects up to reordering of distinct labels, and allow absent labels to be freely added to or removed from the right of effect contexts. For instance, $\ell : P, \ell' : -$ is equivalent to $\ell : P$. We can think of an effect context as denoting a map from labels to infinite sequences of signatures where a cofinite tail of each sequence contains only $-$.

Extensions D and masks L are used respectively to extend effect contexts with more labels or removes some labels from them. Extensions are like effect contexts except that we do not ignore labels with absent signatures in their equivalence relation, so $\ell : P, \ell' : -$ and $\ell : P$ are distinct.

We define a sub-effecting relation on effect contexts $E \leqslant E'$ if we can replace the absent signatures in E with proper signatures to obtain E'. We also have a subtyping relation on extensions $D \leqslant D'$. Different from sub-effecting, it requires D and D' to contain the same row of labels, but allows absent signatures in D to be replaced by other signatures in D'. We give the full rules for type equivalence and sub-effecting in Appendix A.3.

Masks L are simply multisets of labels without signatures; we do not need signatures when removing labels from effect contexts. We define three operations $D + E$, $E - L$, and $L \bowtie D$ as follows.

$$D + E = D, E \qquad \cdot - L = \cdot \qquad (\ell : P, E) - L = \begin{cases} E - L' & \text{if } L \equiv \ell, L' \\ \ell : P, (E - L) & \text{otherwise} \end{cases}$$

$$L \bowtie \cdot = (L, \cdot) \qquad L \bowtie (\ell : P, D) = \begin{cases} L' \bowtie D & \text{if } L \equiv \ell, L' \\ (L', (\ell : P, D')) & \text{otherwise} \\ \quad \text{where } (L', D') = L \bowtie D \end{cases}$$

The operation $D + E$ extends E with D. The operation $E - L$ removes the labels in L from E. The operation $L \bowtie D = (L', D')$ gives the difference between L and D. The L' are those labels in L not appearing in the domain of D, and the D' are those entries in D with labels not in L.

4.3 Modalities Manipulating Effect Contexts

Components of types and terms may have different effect contexts from the ambient one. We use *modalities* to manipulate effect contexts. For the modal type μA, the effect context for A is derived from the ambient effect context manipulated by the modality μ as follows.

$$[E](F) = E \qquad \langle L | D \rangle(F) = D + (F - L)$$

The absolute modality $[E]$ completely replaces the effect context F with E, similar to effect annotations on function types in traditional effect systems. The relative modality $\langle L | D \rangle$ is the key novelty of Met. It specifies a transformation on the input effect context. It masks the labels L in F before extending the resulting context with D. We call $\langle | \rangle$ the identity modality and write $\mathbb{1}$ for it. Modalities are monotone total functions on effect contexts. If $E \leqslant F$, we have $\mu(E) \leqslant \mu(F)$.

We write μ_F for the pair of μ and F where F is the effect context that μ acts on. We refer to such a pair as a concrete modality. We write $\mu_F : E \longrightarrow F$ if $\mu(F) = E$. The arrow goes from E to F instead of the other direction to be consistent with MTT. Note that our terminology here differs slightly from that of MTT introduced in Section 3. Concrete modalities μ_F correspond to the notion of modalities in MTT and our modalities μ are actually indexed families of modalities in MTT.

Modality Composition. We can compose the actions of modalities in the intuitive way.

$$\begin{array}{rcll} \mu & \circ & [E] & = & [E] \\ [E] & \circ & \langle L | D \rangle & = & [D + (E - L)] \\ \langle L_1 | D_1 \rangle & \circ & \langle L_2 | D_2 \rangle & = & \langle L_1 + L | D_2 + D \rangle \quad \text{where } (L, D) = L_2 \bowtie D_1 \end{array}$$

To keep close to MTT, our composition reads from left to right. First, an absolute modality completely specifies the new effect context, thus shadowing any other modality μ. Second, replacing the effect context with E and then masking L and extending with D is equivalent to just replacing with $D + (E - L)$. Third, sequential masking and extending can be combined into one by using $L_2 \bowtie D_1$ to cancel the overlapping part of L_2 and D_1. For instance, we have $\langle\!|\texttt{yield} : \texttt{Int} \twoheadrightarrow 1\rangle \circ \langle \texttt{yield} |\!\rangle = \mathbb{1}$.

Composition is well-defined since composing followed by applying is equivalent to sequentially applying $(\mu \circ \nu)(E) = \nu(\mu(E))$. We also have associativity $(\mu \circ \nu) \circ \xi = \mu \circ (\nu \circ \xi)$ and identity $\mathbb{1}$. The definition of composition naturally generalises to indexed modalities μ_F. We can compose $\mu_F : E \longrightarrow F$ and $\nu_E : E' \longrightarrow E$ to get $\mu_F \circ \nu_E : E' \longrightarrow F$ which is defined as $(\mu \circ \nu)_F$.

Modality Transformations. Just as modalities allow us to manipulate effect contexts, we need a transformation relation that tells us when we can change modalities.

In Met, there could only be at most one transformation between any two modalities. As a result, we do not need to give names to transformations. We write $\mu_F \Rightarrow \nu_F$ for a transformation between indexed modalities $\mu_F : E \longrightarrow F$ and $\nu_F : E' \longrightarrow F$. Intuitively, such a transformation indicates that under ambient effect context F, the action of μ can be replaced by the action of ν. This relation is used to control variable access as we have demonstrated in Section 3. For instance, supposing we have a variable of type $\mu(1 \longrightarrow 1)$ under ambient effect context F, we can rewrap it to a function of type $\nu(1 \longrightarrow 1)$ if $\mu_F \Rightarrow \nu_F$.

Intuitively, $\mu_F \Rightarrow \nu_F$ is safe when $\nu(F)$ is larger than $\mu(F)$ so that we have not lost any operations. Moreover, subeffecting should not break the safety guarantee of transformations. That is, $\mu(F') \leqslant \nu(F')$ should hold for any effect context F' with $F \leqslant F'$. We formally define $\mu_F \Rightarrow \nu_F$ by the transitive closure of the following four rules.

MT-Abs
$$\frac{\mu_F : E' \longrightarrow F \quad E \leqslant E'}{[E]_F \Rightarrow \mu_F}$$

MT-Upcast
$$\frac{D \leqslant D'}{\langle L | D \rangle_F \Rightarrow \langle L | D' \rangle_F}$$

MT-Expand
$$\frac{(F - L) \equiv \ell : A \twoheadrightarrow B, E}{\langle L | D \rangle_F \Rightarrow \langle \ell, L | D, \ell : A \twoheadrightarrow B \rangle_F}$$

MT-Shrink
$$\frac{(F - L) \equiv \ell : P, E}{\langle \ell, L | D, \ell : P \rangle_F \Rightarrow \langle L | D \rangle_F}$$

MT-Abs allows us to transform an absolute modality to any other modality as long as no effect leaks. MT-Upcast allow us to upcast a label with an absent signature in D to an arbitrary signature, since the corresponding operation is unused. Recall that the subtyping relation between extensions only upcasts signatures. MT-Expand allows us to simultaneously mask and extend some present operations given that these operations exist in the ambient effect context F. MT-Shrink allows us to do the reverse for any operations regardless of their presence.

The following lemma shows that the syntactic definition of transformation matches our intuition. The proof is in Appendix B.2.

Lemma 4.1 (Semantics of modality transformation). *We have $\mu_F \Rightarrow \nu_F$ if and only if $\mu(F') \leqslant \nu(F')$ for all F' with $F \leqslant F'$.*

Let us give some examples here. First, $[]_E \Rightarrow \mu_E$ always holds, consistent with the intuition that pure values can be used anywhere safely. Second, $\langle\!|\ell : -\rangle_E \Rightarrow \langle\!|\ell : P\rangle_E$ always holds. Third, we have $\langle \ell | \ell : P \rangle_{\ell:P,E} \Leftrightarrow \mathbb{1}_{\ell:P,E}$ in both directions. Last, $\mathbb{1}_E \Rightarrow \langle\!|\ell : P\rangle_E$ does not hold for any E.

4.4 Kinds and Contexts

$\boxed{\Gamma \vdash A : K}$ $\boxed{\Gamma \vdash P}$ $\boxed{\Gamma \vdash (\mu, A) \Rightarrow \nu \mathbin{@} F}$

$$\frac{}{\Gamma \vdash 1 : \mathsf{Abs}} \qquad \frac{\Gamma \vdash A : \mathsf{Abs}}{\Gamma \vdash A : \mathsf{Any}} \qquad \frac{\Gamma \vdash [E] \qquad \Gamma \vdash A : \mathsf{Any}}{\Gamma \vdash [E]A : \mathsf{Abs}} \qquad \frac{\Gamma \vdash \langle L | D \rangle \qquad \Gamma \vdash A : K}{\Gamma \vdash \langle L | D \rangle A : K}$$

$$\frac{\Gamma \vdash A : \mathsf{Any} \quad \Gamma \vdash B : \mathsf{Any}}{\Gamma \vdash A \to B : \mathsf{Any}} \qquad \frac{\Gamma \vdash A : \mathsf{Abs} \quad \Gamma \vdash B : \mathsf{Abs}}{\Gamma \vdash A \twoheadrightarrow B} \qquad \frac{\Gamma \vdash A : \mathsf{Abs}}{\Gamma \vdash (\mu, A) \Rightarrow \nu \mathbin{@} F} \qquad \frac{\mu_F \Rightarrow \nu_F}{\Gamma \vdash (\mu, A) \Rightarrow \nu \mathbin{@} F}$$

$\boxed{\Gamma \mathbin{@} E}$

$$\frac{}{\cdot \mathbin{@} E} \qquad \frac{\Gamma \mathbin{@} F \qquad \mu_F : E \to F \qquad \Gamma \vdash A : K}{\Gamma, x :_{\mu_F} A \mathbin{@} F} \qquad \frac{\Gamma \mathbin{@} F \qquad \mu_F : E \to F}{\Gamma, \mathbin{🔒}_{\mu_F} \mathbin{@} E}$$

Fig. 1. Representative kinding, well-formedness, and auxiliary rules for Mᴇᴛ.

As illustrated in Section 2.8, we have two kinds Abs and Any. The Abs kind is a sub-kind of the kind of all types Any, and denotes types of values that are guaranteed not to use operations from the ambient effect context. We show the kinding and well-formedness rules for types and signatures in Figure 1, relying on the well-formedness of modalities and effect contexts, which is standard and defined in Appendix A.3. Function arrows have kind Any due to the possibility of using operations from the ambient effect context. A modal type $[E]A$ is absolute as it cannot depend on the ambient effect context. We restrict the kind of the argument and return value of effects to be Abs in order to prevent effect leakage as discussed in Section 2.10.

Contexts are ordered. Each term variable binding $x :_{\mu_F} A$ in contexts is tagged with an concrete modality μ_F. We omit this annotation when μ is identity. Contexts contain locks $\mathbin{🔒}_{\mu_F}$ carrying concrete modalities μ_F. As shown in Section 3, they track the introduction and elimination of modalities and play an important role in controlling variable access. We omitted indexes of modalities on bindings and locks in Section 3 for brevity; they are obvious from the context.

We define the relation $\Gamma \mathbin{@} E$ that context Γ is well-formed at effect context E in Figure 1. For instance, given some modalities $\mu_F : E_1 \to F, \nu_F : E_2 \to F$, and $\xi_E : E_3 \to E$, the following context is well-formed at effect context E. Reading from left to right, the lock $\mathbin{🔒}_{[E]_F}$ switches the effect context from F to E as $[E](F) = E$.

$$x :_{\mu_F} A_1, y :_{\nu_F} A_2, \mathbin{🔒}_{[E]_F}, z :_{\xi_E} A_3 \mathbin{@} E$$

Following MTT, we define $\mathsf{locks}(-)$ to compose all the modalities on the locks in a context.

$$\mathsf{locks}(\cdot) = \mathbb{1} \qquad \mathsf{locks}(\Gamma, \mathbin{🔒}_{\mu_F}) = \mathsf{locks}(\Gamma) \circ \mu_F \qquad \mathsf{locks}(\Gamma, x :_{\mu_F} A) = \mathsf{locks}(\Gamma)$$

Following MTT, we identify contexts up to the following two equations.

$$\Gamma, \mathbin{🔒}_{\mathbb{1}_E} \mathbin{@} E = \Gamma \mathbin{@} E \qquad \Gamma, \mathbin{🔒}_{\mu_F}, \mathbin{🔒}_{\nu_{F'}} \mathbin{@} E = \Gamma, \mathbin{🔒}_{\mu_F \circ \nu_{F'}} \mathbin{@} E$$

4.5 Typing

The typing rules of Met are shown in Figure 2. The typing judgement $\Gamma \vdash M : A \mathbin{@} E$ means that the term M has type A under context Γ and effect context E. As usual, we require $\Gamma \mathbin{@} E$, $\Gamma \vdash E$, $\Gamma \vdash A : K$ for some K, and well-formedness for type annotations as well-formedness conditions. We explain the interesting rules, which are highlighted in grey; the other rules are standard.

$\boxed{\Gamma \vdash M : A \mathbin{@} E}$

T-Var
$$\frac{\nu_F = \mathsf{locks}(\Gamma') : E \longrightarrow F}{\Gamma, x :_{\mu_F} A, \Gamma' \vdash x : A \mathbin{@} E}$$

T-Mod
$$\frac{\mu_F : E \longrightarrow F \qquad \Gamma, \text{🔒}_{\mu_F} \vdash V : A \mathbin{@} E}{\Gamma \vdash \mathbf{mod}_\mu\, V : \mu A \mathbin{@} F}$$

T-Letmod
$$\frac{\nu_F : E \longrightarrow F \qquad \Gamma, \text{🔒}_{\nu_F} \vdash V : \mu A \mathbin{@} E \qquad \Gamma, x :_{\nu_F \circ \mu_E} A \vdash M : B \mathbin{@} F}{\Gamma \vdash \mathbf{let}_\nu\, \mathbf{mod}_\mu\, x = V\, \mathbf{in}\, M : B \mathbin{@} F}$$

T-Abs
$$\frac{\Gamma, x : A \vdash M : B \mathbin{@} E}{\Gamma \vdash \lambda x^A.M : A \longrightarrow B \mathbin{@} E}$$

T-App
$$\frac{\Gamma \vdash M : A \longrightarrow B \mathbin{@} E \qquad \Gamma \vdash N : A \mathbin{@} E}{\Gamma \vdash M\, N : B \mathbin{@} E}$$

T-Do
$$\frac{E = \ell : A \twoheadrightarrow B, F \qquad \Gamma \vdash N : A \mathbin{@} E}{\Gamma \vdash \mathbf{do}\, \ell\, N : B \mathbin{@} E}$$

T-Mask
$$\frac{\Gamma, \text{🔒}_{\langle L \rangle_F} \vdash M : A \mathbin{@} F - L}{\Gamma \vdash \mathbf{mask}_L\, M : \langle L \rangle A \mathbin{@} F}$$

T-Handler
$$\frac{\begin{array}{c} H = \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i \\ \Gamma, \text{🔒}_{\langle\!| D \rangle_F} \vdash M : A \mathbin{@} D + F \qquad \Gamma, x : \langle\!| D \rangle A \vdash N : B \mathbin{@} F \\ D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad [\Gamma, p_i : A_i, r_i : B_i \longrightarrow B \vdash N_i : B \mathbin{@} F]_i \end{array}}{\Gamma \vdash \mathbf{handle}\ M\ \mathbf{with}\ H : B \mathbin{@} F}$$

Fig. 2. Typing rules for Met.

Modality Introduction and Elimination. Modalities are introduced by T-Mod and eliminated by T-Letmod. The term $\mathbf{mod}_\mu\, V$ introduces modality μ to the type of the conclusion and lock 🔒_{μ_F} into the context of the premise, and requires the value V to be well-typed under the new effect context E manipulated by μ. The lock 🔒_{μ_F} tracks the change to the effect context. Specialising the modality μ to either absolute or relative modalities, we get the following two rules.

$$\frac{\Gamma, \text{🔒}_{[E]_F} \vdash V : A \mathbin{@} E}{\Gamma \vdash \mathbf{mod}_{[E]}\, V : [E]A \mathbin{@} F} \qquad \frac{\Gamma, \text{🔒}_{\langle L | D \rangle_F} \vdash V : A \mathbin{@} D + (F - L)}{\Gamma \vdash \mathbf{mod}_{\langle L | D \rangle}\, V : \langle L | D \rangle A \mathbin{@} F}$$

Note that we use modality μ instead of concrete modality μ_F in types and terms, because the index can always be inferred from the effect context. We restrict **mod** to values to avoid effect leakage [2, 35]. Otherwise, a term such as $\mathbf{mod}_{\langle\!| \ell : P \rangle}\ (\mathbf{do}\ \ell\, V)$ would type check under the empty effect context but get stuck due to the unhandled operation ℓ.

The term $\mathbf{let}_\nu\ \mathbf{mod}_\mu\ x = V\ \mathbf{in}\ M$ moves the modality μ from the type of V to the binding of x. As with boxing, unboxing is restricted to values. Following MTT, we use let-style modality elimination which takes another modality ν in addition to the modality μ that is eliminated from V. This is crucial for sequential unboxing. For instance, the following term sequentially unboxes $x : \nu\mu A$. The variables y and z are bound as $y :_\nu \mu A$ and $z :_{\nu \circ \mu} A$, respectively.

$$\mathbf{let}\ \mathbf{mod}_\nu\ y = x\ \mathbf{in}\ \mathbf{let}_\nu\ \mathbf{mod}_\mu\ z = y\ \mathbf{in}\ M$$

Masking and Handling. Masking and handling also introduce relative modalities. Unlike **mod**, these constructs can apply to computations as they perform masking and handling semantically. In T-Mask, the mask $\mathbf{mask}_L\ M$ removes effects L from the ambient effect context for M. For the return value of M, we need to box it with $\langle L \rangle$ to reconcile the mismatch between $F - L$ and F. In T-Handler, the handler $\mathbf{handle}\ M\ \mathbf{with}\ H$ extends the ambient effect context with effects D for M. For the return value of M which is bound as x in the return clause, we need to box it with $\langle D \rangle$ to reconcile the mismatch between $D + F$ and F. The other parts of the handler rule are standard.

Accessing Variables. The T-Var rule uses the auxiliary judgement $\Gamma \vdash (\mu, A) \Rightarrow v \mathbin{@} F$ defined in Figure 1. Variables of absolute types can always be used as they do not depend on the effect context. For a non-absolute term variable binding $x :_{\mu_F} A$ from context $\Gamma, x :_{\mu_F} A, \Gamma'$, we must guarantee that it is safe to use x in the current effect context. The term bound to x is defined inside μ under the effect context F. As we track all transformations on effect contexts up to the binding of x as locks in Γ', the current effect context E is obtained by applying $\mathrm{locks}(\Gamma')$ to F. Thus, we need the transformation $\mu_F \Rightarrow \mathrm{locks}(\Gamma')_F$ to hold for effect safety.

Subeffecting. Subeffecting is incorporated into the T-Var rule within the transformation relation $\mu_F \Rightarrow \nu_F$. We have seen how subeffecting works in Section 2.4. We give another example here which upcasts the empty effect context to E. It is well-typed because $[]. \Rightarrow [E].$ holds.

$$\lambda x^{[](\texttt{Int}\to\texttt{Int})}.\mathbf{let}\ \mathbf{mod}_{[]}\ y = x\ \mathbf{in}\ \mathbf{mod}_{[E]}\ y : []\texttt{(Int} \to \texttt{Int)} \to [E]\texttt{(Int} \to \texttt{Int)}$$

4.6 Operational Semantics

The operational semantics for Met is quite standard [24]. We first define evaluation contexts $\mathcal{E}$:

$$\text{Evaluation contexts} \quad \mathcal{E} ::= [\,] \mid \mathcal{E}\ N \mid V\ \mathcal{E} \mid \mathbf{do}\ \ell\ \mathcal{E} \mid \mathbf{mask}_L\ \mathcal{E} \mid \mathbf{handle}\ \mathcal{E}\ \mathbf{with}\ H$$

The reduction rules are as follows.

$$\begin{array}{lr}
\textsc{E-App} & (\lambda x^A.M)\ V \rightsquigarrow M[V/x] \\
\textsc{E-Letmod} & \mathbf{let}_\nu\ \mathbf{mod}_\mu\ x = \mathbf{mod}_\mu\ V\ \mathbf{in}\ M \rightsquigarrow M[V/x] \\
\textsc{E-Mask} & \mathbf{mask}_L\ V \rightsquigarrow \mathbf{mod}_{\langle L \rangle}\ V \\
\textsc{E-Ret} & \mathbf{handle}\ V\ \mathbf{with}\ H \rightsquigarrow N[(\mathbf{mod}_{\langle D \rangle}\ V)/x], \text{where } (\mathbf{return}\ x \mapsto N) \in H \\
\textsc{E-Op} & \mathbf{handle}\ \mathcal{E}[\mathbf{do}\ \ell\ V]\ \mathbf{with}\ H \rightsquigarrow N[V/p, (\lambda y.\mathbf{handle}\ \mathcal{E}[y]\ \mathbf{with}\ H)/r], \\
 & \qquad \text{where } 0\text{–free}(\ell, \mathcal{E}) \text{ and } (\ell\ p\ r \mapsto N) \in H \\
\textsc{E-Lift} & \mathcal{E}[M] \rightsquigarrow \mathcal{E}[N], \qquad \text{if } M \rightsquigarrow N
\end{array}$$

The only slightly non-standard aspect of the rules is the boxing of values escaping masks and handlers in E-Mask and E-Ret. They coincide with the typing rules for masks and handlers. In E-Ret, we assume handlers are decorated with the operations D that they handle as in Section 2.

Following Biernacki et al. [5], the predicate $n\text{–free}(\ell, \mathcal{E})$ is defined inductively on evaluation contexts as follows. We have $n\text{–free}(\ell, \mathcal{E})$ if there are n unmasked handlers for ℓ in $\mathcal{E}$. The meta function $\mathrm{count}(\ell; L)$ yields the number of ℓ labels in L. We omit the inductive cases that do not change n. Notice that the cases for introduction and elimination of modalities fall into this category as they require values which cannot be of the form $\mathbf{do}\ \ell\ V$.

$$\frac{}{0\text{–free}(\ell, [\,])} \qquad \frac{n\text{–free}(\ell, \mathcal{E})}{n\text{–free}(\ell, \mathbf{do}\ \ell'\ \mathcal{E})} \qquad \frac{n\text{–free}(\ell, \mathcal{E}) \qquad \mathrm{count}(l; L) = m}{(n+m)\text{–free}(\ell, \mathbf{mask}_L\ \mathcal{E})}$$

$$\frac{(n+1)\text{–free}(\ell, \mathcal{E}) \qquad \ell \in \mathrm{dom}(H)}{n\text{–free}(\ell, \mathbf{handle}\ \mathcal{E}\ \mathbf{with}\ H)} \qquad \frac{n\text{–free}(\ell, \mathcal{E}) \qquad \ell \notin \mathrm{dom}(H)}{n\text{–free}(\ell, \mathbf{handle}\ \mathcal{E}\ \mathbf{with}\ H)}$$

4.7 Type Soundness and Effect Safety

We prove type soundness and effect safety for Met. Our proofs cover the extensions in Section 6.

Met enjoys substitution properties along the lines of Kavvos and Gratzer [30]. For example, we have the following rule for substituting values with modalities into terms.

$$\frac{\Gamma, \text{🔒}_{\mu_F} \vdash V : A \mathbin{@} F' \qquad \Gamma, x :_{\mu_F} A, \Gamma' \vdash M : B \mathbin{@} E}{\Gamma, \Gamma' \vdash M[V/x] : B \mathbin{@} E}$$

We state and prove the relevant properties in Appendix B.3.

To state syntactic type soundness, we first define normal forms.

Definition 4.2 (Normal Forms). We say a term M is in a normal form with respect to effect type E, if it is either in value normal form $M = U$ or of form $M = \mathcal{E}[\mathbf{do}\ \ell\ U]$ for $\ell \in E$ and $n\text{–free}(\ell, \mathcal{E})$.

The following together give type soundness and effect safety (proofs in Appendices B.4 and B.5).

Theorem 4.3 (Progress). *If $\vdash M : A \mathbin{@} E$, then either there exists N such that $M \rightsquigarrow N$ or M is in a normal form with respect to E.*

Theorem 4.4 (Subject Reduction). *If $\Gamma \vdash M : A \mathbin{@} E$ and $M \rightsquigarrow N$, then $\Gamma \vdash N : A \mathbin{@} E$.*

5 Encoding Effect Polymorphism in Met

Even without effect variables, Met is sufficiently expressive to encode programs from conventional row-based effect systems provided effect variables on function arrows always refer to the lexically closest one. This is an important special case, since most functions in practice use at most one effect variable. For example, as of July 2024, the Koka repository contains 520 effectful functions across 112 files but only 86 functions across 5 files use more than one effect variable, almost all of them internal primitives for handlers not exposed to programmers. Moreover, almost all programs in the Frank repository make no mention of effect variables, relying on syntactic sugar to hide the single effect variable. We formally characterise and prove this intuition on the expressiveness of Met.

5.1 Row Effect Types with a Single Effect Variable

We first define $\mathrm{F}^1_{\mathrm{eff}}$, a core calculus with row-based effect types in the style of Koka [32], but where each scope can only refer to the lexically closest effect variable.

$$\begin{array}{lll}
\text{Types} & A, B ::= 1 \mid A \to^{\{E \mid \varepsilon\}} B \mid \forall \varepsilon. A & \text{Terms} \quad M, N ::= () \mid x \mid \lambda^{\{E \mid \varepsilon\}} x^A . M \mid M\, N \\
\text{Effects} & L, D, E, F ::= \cdot \mid \ell, E & \mid\ \Lambda \varepsilon. V \mid M \spadesuit \{E \mid \varepsilon\} \mid \mathbf{do}\ \ell\ M \\
\text{Contexts} & \Gamma ::= \cdot \mid \Gamma, x :_\varepsilon A \mid \Gamma, \blacklozenge_E \mid \Gamma, \blacklozenge^\Lambda_E & \mid\ \mathbf{mask}_L\ M \mid \mathbf{handle}\ M\ \mathbf{with}\ H \\
\text{Values} & V, W ::= x \mid \lambda^{\{E \mid \varepsilon\}} x^A . M \mid \Lambda \varepsilon. V & \text{Handlers} \quad H ::= \{\mathbf{return}\ x \mapsto M\} \mid \{\ell\ p\ r \mapsto M\} \uplus H
\end{array}$$

In types we include units, effectful functions, and effect abstraction $\forall \varepsilon. A$. As we consider only one effect variable at a time, we need not track effect variables on function types and effect abstraction. Nonetheless, we include them in grey font for easier comparison with existing calculi. In Γ, each term variable is annotated with the effect variable ε that was referred to at the time of its introduction. Further, we add markers $\blacklozenge_E$ and $\blacklozenge^\Lambda_E$ to the context, which track the change of effects due to functions, masks, handlers, and effect abstraction. These markers are not needed by the typing rules but help with the encoding. As with Met, we require contexts to be ordered. For simplicity we assume operation signatures come from a global context $\Sigma = \{\overline{\ell : A \twoheadrightarrow B}\}$, thus unifying extensions D, masks L, and effects (effect contexts) E into one syntactic category. Mirroring our kind restriction for operation signatures in Met, we assume that these A and B are not function arrows, but they can be effect abstractions (which may themselves contain function arrows).

$\boxed{\Gamma \vdash M : A \,!\, \{E|\varepsilon\}}$

R-Var
$$\frac{\varepsilon = \varepsilon' \text{ or } A = \forall \varepsilon''.A' \text{ or } A = 1}{\Gamma_1, x :_{\varepsilon'} A, \Gamma_2 \vdash x : A \,!\, \{E|\varepsilon\}}$$

R-Abs
$$\frac{\Gamma, \blacklozenge_E, x :_\varepsilon A \vdash M : B \,!\, \{F|\varepsilon\}}{\Gamma \vdash \lambda^{\{F|\varepsilon\}} x^A.M : A \to^{\{F|\varepsilon\}} B \,!\, \{E|\varepsilon\}}$$

R-App
$$\frac{\Gamma \vdash M : A \to^{\{E|\varepsilon\}} B \,!\, \{E|\varepsilon\} \qquad \Gamma \vdash N : A \,!\, \{E|\varepsilon\}}{\Gamma \vdash M\,N : B \,!\, \{E|\varepsilon\}}$$

R-EAbs
$$\frac{\varepsilon' \notin \mathrm{ftv}(\Gamma) \qquad \Gamma, \blacklozenge^{\Lambda}_E \vdash V : A \,!\, \{\cdot|\varepsilon'\}}{\Gamma \vdash \Lambda\varepsilon'.V : \forall \varepsilon'.A \,!\, \{E|\varepsilon\}}$$

R-EApp
$$\frac{\Gamma \vdash M : \forall \varepsilon'.A \,!\, \{E|\varepsilon\}}{\Gamma \vdash M \spadesuit \{E|\varepsilon\} : A[\{E|\varepsilon\}/] \,!\, \{E|\varepsilon\}}$$

R-Mask
$$\frac{\Gamma, \blacklozenge_{L+E} \vdash M : A \,!\, \{E|\varepsilon\}}{\Gamma \vdash \mathbf{mask}_L\ M : A \,!\, \{L + E|\varepsilon\}}$$

R-Do
$$\frac{(\ell : A \twoheadrightarrow B) \in \Sigma \qquad \Gamma \vdash M : A \,!\, \{\ell, E|\varepsilon\}}{\Gamma \vdash \mathbf{do}\ \ell\ M : B \,!\, \{\ell, E|\varepsilon\}}$$

R-Handler
$$\frac{\Gamma, \blacklozenge_E \vdash M : A \,!\, \{\overline{\ell_i}, E|\varepsilon\} \qquad \Gamma, x :_\varepsilon A \vdash N : B \,!\, \{E|\varepsilon\} \qquad \{\ell_i : A_i \twoheadrightarrow B_i\} \subseteq \Sigma \qquad [\Gamma, p_i :_\varepsilon A_i, r_i :_\varepsilon B_i \to^E B \vdash N_i : B \,!\, \{E|\varepsilon\}]_i}{\Gamma \vdash \mathbf{handle}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i : B \,!\, \{E|\varepsilon\}}$$

Fig. 3. Typing rules of $\mathrm{F}^1_{\mathrm{eff}}$.

Figure 3 gives the typing rules of $\mathrm{F}^1_{\mathrm{eff}}$. The judgement $\Gamma \vdash M : A \,!\, \{E|\varepsilon\}$ states that in context Γ, the term M has type A and might use concrete effects E extended with effect variable ε. The typing rules are mostly standard for row-based effect systems. The R-Var rule ensures that either the current effect variable matches the effect variable at which the variable was introduced or that the value is an effect abstraction or unit. These constraints guarantee that programs can only refer to one effect variable in one scope. The R-App, R-Do, R-Mask, and R-Handler rules are standard. The R-Abs rule is standard except for requiring the effect variable to remain unchanged. The R-EAbs rule introduces a new effect variable ε' and the R-EApp rule instantiates an effect abstraction. While conventional systems allow instantiating with any effect row, R-EApp only allows instantiation with the ambient effects E and effect variable ε. The instantiation operator $[\{E|\varepsilon\}/]$ implements standard type substitution for the single effect variable as follows.

$$\begin{aligned} 1[\{E|\varepsilon\}/] &= 1 \qquad & (\forall \varepsilon'.A)[\{E|\varepsilon\}/] &= \forall \varepsilon'.A \\ (A \to^{\{F|\varepsilon'\}} B)[\{E|\varepsilon\}/] &= A[\{E|\varepsilon\}/] \to^{\{F,E|\varepsilon\}} B[\{E|\varepsilon\}/] \end{aligned}$$

For example, the following $\mathrm{F}^1_{\mathrm{eff}}$ function (grey parts omitted) sums up all yielded integers.

$$\begin{aligned} &\mathtt{asSum} : \forall.(1 \to^{\mathtt{Gen\ Int}} 1) \to \mathtt{Int} \\ &\mathtt{asSum} = \Lambda.\lambda f.\mathbf{handle}\ f\ ()\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto 0, \mathtt{yield}\ x\ r \mapsto x + r\ ()\} \end{aligned}$$

5.2 Encoding

We now give compositional translations for types and contexts of $\mathrm{F}^1_{\mathrm{eff}}$ into Met. We transform $\mathrm{F}^1_{\mathrm{eff}}$ types at effect context E to modal types in Met by the translation $[\![-]\!]_E$.

$$\begin{aligned} [\![1]\!]_E &= \mathbb{1}1 \\ [\![A \to^F B]\!]_E &= \langle E - F | F - E \rangle([\![A]\!]_F \to [\![B]\!]_F) \\ [\![\forall.A]\!]_E &= []\,[\![A]\!]. \end{aligned} \qquad \begin{aligned} [\![\cdot]\!]_E &= \cdot \\ [\![\Gamma, x : A]\!]_E &= [\![\Gamma]\!]_E, x :_{\mu_E} A' \text{ for } \mu A' = [\![A]\!]_E \\ [\![\Gamma, \blacklozenge_F]\!]_E &= [\![\Gamma]\!]_F, \text{🔒}_{\langle F-E|E-F\rangle} \\ [\![\Gamma, \blacklozenge^{\Lambda}_F]\!]. &= [\![\Gamma]\!]_F, \text{🔒}_{[]} \end{aligned}$$

For the unit type, we insert the identity modality for the uniformity of the translation. For a function arrow $A \to^F B$, we use a relative modality $\langle E - F | F - E \rangle$ to wrap the function arrow. This modality transforms the outside effect context E to F which is specified by the original function arrow $A \to^F B$. For effect abstraction, we use an empty absolute modality to wrap the type, simulating entering a new scope with different effect variables. We translate contexts by translating each type and moving top-level modalities to their bindings. For each marker, we insert a corresponding lock to reflect the changes of effect context.

As an example of type translation, the type of `asSum` in $\mathrm{F}^1_{\mathrm{eff}}$ from Section 5.1 is translated to

$$\texttt{asSum} \;:\; [\,](\langle\!|\texttt{Gen Int}\rangle(\mathbb{1}1 \to \mathbb{1}1) \to \mathbb{1}\texttt{Int}).$$

Observe that not every valid typing judgement in $\mathrm{F}^1_{\mathrm{eff}}$ can be transformed to a valid typing judgement in Met, because the translation depends on markers in contexts, while the typing of $\mathrm{F}^1_{\mathrm{eff}}$ does not. We define well-scoped typing judgements, which characterise the typing judgements for which our encoding is well-defined, as follows.

Definition 5.1 (Well-scoped). A typing judgement $\Gamma_1, x :_\varepsilon A, \Gamma_2 \vdash M : B \,!\, E$ is *well-scoped for* x if either $x \notin \mathrm{fv}(M)$ or $\blacklozenge^{\Lambda}_{F} \notin \Gamma_2$ or $A = \forall .A'$. A typing judgement $\Gamma \vdash M : A \,!\, E$ is *well-scoped* if it is well-scoped for all $x \in \Gamma$.

In particular, if the judgement at the leaf of a derivation tree is well-scoped, then every judgement in the derivation tree is well-scoped. Also note that contexts with no markers are always well-scoped. Consequently, restricting judgements to well-scoped ones in $\mathrm{F}^1_{\mathrm{eff}}$ does not reduce expressive power, as we can always ensure that contexts at the leaves have no markers.

We write $M : A \,!\, E \dashrightarrow M'$ for the translation of a $\mathrm{F}^1_{\mathrm{eff}}$ term M of type A with effect context E to a Met term M'. We have the following type preservation theorem. The translation of terms and proof of type preservation can be found in Appendix C.

Lemma 5.2 (Type preservation of encoding). *If* $\Gamma \vdash M : A \,!\, \{E|\varepsilon\}$ *is well-scoped, then* $M : A \,!\, E \dashrightarrow M'$ *and* $[\![\Gamma]\!]_E \vdash M' : [\![A]\!]_E \;@\; E$.

Intuitively, the term translation inserts boxing and unboxing constructs in appropriate places in order to realise transition between effect contexts as embodied by the type translation. Specifically, for variables we unbox them immediately after they are bound, and re-box them when they are used. As an example of term translation, we can translate the `asSum` handler in $\mathrm{F}^1_{\mathrm{eff}}$ defined in Section 5.1 as follows into Met, omitting boxing and unboxing of the identity modality $\mathbb{1}$.

$$\begin{aligned}\texttt{asSum} = \mathbf{mod}_{[\,]}(\lambda f.&\mathbf{let\ mod}_{\langle\!|\texttt{Gen Int}\rangle}\ f = f\ \mathbf{in\ handle}\ f\ ()\ \mathbf{with}\\ &\{\mathbf{return}\ x \mapsto \mathbf{let\ mod}_{\langle\!|\texttt{Gen Int}\rangle}\ x = x\ \mathbf{in}\ 0, \texttt{yield}\ x\ r \mapsto x + r\ ()\})\end{aligned}$$

The explicit boxing and unboxing, as well as the identity modalities, here are only generated to keep the encoding systematic. We need not write them in practice as illustrated in Section 2.

Our encoding focuses on presenting the core idea and does not consider advanced features including data types and polymorphism. In Section 6 we show how to extend Met with these features and in Appendix C.3 we briefly outline how to extend the encoding to cover them.

6 Extensions

In this section we demonstrate that Met scales to support data types and polymorphism including both value and effect polymorphism. Effect polymorphism helps deal with situations in which it is useful to refer to one or more effect contexts that differ from the ambient one (such as the higher-order fork operation in Section 2.10), recovering the full expressive power of row-based effect systems. We only discuss the key ideas of extensions here; their full specification as well

as more extensions including shallow handlers [23, 27] are given in Appendix A. We prove type soundness and effect safety for all extensions in Appendix B.

6.1 Making Data Types Crisp

We demonstrate the extensibility of Met with data types by extending it with pair and sum types. We expect no extra challenge to extend Met with algebraic data types. The syntax and typing rules are shown as follows.

$$\textsc{T-Pair}\ \frac{\Gamma \vdash M : A \ @\ E \qquad \Gamma \vdash N : B \ @\ E}{\Gamma \vdash (M, N) : A * B \ @\ E}$$

$$\textsc{T-Inl}\ \frac{\Gamma \vdash M : A \ @\ E}{\Gamma \vdash \mathbf{inl}\ M : A + B \ @\ E}$$

$$\textsc{T-Inr}\ \frac{\Gamma \vdash M : B \ @\ E}{\Gamma \vdash \mathbf{inr}\ M : A + B \ @\ E}$$

$$\textsc{T-CrispPair}\ \frac{\nu_F : E \longrightarrow F \qquad \Gamma, \text{🔒}_{\nu_F} \vdash V : A * B \ @\ E \qquad \Gamma, x :_{\nu_F} A, y :_{\nu_F} B \vdash M : A' \ @\ F}{\Gamma \vdash \mathbf{case}_{\nu}\ V\ \mathbf{of}\ (x, y) \mapsto M : A' \ @\ F}$$

$$\textsc{T-CrispSum}\ \frac{\nu_F : E \longrightarrow F \qquad \Gamma, \text{🔒}_{\nu_F} \vdash V : A + B \ @\ E \qquad \Gamma, x :_{\nu_F} A \vdash M_1 : A' \ @\ F \qquad \Gamma, y :_{\nu_F} B \vdash M_2 : A' \ @\ F}{\Gamma \vdash \mathbf{case}_{\nu}\ V\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1, \mathbf{inr}\ y \mapsto M_2\} : A' \ @\ F}$$

The T-Pair, T-Inl, and T-Inr are standard introduction rules. The elimination rules T-CrispPair and T-CrispSum are more interesting. In addition to normal pattern matching, they interpret the value V under the effect context transformed by certain modalities ν, which can then be tagged to the variable bindings in case clauses. They follow the crisp induction principles of multimodal type theory [20, 21, 48]. These crisp elimination rules provide extra expressiveness. For example, we can write the following function which transforms a sum of type $\mu(A + B)$ to another sum of type $(\mu A + \mu B)$. This function is not expressible without crisp elimination rules.

$$\lambda x^{\mu(A+B)}.\mathbf{let}\ \mathbf{mod}_{\mu}\ y = x\ \mathbf{in}\ \mathbf{case}_{\mu}\ y\ \mathbf{of}\ \{\mathbf{inl}\ x_1 \mapsto \mathbf{inl}\ (\mathbf{mod}_{\mu}\ x_1), \mathbf{inr}\ x_2 \mapsto \mathbf{inr}\ (\mathbf{mod}_{\mu}\ x_2)\}$$

6.2 Polymorphism for Values and Effects

The extensions to syntax and typing rules with polymorphism are as follows.

$$\begin{array}{lll} \text{Types} & A, B ::= \cdots \mid \forall \alpha^K.A & \qquad \text{Contexts} \quad \Gamma ::= \cdots \mid \Gamma, \alpha : K \\ \text{Effects} & E ::= \cdots \mid \varepsilon \mid E \backslash L & \qquad \text{Terms} \quad M, N ::= \cdots \mid \Lambda \alpha^K.V \mid M\,A \\ \text{Kinds} & K ::= \cdots \mid \text{Effect} & \qquad \text{Values} \quad V, W ::= \cdots \mid \Lambda \alpha^K.V \mid V\,A \end{array}$$

$$\textsc{T-TAbs}\ \frac{\Gamma, \alpha : K \vdash V : A \ @\ E}{\Gamma \vdash \Lambda \alpha^K.V : \forall \alpha^K.A \ @\ E}$$

$$\textsc{T-TApp}\ \frac{\Gamma \vdash M : \forall \alpha^K.B \ @\ E \qquad \Gamma \vdash A : K}{\Gamma \vdash M\,A : B[A/\alpha] \ @\ E}$$

It may appear surprising that we treat type application $V\,A$ as values. This is useful in practice to allow instantiation inside boxes. We also extend the semantics to allow reduction in values.

To support effect polymorphism, we extend the syntax of effect contexts E with effect variables ε and introduce a new kind Effect for them. As is typical for row polymorphism, we restrict each effect type to contain at most one effect variable. We also extend the syntax with effect masking $E\backslash L$, which means the effect types given by masking L from E. The latter is needed to keep the syntax of effect contexts closed under the masking operation $E - L$; otherwise we cannot define $\varepsilon - L$. In other words, the syntax of effects is the free algebra generated from extending D, E and masking $E\backslash L$ with base elements $\cdot$ and ε.

The effect equivalence and subeffecting rules are extended in a relatively standard way.

$$\frac{}{E\backslash\cdot \equiv E} \qquad \frac{}{\cdot\backslash L \equiv \cdot} \qquad \frac{}{(\ell : P, E)\backslash(\ell, L) \equiv E\backslash L} \qquad \frac{\ell \notin L}{(\ell : P, E)\backslash L \equiv \ell : P, E\backslash L}$$

$$\frac{}{(\varepsilon\backslash L)\backslash L' \equiv \varepsilon\backslash(L, L')} \qquad \frac{}{\varepsilon\backslash L \equiv \varepsilon\backslash L} \qquad \frac{}{\cdot \leqslant \varepsilon\backslash L} \qquad \frac{}{\varepsilon\backslash L \leqslant \varepsilon\backslash L}$$

We do not allow non-trivial equivalence or subtyping between different effect variables. We always identify effects up to the equivalence relation. That is, we can directly treat syntax of effects as the free algebra quotiented by the equivalence relation $E \equiv F$. Observe that using the equivalence relation, all open effect types with effect variable ε can be simplified to an equivalent normal form $D, \varepsilon\backslash L$. We assume the operation $E - L$ is defined for effects E in normal form and extend it with one case for effect variables as $\varepsilon\backslash L - L' = \varepsilon\backslash(L, L')$.

7 Simple Bidirectional Type Checking and Elaboration

In this section we outline the design of MetL, a basic surface language on top of Met, which uses a simple bidirectional typing strategy to infer all boxing and unboxing [16, 43].

The bidirectional typing rules for simply-typed λ-calculus and modalities of MetL and its type-directed elaboration to Met are shown in Figure 4. We show the full typing and elaboration rules and discuss our implementation in Appendix D.

$\boxed{\Gamma \vdash M \Rightarrow A \mathbin{@} E \dashrightarrow N}$ $\boxed{\Gamma \vdash M \Leftarrow A \mathbin{@} E \dashrightarrow N}$

B-Var
$$\frac{\nu_F = \mathsf{locks}(\Gamma') : E \to F \qquad \overline{\zeta} G = \mathsf{across}(\Gamma; A; \nu; F)}{\Gamma, x : \overline{\mu} G, \Gamma' \vdash x \Rightarrow \overline{\zeta} G \mathbin{@} E \dashrightarrow \mathbf{mod}_{\overline{\zeta}}\, x}$$

B-Mod
$$\frac{\mu_F : E \to F \qquad \Gamma, \text{🔒}_{\mu_F} \vdash V \Leftarrow A \mathbin{@} E \dashrightarrow V'}{\Gamma \vdash V \Leftarrow \mu A \mathbin{@} F \dashrightarrow \mathbf{mod}_{\mu}\, V'}$$

B-Abs
$$\frac{\Gamma, x : \overline{\mu} G \vdash M \Leftarrow B \mathbin{@} E \dashrightarrow M'}{\Gamma \vdash \lambda x.M \Leftarrow \overline{\mu} G \to B \mathbin{@} E \dashrightarrow \lambda x.\mathbf{let}\ \mathbf{mod}_{\overline{\mu}}\, \hat{x} = x\ \mathbf{in}\ M'}$$

B-App
$$\frac{\Gamma \vdash M \Rightarrow \overline{\mu}(A \to B) \mathbin{@} E \dashrightarrow M' \qquad \overline{\mu}_E \Rightarrow \mathbb{1}_E \qquad \Gamma \vdash N \Leftarrow A \mathbin{@} E \dashrightarrow N'}{\Gamma \vdash M\ N \Rightarrow B \mathbin{@} E \dashrightarrow (\mathbf{let}\ \mathbf{mod}_{\overline{\mu}}\, x = M'\ \mathbf{in}\ x)\ N'}$$

B-Annotation
$$\frac{\Gamma \vdash V \Leftarrow A \mathbin{@} E \dashrightarrow V'}{\Gamma \vdash V : A \Rightarrow A \mathbin{@} E \dashrightarrow V'}$$

B-Switch
$$\frac{\Gamma \vdash M \Rightarrow \overline{\mu} G \mathbin{@} E \dashrightarrow M' \qquad \Gamma \vdash (\overline{\mu}, G) \Rightarrow \overline{\nu} \mathbin{@} E}{\Gamma \vdash M \Leftarrow \overline{\nu} G \mathbin{@} E \dashrightarrow \mathbf{let}\ \mathbf{mod}_{\overline{\mu}}\, x = M'\ \mathbf{in}\ \mathbf{mod}_{\overline{\nu}}\, x}$$

Fig. 4. Representative bidirectional typing and elaboration rules for MetL.

As with usual bidirectional typing, we have inference mode $\Gamma \vdash M \Rightarrow A \mathbin{@} E$ and checking mode $\Gamma \vdash M \Leftarrow A \mathbin{@} E$. They both additionally take the effect context E as an input. The highlighted part $\dashrightarrow N$ gives the elaborated term N in Met.

We write G for guarded types which do not have top-level modalities, and $\overline{\mu}$ for a sequence of modalities which can be empty. As a result, every type is in form $\overline{\mu} G$. We use the following syntactic sugar for boxing and unboxing modality sequences to simplify the elaboration.

$$\begin{aligned} \mathbf{mod}_{\overline{\mu}}\, V &\doteq \overline{\mathbf{mod}_{\mu}\, V} \\ \mathbf{let}\ \mathbf{mod}_{\overline{\mu}}\, x = M\ \mathbf{in}\ N &\doteq (\lambda x.\overline{\mathbf{let}\ \mathbf{mod}_{\mu}\, x = x\ \mathbf{in}}\ N)\ M \end{aligned}$$

Our bidirectional typing rules are mostly simple and standard. The most novel part is the usage of an auxiliary function across in B-Var defined as follows.

$$\text{across}(\Gamma, A, \nu, F) \quad = \quad \begin{cases} A, & \text{if } \Gamma \vdash A : \text{Abs} \\ \zeta G, & \text{otherwise, where } A = \overline{\mu} G \text{ and } \nu_F \backslash \overline{\mu}_F = \zeta_E \end{cases}$$

When A is absolute, we can always access the variable. Otherwise, in order to know how far we should unbox the modalities $\overline{\mu}$ of the variable, we define a right residual operation $\nu_F \backslash \mu_F$ for the modality transformation relation. Given $\mu_F : E \longrightarrow F$ and $\nu_F : F' \longrightarrow F$, the partial operation $\nu_F \backslash \mu_F$ fails if there does not exist $\zeta_{F'}$ such that $\mu_F \Rightarrow \nu_F \circ \zeta_{F'}$. Otherwise, it gives an indexed modality such that $\mu_F \Rightarrow \nu_F \circ (\nu_F \backslash \mu_F)$ and for any $\zeta_{F'}$ with $\mu_F \Rightarrow \nu_F \circ \zeta_{F'}$, we have $\nu_F \backslash \mu_F \Rightarrow \zeta_{F'}$. Intuitively, $\nu_F \backslash \mu_F$ gives the best solution $\zeta_{F'}$ for the transformation $\mu_F \Rightarrow \nu_F \circ \zeta_{F'}$ to hold. The concrete definition of $\nu_F \backslash \mu_F$ is given in Appendix D.1.

B-App unboxes M when it has top-level modalities and inserts explicit unboxing in the elaborated term. B-Mod introduces a lock into the context and inserts explicit boxing in the elaborated term. B-Annotation is standard for bidirectional typing. B-Switch not only switches the direction from checking to inference, but also transforms the top-level modalities when there is a mismatch by inserting unboxing and re-boxing. It uses the judgement $\Gamma \vdash (\mu, A) \Rightarrow \nu \mathbin{@} E$ defined in Section 4.4.

Though incorporating polymorphic type inference is beyond the scope of this paper, we are confident that modal effect types are compatible with it. The key observation here is that in the presence of polymorphism, the problem of automatically boxing and unboxing is closely related to that of inferring first-class polymorphism. Modality introduction is analogous to type abstraction (which type inference algorithms realise through generalisation). Modality elimination is analogous to type application (which type inference algorithms realise through instantiation). As such, one can adapt any of the myriad techniques for combining first-class polymorphism with Hindley-Milner type inference. As we adopt a bidirectional type system, we could simply follow the literature on extending bidirectional typing with sound and complete inference for higher-rank polymorphism [17], first-class polymorphism [60] and bounded quantification [14].

In the future, we plan to further explore type inference for modal effect types and in particular design an extension to OCaml, building on and complementing recent work on modal types for OCaml [35] and making use of existing techniques for supporting first-class polymorphism.

8 Related and Future Work

8.1 Row-based Effect Systems

Row polymorphism is one popular approach to implementing effect systems for effect handlers. Links [22] uses Rémy-style row polymorphism with presence types [46], whereas Koka [32] and Frank [34] use scoped rows [31] which allow duplicated labels. Morris and McKinna [37] propose a general framework for comparing different styles of row types, and Yoshioka et al. [58] propose a similar framework focusing on comparing effect rows. Met adopts Leijen-style scoped rows, but also supports absent operation signatures in the spirit of Rémy-style presence types.

8.2 Capability-based Effect Systems

Capability-based effect systems such as Effekt [8, 9] and $\text{CC}_{<:\Box}$ [7] interpret effects as capabilities and offer a form of effect polymorphism through capability passing.

For instance, in Effekt the `asList` handler in Section 2.3 has the following type:

```
def asList{ f: Unit ⇒ List[Int] / { Gen[Int] } }: List[Int] / {}
```

The special *block parameter* (or *capability*) `f` can use the effect `Gen[Int]` in addition to those from the context. The annotation `{ Gen[Int] }` is similar to our relative modalities.

A key difference between Effekt and MET is that Effekt requires blocks to be second-class, whereas MET supports first-class functions by default. For instance, consider the standard composition function: `compose f g x = g (f x)`. We cannot write this function naively in Effekt as it relies on first-class functions. One solution is to uncurry it.

```
def composeUncurried[A, B, C](x: A){ f: A ⇒ B }{ g: B ⇒ C }: C / {}
```

Note that we move `x` to be the first argument, as Effekt requires value parameters to appear before block parameters. We cannot partially apply `composeUncurried`.

Brachthäuser et al. [8] recover first-class functions in Effekt by boxing blocks. However, such boxed blocks can only use those capabilities specified in the box types, similarly to our absolute modalities. With boxes, we can write the curried `compose` with the following type signature

```
def compose[A, B, C]{ f: A ⇒ B }{ g: B ⇒ C }: A ⇒ C at {f, g} / {}
```

which returns a value of type `A ⇒ C at {f, g}` — a first-class boxed block. The annotation `{f, g}` indicates that this block captures the capabilities `f` and `g`. This kind of annotation is reminiscent of effect variables, and indeed such examples illustrate why MET without effect polymorphism is not expressive enough to encode all of Effekt. If we extend MET with effect variables (Section 6.2) then it is possible to encode capability variables like `f` and `g`.

Another key difference between MET and Effekt is that Effekt uses named handlers [6, 55, 59], in which operations are dispatched to a specific named handler, whereas MET uses Plotkin and Pretnar [44]-style handlers, in which operations are dispatched to the first matching handler in the dynamic context. Named handlers also provide a form of effect generativity. In future it would be interesting to explore variants of modal effect types with named handlers and generative effects [15].

$CC_{<:\square}$ [7], the basis for capture tracking in Scala 3, also provides succinct types for uncurried higher-order functions like `composeUncurried`. As in Effekt, the curried version requires the result function to be explicitly annotated with its capture set `{f,g}`. Though $CC_{<:\square}$ is a good fit for Scala 3, it does rely on existing advanced features like path-dependent types (especially the ability of using term variables in types) and implicit parameters. Modal effect types do not require the language to support such advanced features.

8.3 Direct Comparison between METL, Koka, and Effekt

In Section 2, we compare the types of `iter`, `asList`, `state`, and `schedule` in MET with their counterparts in a row-based effect system similar to Koka. In Section 8.2, we discuss the differences between modal effect types and capability-based effect systems such as Effekt. Here we present a more direct comparison of the type signatures of typical programs in METL, Koka, and Effekt, in order to demonstrate the practicality of modal effect types.

Invoking Effects. An effect system tracks which effects are invoked. Consider a function `foo x = do yield (x + 42)` which uses the `yield` operation from Section 2.2. Its types in METL, Koka, and Effekt are as follows.

```
foo : [Gen Int](Int → 1)                    # METL
foo : forall<e>. (x : int) → <gen<int>|e> ()  # Koka
def foo(x : Int) : Unit / { Gen[Int] }      # Effekt
```

Both METL and Effekt support effect subtyping which enables us to apply `foo` with other effects. Koka does not support effect subtyping and uses an effect variable `e` for modularity. Koka supports a special typing rule which implicitly inserts effect variables on function types in covariant position. Consequently, we can in fact simply write the following type in Koka.

```
foo : (x : int) → (gen<int>) ()             # Koka
```

Handling Effects. An effect system tracks which effects are handled. Recall the `asList` handler from Section 2.3. Its types in Metl, Koka, and Effekt are as follows.

```
asList : [](<Gen Int>(1 → 1) → List Int)                              # METL
asList : forall<e>. (action : () → <gen<int>|e> ()) → e list<int>  # Koka
def asList{ f: Unit ⇒ List[Int] / { Gen[Int] } }: List[Int] / {}    # Effekt
```

Both Metl and Effekt allow the argument of `asList` to use the ambient effects in addition to `Gen Int`. In Koka, we must make the argument polymorphic over other effects.

Functions in Data Types. An effect system should be compatible with algebraic data types. Consider a pair of `foo` functions as defined in Section 8.3.

In Metl, we can choose to either share a single absolute modality between both components, or to have a separate absolute modality for each component.

```
pair1 : [Gen Int](Int → 1, Int → 1)
pair2 : ([Gen Int](Int → 1), [Gen Int](Int → 1))
```

The values `pair1` and `pair2` can be easily converted between one another.

In Koka, we must make both functions effect-polymorphic. We can choose either to use the same or different effect variables:

```
pair3 : forall<e,e1>. ((x : int) → <gen<int>|e> (), (y : int) → <gen<int>|e1> ())
pair4 : forall<e>.    ((x : int) → <gen<int>|e> (), (y : int) → <gen<int>|e> ())
```

In Effekt, as discussed in Section 8.2, we cannot express this pair of functions directly, as functions are second-class. We must box the functions to make them first-class.

```
pair5 : Tuple2[Int ⇒ Unit / { Gen[Int] } at {}, Int ⇒ Unit / { Gen[Int] } at {}]
```

The syntax `at {}` means that preceding function type is boxed with no captured capability.

Higher-Order Functions. An effect system should be compatible with higher-order functions. Consider the standard sequential composition function `compose f g x = g (f x)`. Its type in Metl and Koka are as follows.

```
compose : forall a b c . [](a → b) → (b → c) → (a → c))                  # METL
compose : forall<a,b,c,e>. (f : (a) →e b) → (g : (b) →e c) → ((x : a) →e c)# Koka
```

In Effekt, as we discussed in Section 8.2, we must either switch to an uncurried version or box the result with capability variables.

```
def composeUncurried[A, B, C](x: A){ f: A ⇒ B }{ g: B ⇒ C }: C / {}  # Effekt
def compose[A, B, C]{ f: A ⇒ B }{ g: B ⇒ C }: A ⇒ C at {f, g} / {}  # Effekt
```

In summary, compared to Koka, Metl provides more succinct types without effect variables for a rich class of programs. Compared to Effekt, Metl supports first-class functions smoothly with no extra restrictions or capability variables in types.

8.4 Frank

Our absolute and relative modalities are inspired by *abilities* and *adjustments* in Frank [13, 34]. Absolute modalities and abilities specify the whole effect context required to run some computation. Relative modalities and adjustments specify changes to the ambient effect context. The key difference is that Frank is based on a traditional row-based effect system and implicitly inserts effect variables into higher-order programs. This is a fragile syntactic abstraction as discussed in Section 1. In contrast, Met exploits modal types to robustly capture the essence of modular effect programming without effect polymorphism. As demonstrated in Section 5, a core Frank-like calculus with implicit

effect variables is expressible in Met. Frank's *adaptors* are richer than Met's masking, though we expect relative modalities to extend readily to encompass the full power of adaptors.

Unlike adjustments in Frank, modal types are first-class types just like data types and can appear anywhere. For instance, we can put two functions with modal types in a pair.

```
handleTwo : [](<Gen Int>(1 → 1), <State Int>(1 → 1)) → (List Int, 1))
handleTwo (f, g) = (asList f, state g 42)
```

8.5 Relationship Between Met and Multimodal Type Theory

The literature on multimodal type theory organises the structure of modes (objects), modalities (morphisms between objects), and their transformations (2-cells between morphisms) in a *2-category* [19, 20, 30] (or, in the case of a single mode, a semiring [1, 10, 40, 41]). In Met, modes are effect contexts E, modalities are of the form $\mu_F : E \to F$, and transformations are of the form $\mu_F \Rightarrow \nu_F$. However, 2-categories are insufficient in a system that also includes submoding. The extra structure can be captured by moving to *double categories*, which have an additional kind of vertical morphism between objects (in Met, vertical morphisms are given by the subeffecting relation $E \leqslant F$), as also proposed by Katsumata [29]. Consequently, the transformations do not strictly require the two modalities to have the same sources and targets, enabling us to have $[]_F \Rightarrow [E]_F$ in Met. The relationship between Met and MTT is explained in more detail in Appendix B.1.

8.6 Other Related Work

We discuss other related work on effect systems and modal types.

Subtyping-based Effect Systems. Eff [3, 45] is equipped with an effect system with both effect variables and sub-effecting based on the type inference and elaboration described in Karachalias et al. [28]. The effect system of Helium [6] is based on finite sets, offering a natural sub-effecting relation corresponding to set-inclusion. As such, their system aligns closely with Lucassen and Gifford [36]-style effect systems. Tang et al. [51] propose a calculus for effect handlers with effect polymorphism and sub-effecting via qualified types [26, 37].

Modal Types and Effects. Choudhury and Krishnaswami [11] proposes to use the necessity modality to recover purity from an effectful calculus, which is similar to our empty absolute modality. Zyuzin and Nanevski [61] extend contextual modal types [38] to algebraic effects and handlers. Their system lacks anything like our relative modality and thus cannot benefit from ambient effect contexts due to strict syntactic restrictions. Consequently, they cannot provide concise modular types for higher-order functions and handlers as Met does.

Effects in Call-By-Push-Value. In CBPV [33] effects are typically tracked only on typing judgements for computations and captured into types when switching to values [18, 27, 52]. In contrast, Met tracks effect contexts as modes on typing judgements for all kinds of term, in order to fully exploit ambient effects.

8.7 Future Work

Future work includes: implementing our system as an extension to OCaml; exploring extensions of modal effect types with Fitch-style unboxing, named handlers, generative effects, and capabilities; combining modal effect types with control-flow linearity; and developing a denotational semantics.

Data-Availability Statement

We have implemented a prototype of our surface language Metl. The prototype implements and extends the bidirectional type checker outlined in Section 7 and appendix D. It type checks all examples in the paper. We plan to submit the prototype for Artifact Evaluation.

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A Full Specification of Met with Extensions

We provide the full specification of Met including all extensions.

A.1 More Extensions

We first present three more extensions of modal effect types: richer forms of handlers, boxing pure computations, and commuting modalities with type abstraction. We discuss the key ideas of these extensions below and show their full specification in the following sub-sections.

Absolute and Shallow Handlers. Up to now we have considered only *deep* handlers of the form **handle** M **with** H where M depends on the ambient effect contexts. Deep handlers automatically wrap the handler around the body of the continuation r captured in a handler clause, and thus r depends on the ambient effect context. Though this usually suffices in practice, in some cases we may want the computation M or the continuation to be absolute, i.e., independent from the ambient effect context. We call such handlers *absolute* handlers. This situation is more prevalent with effect polymorphism.

To support absolute handlers, we extend the handler syntax and typing rules as follows.

$$\textsc{T-Handler}^{\mathbb{A}}\ \frac{\begin{array}{c} D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \blacksquare_{[D+E]_F} \vdash M : A \mathbin{@} D + E \\ \Gamma, \blacksquare_{[E]_F}, x : [D+E]A \vdash N : B \mathbin{@} F \\ [\Gamma, \blacksquare_{[E]_F}, p_i : A_i, r_i : [E](B_i \to B) \vdash N_i : B \mathbin{@} E]_i \qquad [E]_F \Rightarrow \mathbb{1}_F \end{array}}{\Gamma \vdash \mathbf{handle}^{\mathbb{A}}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i : B \mathbin{@} F}$$

The T-Handler$^{\mathbb{A}}$ rule extends the context with an absolute lock $\blacksquare_{[D+E]_F}$ specifying the effect context for M, and boxes the continuation r with the absolute modality $[E]$, where E exactly gives the effect context after handling. We put the lock $\blacksquare_{[E]_F}$ in handler clauses as deep handlers capture themselves into continuations. We also extend the handler syntax with *shallow* handlers $\mathbf{handle}^{\dagger}\ M\ \mathbf{with}\ H$, in which the handler is not automatically wrapped around the body of continuations, and *absolute shallow* handlers $\mathbf{handle}^{\mathbb{A}\dagger}\ M\ \mathbf{with}\ H$ [23, 27].

Boxing Computations under Empty Effect Contexts. We have restricted boxes to values in order to guarantee effect safety. This restriction is not essential for $[]$. For example, suppose we have $f :_{[]_F} (A \to B)$ and $x :_{[]_F} A$, it is sound to treat $\mathbf{mod}_{[]}\ (f\ x)$ as a computation which returns a value of type $[]B$. As $f\ x$ is evaluated under the empty effect context, we can guarantee that it cannot get stuck on unhandled operations.

We extend the introduction rule for the empty absolute modality to allow non-value terms with the following typing rule.

$$\textsc{T-ModAbs}\ \frac{\Gamma, \blacksquare_{[]_F} \vdash M : A \mathbin{@} \cdot}{\Gamma \vdash \mathbf{mod}_{[]}\ M : []A \mathbin{@} F}$$

As an example, we can write the following *app* function.

$$\begin{array}{lcl} app & : & \forall\alpha.\forall\beta.[](\alpha \to \beta) \to []\alpha \to []\beta \\ app & = & \Lambda\alpha.\Lambda\beta.\lambda f.\lambda x.\mathbf{let}\ \mathbf{mod}_{[]}\ f = f\ \mathbf{in}\ \mathbf{let}\ \mathbf{mod}_{[]}\ x = x\ \mathbf{in}\ \mathbf{mod}_{[]}\ (f\ x) \end{array}$$

The formula corresponding to the type of this function is commonly referred to as Axiom K in modal logic and is also satisfied by other similar modalities such as the safety modality of Choudhury and Krishnaswami [11].

Commuting Modalities and Type Abstraction. Crisp elimination rules in Section 6.1 allow us to commute modalities and data types. Similarly, it is also sound and useful to commute type abstractions and modalities. However, the current modality elimination rule cannot do so, for a similar reason to why it is not possible to transform $\forall\alpha.A + B$ to $(\forall\alpha.A) + (\forall\alpha.B)$ in System F. We extend modality elimination to the form $\mathbf{let}_\nu\ \mathbf{mod}_\mu\ \Lambda\overline{\alpha^K}x = V\ \mathbf{in}\ M$ which allows V to use additional type variables in $\overline{\alpha^K}$ which are abstracted when bound to x. The extended typing and reduction rules are as follows.

$$\text{T-LetMod'}\quad \frac{\nu_F : E \longrightarrow F \qquad \Gamma, \blacksquare_{\nu_F}, \overline{\alpha : K} \vdash V : \mu A \mathbin{@} E \qquad \Gamma, x :_{\nu_F \circ \mu_E} \forall\overline{\alpha^K}.A \vdash M : B \mathbin{@} F}{\Gamma \vdash \mathbf{let}_\nu\ \mathbf{mod}_\mu\ \Lambda\overline{\alpha^K}.x = V\ \mathbf{in}\ M : B \mathbin{@} F}$$

$$\text{E-LetMod'}\quad \mathbf{let}_\nu\ \mathbf{mod}_\mu\ \Lambda\overline{\alpha^K}.x = \mathbf{mod}_\mu\ U\ \mathbf{in}\ M \rightsquigarrow M[(\Lambda\overline{\alpha^K}.U)/x]$$

For instance, we can now write a function of type $\forall\alpha^K.\mu A \to \mu(\forall\alpha.A)$ where $\alpha \notin \mathrm{ftv}(\mu)$ as follows.

$$\lambda x^{\forall\alpha^K.\mu A}.\mathbf{let}\ \mathbf{mod}_\mu\ \Lambda\alpha^K.y = x\ \alpha\ \mathbf{in}\ \mathbf{mod}_\mu\ y$$

A.2 Syntax

Figure 5 gives the syntax of Met with all extensions including data types, polymorphism, and enriched handlers. We highlight the syntax not present in core Met.

Types	$A, B ::= 1 \mid A \to B \mid \mu A \mid \alpha \mid \forall\alpha^K.A \mid A * B \mid A + B$
Masks	$L ::= \cdot \mid \ell, L$
Extensions	$D ::= \cdot \mid \ell : P, D$
Effect Contexts	$E, F ::= \cdot \mid \ell : P, E \mid \varepsilon \mid E \backslash L$
Signatures	$P ::= A \twoheadrightarrow B \mid -$
Modalities	$\mu ::= [E] \mid \langle L \mid D \rangle$
Kinds	$K ::= \mathsf{Abs} \mid \mathsf{Any} \mid \mathsf{Effect}$
Contexts	$\Gamma ::= \cdot \mid \Gamma, x :_{\mu_F} A \mid \Gamma, \blacksquare_{\mu_F} \mid \Gamma, \alpha : K$
Terms	$M, N ::= x \mid \lambda x^A.M \mid M\,N \mid \Lambda\alpha^K.V \mid M\,A$
	$\mid\ \mathbf{mod}_\mu\ V \mid \mathbf{let}_\nu\ \mathbf{mod}_\mu\ x = V\ \mathbf{in}\ M$
	$\mid\ \mathbf{do}\ \ell\ M \mid \mathbf{mask}_L\ M \mid \mathbf{handle}^{\delta}\ M\ \mathbf{with}\ H$
	$\mid\ (M, N) \mid \mathbf{case}_\nu\ V\ \mathbf{of}\ (x, y) \mapsto M$
	$\mid\ \mathbf{inl}\ M \mid \mathbf{inr}\ M \mid \mathbf{case}_\nu\ V\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M, \mathbf{inr}\ y \mapsto N\}$
	$\mid\ \mathbf{let}_\nu\ \mathbf{mod}_\mu\ \Lambda\overline{\alpha^K}.x = V\ \mathbf{in}\ M$
Values	$V, W ::= x \mid \lambda x^A.M \mid \mathbf{mod}_\mu\ V \mid \Lambda\alpha^K.V \mid V\,A \mid (V, W) \mid \mathbf{inl}\ V \mid \mathbf{inr}\ V$
Handlers	$H ::= \{\mathbf{return}\ x \mapsto M\} \mid \{\ell\ p\ r \mapsto M\} \uplus H$
Decorations	$\delta ::= \cdot \mid \mathbb{A} \mid \dagger \mid \mathbb{A}\dagger$

Fig. 5. Syntax of Met with all extensions.

A.3 Kinding, Well-Formedness, Type Equivalence and Sub-effecting

The full kinding and well-formedness rules for Met are shown in Figure 6. The type equivalence and sub-effecting rules are shown in Figure 7. We highlight the special rule that allows us to add or remove absent labels from the right of effect contexts.

A.4 Auxiliary Operations

Since we extend the syntax of effect contexts E, we also need to define a new case for the operation $E - L$ as follows. The definitions of other operations $D + E$ and $L \bowtie D$ remain unchanged from those in Section 4.2. We include the full definition here for easy reference.

$\boxed{D + E}\ \boxed{E - L}\ \boxed{L \bowtie D}$

$$D + E = D, E \qquad \cdot - L = \cdot \qquad (\ell : P, E) - L = \begin{cases} E - L' & \text{if } L \equiv \ell, L' \\ \ell : P, (E - L) & \text{otherwise} \end{cases}$$

$$L \bowtie \cdot = (L, \cdot) \qquad L \bowtie (\ell : P, D) = \begin{cases} L' \bowtie D & \text{if } L \equiv \ell, L' \\ (L', (\ell : P, D')) & \text{otherwise} \\ \quad \text{where } (L', D') = L \bowtie D \end{cases}$$

Since we extend the syntax of contexts, we need to extend $\mathsf{locks}(\Gamma)$ with one extra trivial case.

$$\mathsf{locks}(\cdot) = \mathbb{1} \qquad \mathsf{locks}(\Gamma, \blacksquare_{\mu_F}) = \mathsf{locks}(\Gamma) \circ \mu_F \qquad \mathsf{locks}(\Gamma, x :_{\mu_F} A) = \mathsf{locks}(\Gamma) \qquad \mathsf{locks}(\Gamma, \alpha : K) = \mathsf{locks}(\Gamma)$$

$\boxed{\Gamma \vdash A : K}$

$$\frac{\Gamma \ni \alpha : K}{\Gamma \vdash \alpha : K} \qquad \frac{\Gamma \vdash A : \mathsf{Abs}}{\Gamma \vdash A : \mathsf{Any}} \qquad \frac{\Gamma \vdash [E] \qquad \Gamma \vdash A : \mathsf{Any}}{\Gamma \vdash [E]A : \mathsf{Abs}} \qquad \frac{\Gamma \vdash \langle L | D \rangle \qquad \Gamma \vdash A : K}{\Gamma \vdash \langle L | D \rangle A : K}$$

$$\frac{\Gamma \vdash A : \mathsf{Any} \quad \Gamma \vdash B : \mathsf{Any}}{\Gamma \vdash A \to B : \mathsf{Any}} \qquad \frac{\Gamma, \alpha : K \vdash A : K'}{\Gamma \vdash \forall \alpha^K . A : K'} \qquad \frac{}{\Gamma \vdash 1 : \mathsf{Abs}} \qquad \frac{\Gamma \vdash A : K \quad \Gamma \vdash B : K}{\Gamma \vdash A * B : K} \qquad \frac{\Gamma \vdash A : K \quad \Gamma \vdash B : K}{\Gamma \vdash A + B : K}$$

$\boxed{\Gamma \vdash \mu}\ \boxed{\Gamma \vdash E : K}\ \boxed{\Gamma \vdash L}\ \boxed{\Gamma \vdash D}\ \boxed{\Gamma \vdash P}$

$$\frac{\Gamma \vdash L \qquad \Gamma \vdash D}{\Gamma \vdash \langle L | D \rangle} \qquad \frac{\Gamma \vdash E : \mathsf{Effect}}{\Gamma \vdash [E]} \qquad \frac{}{\Gamma \vdash \cdot : \mathsf{Effect}} \qquad \frac{\Gamma \vdash P \qquad \Gamma \vdash E : \mathsf{Effect}}{\Gamma \vdash \ell : P, E : \mathsf{Effect}}$$

$$\frac{\Gamma \vdash E : \mathsf{Effect} \qquad \Gamma \vdash L}{\Gamma \vdash E \backslash L : \mathsf{Effect}} \qquad \frac{}{\Gamma \vdash L} \qquad \frac{}{\Gamma \vdash \cdot} \qquad \frac{\Gamma \vdash P \qquad \Gamma \vdash D}{\Gamma \vdash \ell : P, D}$$

$$\frac{}{\Gamma \vdash -} \qquad \frac{\Gamma \vdash A : \mathsf{Abs} \qquad \Gamma \vdash B : \mathsf{Abs}}{\Gamma \vdash A \twoheadrightarrow B}$$

$\boxed{\Gamma \vdash (\mu, A) \Rightarrow \nu \ @\ F}$

$$\frac{\Gamma \vdash A : \mathsf{Abs}}{\Gamma \vdash (\mu, A) \Rightarrow \nu \ @\ F} \qquad \frac{\mu_F \Rightarrow \nu_F}{\Gamma \vdash (\mu, A) \Rightarrow \nu \ @\ F}$$

$\boxed{\Gamma \ @\ E}$

$$\frac{}{\cdot \ @\ E} \qquad \frac{\Gamma \ @\ F \qquad \mu_F : E \to F \qquad \Gamma \vdash A : K}{\Gamma, x :_{\mu_F} A \ @\ F} \qquad \frac{\Gamma \ @\ E}{\Gamma, \alpha : K \ @\ E} \qquad \frac{\Gamma \ @\ F \qquad \mu_F : E \to F}{\Gamma, \blacksquare_{\mu_F} \ @\ E}$$

Fig. 6. Kinding, well-formedness, and auxiliary rules for Met.

$\boxed{L \equiv L'}$ $\boxed{D \equiv D'}$

$$\frac{}{\cdot \equiv \cdot} \qquad \frac{L_1 \equiv L_2 \qquad L_2 \equiv L_3}{L_1 \equiv L_3} \qquad \frac{L \equiv L'}{\ell, L \equiv \ell, L'} \qquad \frac{\ell \neq \ell' \qquad L \equiv L'}{\ell, \ell', L \equiv \ell', \ell, L}$$

$$\frac{}{\cdot \equiv \cdot} \qquad \frac{D_1 \equiv D_2 \qquad D_2 \equiv D_3}{D_1 \equiv D_3} \qquad \frac{P \equiv P' \qquad D \equiv D'}{\ell : P, D \equiv \ell : P', D'} \qquad \frac{\ell \neq \ell'}{\ell : P, \ell' : P', D \equiv \ell' : P', \ell : P, D}$$

$\boxed{E \equiv F}$

$$\frac{}{\cdot \equiv \cdot} \qquad \frac{E_1 \equiv E_2 \qquad E_2 \equiv E_3}{E_1 \equiv E_3} \qquad \frac{P \equiv P' \qquad E \equiv E'}{\ell : P, E \equiv \ell : P', E'} \qquad \frac{\ell \neq \ell'}{\ell : P, \ell' : P', E \equiv \ell' : P', \ell : P, E}$$

$$\frac{}{E, \ell : - \equiv E} \qquad \frac{}{\varepsilon \backslash L \equiv \varepsilon \backslash L} \qquad \frac{E \equiv E' \qquad L \equiv L'}{E \backslash L \equiv E' \backslash L'} \qquad \frac{}{E \backslash \cdot \equiv E} \qquad \frac{}{\cdot \backslash L \equiv \cdot}$$

$$\frac{}{(\ell : P, E) \backslash (\ell, L) \equiv E \backslash L} \qquad \frac{\ell \notin L}{(\ell : P, E) \backslash L \equiv \ell : P, E \backslash L} \qquad \frac{}{(\varepsilon \backslash L) \backslash L' \equiv \varepsilon \backslash (L, L')}$$

$\boxed{P \equiv P'}$ $\boxed{\mu \equiv \nu}$

$$\frac{A \equiv A' \qquad B \equiv B'}{A \twoheadrightarrow B \equiv A' \twoheadrightarrow B'} \qquad \frac{}{- \equiv -} \qquad \frac{E \equiv F}{[E] \equiv [F]} \qquad \frac{L \equiv L' \qquad D \equiv D'}{\langle L | D \rangle \equiv \langle L' | D' \rangle}$$

$\boxed{A \equiv B}$

$$\frac{}{\alpha \equiv \alpha} \qquad \frac{\mu \equiv \nu \qquad A \equiv B}{\mu A \equiv \nu B} \qquad \frac{A \equiv A' \qquad B \equiv B'}{A \to B \equiv A' \to B'} \qquad \frac{A \equiv B}{\forall \alpha^K . A \equiv \forall \alpha^K . B}$$

$$\frac{A \equiv A' \qquad B \equiv B'}{A * B \equiv A' * B'} \qquad \frac{A \equiv A' \qquad B \equiv B'}{A + B \equiv A' + B'}$$

$\boxed{E \leqslant F}$

$$\frac{}{\cdot \leqslant E} \qquad \frac{E \equiv F}{E \leqslant F} \qquad \frac{E_1 \equiv \ell : P_1, E_1' \qquad E_2 \equiv \ell : P_2, E_2' \qquad P_1 \leqslant P_2 \qquad E_1' \leqslant E_2'}{E_1 \leqslant E_2}$$

$\boxed{P \leqslant P'}$ $\boxed{D \leqslant D'}$

$$\frac{}{P \leqslant P} \qquad \frac{}{- \leqslant P} \qquad \frac{}{\cdot \leqslant \cdot} \qquad \frac{D_1 \equiv \ell : P_1, D_1' \qquad D_2 \equiv \ell : P_2, D_2' \qquad P_1 \leqslant P_2 \qquad D_1' \leqslant D_2'}{D_1 \leqslant D_2}$$

Fig. 7. Type equivalence and sub-effecting for Met.

A.5 Typing Rules

Figure 8 gives the typing rules of Met. We only show the extended rules with respect to the typing rules of core Met in Figure 2.

$\boxed{\Gamma \vdash M : A \mathbin{@} E}$

$$\textsc{T-TAbs}\ \frac{\Gamma, \alpha : K \vdash V : A \mathbin{@} E}{\Gamma \vdash \Lambda\alpha^K.V : \forall\alpha^K.A \mathbin{@} E} \qquad \textsc{T-TApp}\ \frac{\Gamma \vdash M : \forall\alpha^K.B \mathbin{@} E \qquad \Gamma \vdash A : K}{\Gamma \vdash M\,A : B[A/\alpha] \mathbin{@} E}$$

$$\textsc{T-Pair}\ \frac{\Gamma \vdash M : A \mathbin{@} E \qquad \Gamma \vdash N : B \mathbin{@} E}{\Gamma \vdash (M, N) : A * B \mathbin{@} E} \qquad \textsc{T-Inl}\ \frac{\Gamma \vdash M : A \mathbin{@} E}{\Gamma \vdash \mathbf{inl}\ M : A + B \mathbin{@} E} \qquad \textsc{T-Inr}\ \frac{\Gamma \vdash M : B \mathbin{@} E}{\Gamma \vdash \mathbf{inr}\ M : A + B \mathbin{@} E}$$

$$\textsc{T-CrispPair}\ \frac{\nu_F : E \to F \qquad \Gamma, \blacklock_{\nu_F} \vdash V : A * B \mathbin{@} E \qquad \Gamma, x :_{\nu_F} A, y :_{\nu_F} B \vdash M : A' \mathbin{@} F}{\Gamma \vdash \mathbf{case}_\nu\ V\ \mathbf{of}\ (x, y) \mapsto M : A' \mathbin{@} F}$$

$$\textsc{T-CrispSum}\ \frac{\nu_F : E \to F \qquad \Gamma, \blacklock_{\nu_F} \vdash V : A + B \mathbin{@} E \qquad \Gamma, x :_{\nu_F} A \vdash M_1 : A' \mathbin{@} F \qquad \Gamma, y :_{\nu_F} B \vdash M_2 : A' \mathbin{@} F}{\Gamma \vdash \mathbf{case}_\nu\ V\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1, \mathbf{inr}\ y \mapsto M_2\} : A' \mathbin{@} F}$$

$$\textsc{T-Handler}^{\mathbb{A}}\ \frac{D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \blacklock_{[D+E]_F} \vdash M : A \mathbin{@} D + E \qquad \Gamma, \blacklock_{[E]_F}, x : [D+E]A \vdash N : B \mathbin{@} F \qquad [\Gamma, \blacklock_{[E]_F}, p_i : A_i, r_i : [E](B_i \to B) \vdash N_i : B \mathbin{@} E]_i \qquad [E]_F \Rightarrow \mathbb{1}_F}{\Gamma \vdash \mathbf{handle}^{\mathbb{A}}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i : B \mathbin{@} F}$$

$$\textsc{T-ShallowHandler}\ \frac{D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \blacklock_{\langle\!| D \rangle_F} \vdash M : A \mathbin{@} D + F \qquad \Gamma, x : \langle\!| D \rangle A \vdash N : B \mathbin{@} F \qquad [\Gamma, p_i : A_i, r_i : \langle\!| D \rangle(B_i \to A) \vdash N_i : B \mathbin{@} F]_i}{\Gamma \vdash \mathbf{handle}^{\dagger}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i : B \mathbin{@} F}$$

$$\textsc{T-ShallowHandler}^{\mathbb{A}}\ \frac{D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \blacklock_{[D+E]_F} \vdash M : A \mathbin{@} D + E \qquad \Gamma, x : [D+E]A \vdash N : B \mathbin{@} F \qquad [\Gamma, p_i : A_i, r_i : [D+E](B_i \to A) \vdash N_i : B \mathbin{@} F]_i \qquad [E]_F \Rightarrow \mathbb{1}_F}{\Gamma \vdash \mathbf{handle}^{\mathbb{A}\dagger}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i : B \mathbin{@} F}$$

$$\textsc{T-ModAbs}\ \frac{\Gamma, \blacklock_{[]_F} \vdash M : A \mathbin{@} \cdot}{\Gamma \vdash \mathbf{mod}_{[]}\ M : []A \mathbin{@} F} \qquad \textsc{T-Letmod'}\ \frac{\nu_F : E \to F \qquad \Gamma, \blacklock_{\nu_F}, \overline{\alpha : K} \vdash V : \mu A \mathbin{@} E \qquad \Gamma, x :_{\nu_F \circ \mu_E} \forall\overline{\alpha^K}.A \vdash M : B \mathbin{@} F}{\Gamma \vdash \mathbf{let}_\nu\ \mathbf{mod}_\mu\ \Lambda\overline{\alpha^K}.x = V\ \mathbf{in}\ M : B \mathbin{@} F}$$

Fig. 8. Typing rules for Met (only showing extensions to core Met in Figure 2).

A.6 Operational Semantics

As type application are treated as values and can reduce, we first define value normal forms U that cannot reduce further as follows.

$$\text{Value normal forms} \quad U ::= x \mid \lambda x^A.M \mid \Lambda\alpha^K.V \mid \mathbf{mod}_\mu\, U \mid (U_1, U_2) \mid \mathbf{inl}\ U \mid \mathbf{inr}\ U$$

$$
\begin{array}{ll}
\textsc{E-App} & (\lambda x^A.M)\ U \rightsquigarrow M[U/x] \\
\textsc{E-TApp} & (\Lambda \alpha.V)\ A \rightsquigarrow V[A/\alpha] \\
\textsc{E-Letmod} & \mathbf{let}_\nu\ \mathbf{mod}_\mu\ x = \mathbf{mod}_\mu\ U\ \mathbf{in}\ M \rightsquigarrow M[U/x] \\
\textsc{E-Letmod'} & \mathbf{let}_\nu\ \mathbf{mod}_\mu\ \Lambda\overline{\alpha^K}.x = \mathbf{mod}_\mu\ U\ \mathbf{in}\ M \rightsquigarrow M[(\Lambda\overline{\alpha^K}.U)/x] \\
\textsc{E-Mask} & \mathbf{mask}_L\ U \rightsquigarrow \mathbf{mod}_{\langle L|\rangle}\ U \\
\textsc{E-Pair} & \mathbf{case}_\mu\ (U_1, U_2)\ \mathbf{of}\ (x, y) \mapsto N \rightsquigarrow N[U_1/x, U_2/y] \\
\textsc{E-Inl} & \mathbf{case}_\mu\ \mathbf{inl}\ U\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto N_1, \cdots\} \rightsquigarrow N_1[U/x] \\
\textsc{E-Inr} & \mathbf{case}_\mu\ \mathbf{inr}\ U\ \mathbf{of}\ \{\mathbf{inr}\ y \mapsto N_2, \cdots\} \rightsquigarrow N_2[U/y] \\
\textsc{E-Ret} & \mathbf{handle}\ U\ \mathbf{with}\ H \rightsquigarrow N[(\mathbf{mod}_{\langle|D\rangle}\ U)/x], \\
 & \qquad \text{where } (\mathbf{return}\ x \mapsto N) \in H \\
\textsc{E-Op} & \mathbf{handle}\ \mathcal{E}[\mathbf{do}\ \ell\ U]\ \mathbf{with}\ H \rightsquigarrow N[U/p, (\lambda y.\mathbf{handle}\ \mathcal{E}[y]\ \mathbf{with}\ H)/r], \\
 & \qquad \text{where } 0\text{–free}(\ell, \mathcal{E}) \text{ and } (\ell\ p\ r \mapsto N) \in H \\
\textsc{E-Ret}^{\mathbb{A}} & \mathbf{handle}\ U\ \mathbf{with}\ H \rightsquigarrow N[(\mathbf{mod}_{[D+E]}\ U)/x] \\
 & \qquad \text{where } (\mathbf{return}\ x \mapsto N) \in H \\
\textsc{E-Op}^{\mathbb{A}} & \mathbf{handle}^{\mathbb{A}}\ \mathcal{E}[\mathbf{do}\ \ell\ U]\ \mathbf{with}\ H \rightsquigarrow \\
 & \qquad N[U/p, (\mathbf{mod}_{[E]}\ (\lambda y.\mathbf{handle}^{\mathbb{A}}\ \mathcal{E}[y]\ \mathbf{with}\ H))/r] \\
 & \qquad \text{where } 0\text{–free}(\ell, \mathcal{E}) \text{ and } (\ell\ p\ r \mapsto N) \in H \\
\textsc{E-Ret}^{\dagger} & \mathbf{handle}^{\dagger}\ U\ \mathbf{with}\ H \rightsquigarrow N[(\mathbf{mod}_{\langle|D\rangle}\ U)/x] \\
 & \qquad \text{where } (\mathbf{return}\ x \mapsto N) \in H \\
\textsc{E-Op}^{\dagger} & \mathbf{handle}^{\dagger}\ \mathcal{E}[\mathbf{do}\ \ell\ U]\ \mathbf{with}\ H \rightsquigarrow N[U/p, (\lambda y.\mathcal{E}[y])/r] \\
 & \qquad \text{where } 0\text{–free}(\ell, \mathcal{E}) \text{ and } (\ell\ p\ r \mapsto N) \in H \\
\textsc{E-Ret}^{\mathbb{A}\dagger} & \mathbf{handle}^{\mathbb{A}\dagger}\ U\ \mathbf{with}\ H \rightsquigarrow N[(\mathbf{mod}_{[D+E]}\ U)/x] \\
 & \qquad \text{where } (\mathbf{return}\ x \mapsto N) \in H \\
\textsc{E-Op}^{\mathbb{A}\dagger} & \mathbf{handle}^{\mathbb{A}\dagger}\ \mathcal{E}[\mathbf{do}\ \ell\ U]\ \mathbf{with}\ H \rightsquigarrow N[U/p, (\mathbf{mod}_{[D+E]}\ (\lambda y.\mathcal{E}[y]))/r] \\
 & \qquad \text{where } 0\text{–free}(\ell, \mathcal{E}) \text{ and } (\ell\ p\ r \mapsto N) \in H \\
\textsc{E-Lift} & \mathcal{E}[M] \rightsquigarrow \mathcal{E}[N], \qquad \text{if } M \rightsquigarrow N
\end{array}
$$

Fig. 9. Operational semantics for Met.

We also extend the definition of evaluation contexts. The full definition is given as follows. Notice that we use value normal forms instead of values.

$$
\begin{array}{lll}
\text{Evaluation contexts} & \mathcal{E} ::= [\ ] \mid \mathcal{E}\ A \mid \mathcal{E}\ N \mid U\ \mathcal{E} \mid \mathbf{do}\ \ell\ \mathcal{E} \mid \mathbf{mask}_L\ \mathcal{E} \mid \mathbf{handle}^{\delta}\ \mathcal{E}\ \mathbf{with}\ H \\
 & \quad \mid \mathbf{mod}_\mu\ \mathcal{E} \mid \mathbf{let}_\nu\ \mathbf{mod}_\mu\ x = \mathcal{E}\ \mathbf{in}\ M \mid \mathbf{let}_\nu\ \mathbf{mod}_\mu\ \Lambda\overline{\alpha^K}.x = \mathcal{E}\ \mathbf{in}\ M \\
 & \quad \mid (\mathcal{E}, N) \mid (U, \mathcal{E}) \mid \mathbf{case}_\nu\ \mathcal{E}\ \mathbf{of}\ (x, y) \mapsto M \\
 & \quad \mid \mathbf{inl}\ \mathcal{E} \mid \mathbf{inr}\ \mathcal{E} \mid \mathbf{case}_\nu\ \mathcal{E}\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M, \mathbf{inr}\ y \mapsto N\}
\end{array}
$$

Figure 9 shows the operational semantics of Met.

B Meta Theory and Proofs for Met

We provide meta theory and proofs for Met in Sections 4 and 6 including all extensions.

B.1 The Double Category of Effects

A double category extends a 2-category with an additional kind of morphisms. Alongside the regular morphisms, now called *horizontal* morphisms, there are also *vertical* morphisms that connect the objects of the 2-category. This makes it possible to generalise the 2-cells to transform arbitrary

Fig. 10. 2-cells in a 2-category compared to 2-cells in a double category.

morphisms, whose source and target are connected by vertical morphisms. Figure 10 shows the differences between 2-cells in a 2-category and those in a double category using syntax of Met.

In Met, objects/modes are given by effect contexts, the horizontal morphisms by modalities, the vertical morphisms by the sub-effecting relation, and 2-cells by the modality transformations.

Now we show that it indeed has the structure of a double category.

Since the sub-effecting relation is a preorder, effect contexts (objects) E and sub-effecting (vertical morphisms) $E \leqslant F$ obviously form a category given by the poset.

We repeat the definition of modalities and modality composition from Section 4.3 here for easy reference. We define them directly in terms of morphisms between modes.

$$\begin{array}{rcl} [E]_F & : & E \to F \\ \langle L|D\rangle_F & : & D + (F - L) \to F \end{array}$$

$$\begin{array}{rcll} [E']_F & \circ & [E]_{E'} & = & [E]_F \\ \langle L|D\rangle_F & \circ & [E]_{D+(F-L)} & = & [E]_F \\ [E]_F & \circ & \langle L|D\rangle_E & = & [D + (E - L)]_F \\ \langle L_1|D_1\rangle_F & \circ & \langle L_2|D_2\rangle_{D_1+(F-L_1)} & = & \langle L_1 + L|D_2 + D\rangle_F \quad \text{where } (L, D) = L_2 \bowtie D_1 \end{array}$$

The effect contexts (objects) and modalities (horizontal morphisms) also form a category since modality composition possesses associativity and identity. We have the following lemma.

Lemma B.1 (Modes and modalities form a category). *Modes and modalities form a category with the identity morphism $\mathbb{1}_E = \langle|\rangle_E : E \to E$ and the morphism composition $\mu_F \circ \nu_{F'}$ such that*

(1) Identity: $\mathbb{1}_F \circ \mu_F = \mu_F = \mu_F \circ \mathbb{1}_E$ for $\mu_F : E \to F$.
(2) Associativity: $(\mu_{E_1} \circ \nu_{E_2}) \circ \xi_{E_3} = \mu_{E_1} \circ (\nu_{E_2} \circ \xi_{E_3})$ for $\mu_{E_1} : E_2 \to E_1$, $\nu_{E_2} : E_3 \to E_2$, and $\xi_{E_3} : E \to E_3$.

Proof. By inlining the definitions of modalities and checking each case. □

In Section 4, we only define the modality transformations of shape $\mu_F \Rightarrow \nu_F$ where the targets of μ and ν are required to be the same effect context F. This is enough for presenting the calculus, but we can further extend it to allow $\mu_F \Rightarrow \nu_{F'}$ where $F \leqslant F'$. This is used in the meta theory for Met such as the lock weakening lemma (Lemma B.11.3).

The extended modality transformation relation is defined by the transitive closure of the following rules. Compared to the definition in Section 4.3, the only new rule is MT-Mono.

$$\text{MT-Abs}\ \frac{\mu_F : E' \to F \qquad E \leqslant E'}{[E]_F \Rightarrow \mu_F} \qquad \text{MT-Upcast}\ \frac{D \leqslant D'}{\langle L|D\rangle_F \Rightarrow \langle L|D'\rangle_F} \qquad \text{MT-Expand}\ \frac{(F - L) \equiv \ell : P, E}{\langle \ell, L|D, \ell : P\rangle_F \Leftrightarrow \langle L|D\rangle_F} \qquad \text{MT-Mono}\ \frac{F \leqslant F'}{\mu_F \Rightarrow \mu_{F'}}$$

The following lemmas shows that the transformation $\mu_F \Rightarrow \nu_{F'}$ satisfies the requirement of being 2-cells in the double category of effects with well-defined vertical and horizontal composition.

Lemma B.2 (Modality transformations are 2-cells). *If $\mu_F \Rightarrow \nu_{F'}$, $\mu_F : E \to F$, and $\nu_{F'} : E' \to F'$, then $E \leqslant E'$ and $F \leqslant F'$. Moreover, the transformation relation is closed under vertical and horizontal composition as shown by the following admissible rules.*

$$\frac{\mu_{F_1} \Rightarrow \nu_{F_2} \qquad \nu_{F_2} \Rightarrow \xi_{F_3}}{\mu_{F_1} \Rightarrow \xi_{F_3}} \qquad \frac{\mu_F \Rightarrow \mu'_{F'} \qquad \nu_E \Rightarrow \nu'_{E'} \qquad \mu_F : E \to F \qquad \mu'_{F'} : E' \to F'}{\mu_F \circ \nu_E \Rightarrow \mu'_{F'} \circ \nu'_{E'}}$$

Proof. To make proving easier, we give the resulting rules by taking the transitive closure.

$$\frac{\mu_{F'} : E' \to F' \qquad E \leqslant E' \qquad F \leqslant F'}{[E]_F \Rightarrow \mu_{F'}}$$

$$\frac{L = \mathsf{dom}(D) \qquad D_1 \leqslant D'_1 \qquad (F' - L_1) \equiv D, E \qquad F \leqslant F' \qquad \mathsf{present}(D)}{\langle L_1 | D_1 \rangle_F \Rightarrow \langle L, L_1 | D'_1, D \rangle_{F'}}$$

$$\frac{L = \mathsf{dom}(D) \qquad D_1 \leqslant D'_1 \qquad (F' - L_1) \equiv D, E \qquad F \leqslant F'}{\langle L, L_1 | D_1, D \rangle_F \Rightarrow \langle L_1 | D'_1 \rangle_{F'}}$$

The predicate present(D) checks if all labels in D are present. It is easy to see that sources and targets of morphisms increase. Vertical composition follows directly from the fact that we take the transitive closure. Horizontal compositions follows from case analysis on shapes of modalities being composed. □

More on Relationships between Met and Multimodal Type Theory. In addition to extending to a double category, Met also differs from MTT in the usage of morphism families. In types and terms we use μ, indexed families of morphisms between modes, instead of concrete morphisms μ_F. We do not lose any information. Given a typing judgement $\Gamma \vdash M : A \mathbin{@} E$, the indexes for all modalities in M and A are determined by E. Similarly, given a variable binding $x :_{\mu_F} A$, the indexes of all modalities in A are determined by μ_F.

Using indexed families of modalities in types and terms is very useful to allow term variables to be used flexibly in different effect contexts larger than where they are defined. This greatly simplifies the support of subeffecting; we do not need to update all indexes of modalities in a term or type when upcasting this term or type to a larger effect context. As a result, every type is always well-defined at any modes, which means that we do not need to define the well-formedness judgement $A \mathbin{@} E$ as in MTT. Moreover, one important benefit of having types well-defined at any modes is that when adding polymorphism for values, type quantifiers do not need to carry the additional information about the modes at which the type variables can be used, greatly simplifying the type system. Otherwise, polymorphic types would need to have forms $\forall \alpha^{K \mathbin{@} E}.A$, where E indicates the mode of the type variable α.

In contexts, we still keep concrete morphisms μ_F, which makes the proof trees of terms much more structured than using morphism families.

B.2 Lemmas for Modes and Modalities

Beyond the structure and properties of double categories shown in Appendix B.1, we have some extra properties on modes and modalities in Met.

The most important one is that horizontal morphisms (sub-effecting) act functorially on vertical ones (modalities). In other words, the action of μ on effect contexts gives a total monotone function.

Lemma B.3 (Monotone modalities). *If $\mu_F : E \to F$ and $F \leqslant F'$, then $\mu_{F'} : E' \to F'$ with $E \leqslant E'$.*

Proof. By definition. □

We prove the lemma on the equivalence between syntactic and semantic definition of modality transformation in Section 4.3. This lemma can be generalised to the general form of 2-cells in a double category $\mu_F \Rightarrow \nu_{F'}$ where $F \leqslant F'$.

Lemma 4.1 (Semantics of modality transformation). *We have $\mu_F \Rightarrow \nu_F$ if and only if $\mu(F') \leqslant \nu(F')$ for all F' with $F \leqslant F'$.*

Proof. From left to right, it is obvious that the semantics is preserved after taking the transitive closure. We only need to show the transformation given by each rule satisfies the semantics.

Case MT-Abs. Follow from Lemma B.3.

Case MT-Upcast. Since $D \leqslant D'$, we have $D + (F - L) \leqslant D' + (F - L)$ for any F.

Case MT-Expand. Since $(F - L) \equiv \ell : A \twoheadrightarrow B, E$, for any $F \leqslant F'$ we have $(F' - L) \equiv \ell : A \twoheadrightarrow B, E'$ for some E'. Both sides act on F' give $D, \ell : A \twoheadrightarrow B, E'$. Notice that it is important for ℓ to not be absent here; otherwise, in F' we could upcast the absent signature of ℓ to any concrete signatures, which then breaks the condition $\mu(F') \leqslant \nu(F')$.

Case MT-Shrink. Similar to the above case.

From left to right, we need to show that for all pairs μ_F and ν_F satisfying the semantic definition, we have $\mu_F \Rightarrow \nu_F$ in the transitive closure of the three syntactic rules. This obviously holds for those transformation starting from absolute modalities. For those transformation starting from relative modalities, observe that they can only be transformed other relative modalities by the semantic definition. By taking the transitive closure of the last two rules, we have

$$\text{MT-MultiExpand}\quad \frac{L = \mathsf{dom}(D) \qquad D_1 \leqslant D_1' \qquad (F - L_1) \equiv D, E \qquad \mathsf{present}(D)}{\langle L_1 | D_1 \rangle_F \Rightarrow \langle L, L_1 | D_1', D \rangle_F}$$

$$\text{MT-MultiShrink}\quad \frac{L = \mathsf{dom}(D) \qquad D_1 \leqslant D_1' \qquad (F - L_1) \equiv D, E}{\langle L, L_1 | D_1, D \rangle_F \Rightarrow \langle L_1 | D_1' \rangle_F}$$

The predicate $\mathsf{present}(D)$ checks if all labels in D are present. Suppose $\langle L_1 | D_1 \rangle_F$ and $\langle L_2 | D_2 \rangle_F$ satisfies that $D_1 + (F' - L_1) \leqslant D_2 + (F' - L_2)$ (1) for all $F \leqslant F'$. Case analysis on the relationship between D_1 and D_2.

Case D_2 is longer than D_1. By (1) we have $D_2 \equiv D_1', D$ for $D_1 \leqslant D_1'$. Let $L = \mathsf{dom}(D)$. Using proof by contradiction, we can show that $L_2 \equiv L, L_1$, $\mathsf{present}(D)$, and $(F - L_1) \equiv D, E$ for some E; otherwise, we can always properly set F' to violate (1) meanwhile satisfying $F \leqslant F'$. Thus, this case is covered by MT-MultiExpand.

Case D_1 is longer than D_2. We have $D_1 \equiv D_2', D$ for $D_2' \leqslant D_2$. Similar to the above case, using proof by contradiction we can show that it is covered by MT-MultiShrink.

□

Our proofs for type soundness and effect safety do not use ad-hoc case analysis on shapes of modalities or rely on any specific properties about the definition of composition and transformation (except for the parts about effect handlers since they specify the required modalities in the typing rules). As a result, it should be possible to generalise our calculus and proofs to other mode theories satisfying certain extra properties. We state some properties of the mode theory as the following lemmas for easier references in proofs. Most of them directly follow from the definition.

Lemma B.4 (Vertical composition). *If $\mu_{F_1} \Rightarrow \nu_{F_2}$ and $\nu_{F_2} \Rightarrow \xi_{F_3}$, then $\mu_{F_1} \Rightarrow \xi_{F_3}$.*

Proof. Follow from Lemma B.2 □

Lemma B.5 (Horizontal composition). *If* $\mu_F : E \longrightarrow F$, $\mu'_{F'} : E' \longrightarrow F'$, $\mu_F \Rightarrow \mu'_{F'}$, *and* $\nu_E \Rightarrow \nu'_{E'}$, *then* $\mu_F \circ \nu_E \Rightarrow \mu'_{F'} \circ \nu'_{E'}$.

Proof. Follow from Lemma B.2 □

Lemma B.6 (Monotone modality transformation). *If* $\mu_F \Rightarrow \nu_F$ *and* $F \leqslant F'$, *then* $\mu_{F'} \Rightarrow \nu_{F'}$.

Proof. Follow from Lemma 4.1 □

Lemma B.7 (Asymmetric reflexivity of modality transformation). *If* $F \leqslant F'$ *and* $\mu_F : E \longrightarrow F$, *then* $\mu_F \Rightarrow \mu_{F'}$.

Proof. By definition. □

B.3 Lemmas for the Calculus

We prove structural and substitution lemmas for Met as well as some other auxiliary lemmas for proving type soundness.

Lemma B.8 (Canonical forms).

1. *If* $\vdash U : \mu A \mathbin{@} E$, *then* U *is of shape* $\mathbf{mod}_\mu\, U'$.
2. *If* $\vdash U : A \longrightarrow B \mathbin{@} E$, *then* U *is of shape* $\lambda x^A.M$.
3. *If* $\vdash U : \forall \alpha.A \mathbin{@} E$, *then* U *is of shape* $\Lambda \alpha.V$.
4. *If* $\vdash U : (A, B) \mathbin{@} E$, *then* U *is of shape* (U_1, U_2).
5. *If* $\vdash U : A + B \mathbin{@} E$, *then* U *is either of shape* $\mathbf{inl}\ U'$ *or of shape* $\mathbf{inr}\ U'$.

Proof. Directly follows from the typing rules. □

In order to define the lock weakening lemma, we first define a context update operation $(\!|\Gamma|\!)_{F'}$ which gives a new context derived from updating the indexes of all locks and variable bindings in Γ such that $(\!|\Gamma|\!)_{F'} \mathbin{@} F'$.

$$
\begin{array}{rcl}
(\!|\cdot|\!)_F & = & \cdot \\
(\!|\lock_{[E]_{F'}}, \Gamma'|\!)_F & = & \lock_{[E]_F}, \Gamma' \\
(\!|\lock_{\langle L \mid D \rangle_{F'}}, \Gamma'|\!)_F & = & \lock_{\langle L \mid D \rangle_F}, (\!|\Gamma'|\!)_{D+(F-L)} \\
(\!|x :_{\mu_{F'}} A, \Gamma'|\!)_F & = & x :_{\mu_F} A, (\!|\Gamma'|\!)_F \\
(\!|\alpha : K, \Gamma'|\!)_F & = & \alpha : K, (\!|\Gamma'|\!)_F
\end{array}
$$

The have the following lemma showing that the index update operation preserves the $\mathsf{locks}(-)$ operation except for updating the index.

Lemma B.9 (Index update preserves composition). *If* $\mu_F = \mathsf{locks}(\Gamma) : E \longrightarrow F$, $F \leqslant F'$, *and* $\mathsf{locks}((\!|\Gamma|\!)_{F'}) : E' \longrightarrow F'$, *then* $\mathsf{locks}((\!|\Gamma|\!)_{F'}) = \mu_{F'}$.

Proof. By straightforward induction on the context and using the property that $(\mu \circ \nu)_F = \mu_F \circ \nu_E$ for $\mu_F : E \longrightarrow F$. □

Corollary B.10 (Index update preserves transformation). *If* $\mathsf{locks}(\Gamma) : E \longrightarrow F$, $F \leqslant F'$, *and* $\mathsf{locks}((\!|\Gamma|\!)_{F'}) : E' \longrightarrow F'$, *then* $\mathsf{locks}(\Gamma) \Rightarrow \mathsf{locks}((\!|\Gamma|\!)_{F'})$.

Proof. Immediately follow from Lemma B.9 and Lemma B.7. □

We have the following structural lemmas.

Lemma B.11 (Structural rules). *The following structural rules are admissible.*

1. *Variable weakening.*

$$\frac{\Gamma, \Gamma' \vdash M : B \mathbin{@} E \qquad \Gamma, x :_{\mu_F} A, \Gamma' \mathbin{@} E}{\Gamma, x :_{\mu_F} A, \Gamma' \vdash M : B \mathbin{@} E}$$

2. *Variable swapping.*

$$\frac{\Gamma, x :_{\mu_F} A, y :_{\nu_F} B, \Gamma' \vdash M : A' \mathbin{@} E}{\Gamma, y :_{\nu_F} B, x :_{\mu_F} A, \Gamma' \vdash M : A' \mathbin{@} E}$$

3. *Lock weakening.*

$$\frac{\Gamma, \text{🔒}_{\mu_F}, \Gamma' \vdash M : A \mathbin{@} E \qquad \mu_F \Rightarrow \nu_F \qquad \nu_F : F' \to F \qquad \mathsf{locks}(\langle\!|\Gamma'|\!\rangle_{F'}) : E' \to F'}{\Gamma, \text{🔒}_{\nu_F}, \langle\!|\Gamma'|\!\rangle_{F'} \vdash M : A \mathbin{@} E'}$$

4. *Type variable weakening.*

$$\frac{\Gamma, \Gamma' \vdash M : B \mathbin{@} E}{\Gamma, \alpha : K, \Gamma' \vdash M : B \mathbin{@} E}$$

5. *Type variable swapping.*

$$\frac{\Gamma_1, \Gamma_2, \alpha : K, \Gamma_3 \vdash M : A \mathbin{@} E}{\Gamma_1, \alpha : K, \Gamma_3 \vdash M : A \mathbin{@} E} \qquad \frac{\alpha \notin \mathsf{ftv}(\Gamma_2) \qquad \Gamma_1, \alpha : K, \Gamma_3 \vdash M : A \mathbin{@} E}{\Gamma_1, \Gamma_2, \alpha : K, \Gamma_3 \vdash M : A \mathbin{@} E}$$

Proof. 1, 2, 4, and 5 follow from straightforward induction on the typing derivation. For 3, we also proceed by induction on the typing derivation. The most interesting case is T-Var. Other cases mostly follow from IHs.

Case

$$\frac{\text{T-Var}\quad \nu'_{F_1} = \mathsf{locks}(\Gamma_2) : E \to F_1 \qquad \mu'_{F_1} \Rightarrow \nu'_{F_1}\ (1) \text{ or } \Gamma \vdash A : \mathsf{Abs}}{\Gamma_1, x :_{\mu'_{F_1}}, \Gamma_2 \vdash x : A \mathbin{@} E}$$

Trivial when A is pure. Otherwise, case analysis on where the lock weakening happens.

Case Γ. Supposing $\Gamma_1 = \Gamma, \text{🔒}_{\mu_F}, \Gamma_0$ and after lock weakening we have $\Gamma, \text{🔒}_{\nu_F}, \Gamma'_0, x :_{\mu'_{F'_1}}, \Gamma'_2$ where $\Gamma'_2 = \langle\!|\Gamma_2|\!\rangle_{F'_1} : E' \to F'_1$ and $\Gamma'_0 = \langle\!|\Gamma_0|\!\rangle_{F'} : F'_1 \to F'$. By Lemma B.9 on Γ_0, $F \leqslant F'$, and Lemma B.3, we have $F_1 \leqslant F'_1$. Then by (1) and Lemma B.6, we have $\mu'_{F'_1} \Rightarrow \nu'_{F'_1}$. Then by Lemma B.9 we have $\nu'_{F'_1} = \mathsf{locks}(\Gamma'_2)$. Finally by T-Var we have

$$\Gamma, \text{🔒}_{\nu_F}, \Gamma'_0, x :_{\mu'_{F'_1}}, \Gamma'_2 \vdash x : A \mathbin{@} E'$$

Case Γ_2. Suppose $\Gamma_2 = \Gamma_0, \text{🔒}_{\mu_F}, \Gamma'$. is weakened to $\Gamma'_2 = \Gamma_0, \text{🔒}_{\nu_F}, \langle\!|\Gamma'|\!\rangle_{F'}$. By Corollary B.10 we have $\mathsf{locks}(\Gamma') \Rightarrow \mathsf{locks}(\langle\!|\Gamma'|\!\rangle_{F'})$. Then by Lemma B.5 we have we have $\mathsf{locks}(\Gamma_2) \Rightarrow \mathsf{locks}(\Gamma'_2)$. By Lemma B.4 and (1), we have $\mu'_{F_1} \Rightarrow \mathsf{locks}(\Gamma'_2)$. Finally by T-Var we have

$$\Gamma, x :_{\mu'_{F_1}}, \Gamma'_2 \vdash x : A \mathbin{@} E'$$

Case

$$\frac{\text{T-Mod}\quad \mu'_E : F_1 \to E \qquad \Gamma, \text{🔒}_{\mu_F}, \Gamma', \text{🔒}_{\mu'_E} \vdash V : A \mathbin{@} F_1\ (1)}{\Gamma, \text{🔒}_{\mu_F}, \Gamma' \vdash \mathbf{mod}_{\mu'}\ V : \mu' A \mathbin{@} E}$$

We have

$$\langle\!|\Gamma', \text{🔒}_{\mu'_E}|\!\rangle_{F'} = \langle\!|\Gamma'|\!\rangle_{F'}, \langle\!|\text{🔒}_{\mu'_E}|\!\rangle_{E'} = \langle\!|\Gamma'|\!\rangle_{F'}, \text{🔒}_{\mu'_{E'}}.$$

Supposing $\mu'_{E'} : F'_1 \longrightarrow E'$, by $\mathsf{locks}(\langle\!|\Gamma'|\!\rangle_{F'}, \blacksquare_{\mu'_{E'}}) : F'_1 \longrightarrow F'$ and IH on (1), we have

$$\Gamma, \blacksquare_{\mu_F}, \langle\!|\Gamma'|\!\rangle_{F'}, \blacksquare_{\mu'_{E'}} \vdash V : A \mathbin{@} F'_1.$$

Then by T-Mod we have

$$\Gamma, \blacksquare_{\mu_F}, \langle\!|\Gamma'|\!\rangle_{F'} \vdash \mathbf{mod}_{\mu'}\ V : \mu' A \mathbin{@} E'.$$

Case

T-Letmod

$$\frac{\nu'_E : F_1 \longrightarrow E \qquad \Gamma, \blacksquare_{\mu_F}, \Gamma', \blacksquare_{\nu'_E} \vdash V : \mu' A \mathbin{@} F_1\ (1) \qquad \Gamma, \blacksquare_{\mu_F}, \Gamma', x :_{\nu'_E \circ \mu'_{F_1}} A \vdash M : B \mathbin{@} E\ (2)}{\Gamma, \blacksquare_{\mu_F}, \Gamma' \vdash \mathbf{let}_{\nu'}\ \mathbf{mod}_{\mu'}\ x = V\ \mathbf{in}\ M : B \mathbin{@} E}$$

By IH on (1), we have

$$\Gamma, \blacksquare_{\nu_F}, \langle\!|\Gamma'|\!\rangle_{F'}, \blacksquare_{\nu'_{E'}} \vdash V : \mu' A \mathbin{@} F'_1$$

where $\nu'_{E'} : F'_1 \longrightarrow E'$. By IH on (2), we have

$$\Gamma, \blacksquare_{\nu_F}, \langle\!|\Gamma'|\!\rangle_{F'}, x :_{\nu'_{E'} \circ \mu'_{F'_1}} A \vdash M : B \mathbin{@} E'.$$

Then by T-Letmod, we have

$$\Gamma, \blacksquare_{\mu_F}, \langle\!|\Gamma'|\!\rangle_{F'} \vdash \mathbf{let}_{\nu'}\ \mathbf{mod}_{\mu'}\ x = V\ \mathbf{in}\ M : B \mathbin{@} E'$$

Case

T-Letmod'

$$\frac{\nu'_E : F_1 \longrightarrow E \qquad \Gamma, \blacksquare_{\mu_F}, \Gamma', \blacksquare_{\nu'_E}, \overline{\alpha : K} \vdash V : \mu' A \mathbin{@} F_1\ (1) \qquad \Gamma, \blacksquare_{\mu_F}, \Gamma', x :_{\nu'_E \circ \mu'_{F_1}} \forall \overline{\alpha^K}.A \vdash M : B \mathbin{@} E\ (2)}{\Gamma, \blacksquare_{\mu_F}, \Gamma' \vdash \mathbf{let}_{\nu'}\ \mathbf{mod}_{\mu'}\ \Lambda \overline{\alpha^K}.x = V\ \mathbf{in}\ M : B \mathbin{@} E}$$

Similar to the case for T-Letmod. By IH on (1), we have

$$\Gamma, \blacksquare_{\nu_F}, \langle\!|\Gamma'|\!\rangle_{F'}, \blacksquare_{\nu'_{E'}}, \overline{\alpha : K} \vdash V : \mu' A \mathbin{@} F'_1$$

where $\nu'_{E'} : F'_1 \longrightarrow E'$. By IH on (2), we have

$$\Gamma, \blacksquare_{\nu_F}, \langle\!|\Gamma'|\!\rangle_{F'}, x :_{\nu'_{E'} \circ \mu'_{F'_1}} \forall \overline{\alpha^K}.A \vdash M : B \mathbin{@} E'.$$

Then by T-Letmod', we have

$$\Gamma, \blacksquare_{\nu_F}, \langle\!|\Gamma'|\!\rangle_{F'} \vdash \mathbf{let}_{\nu'}\ \mathbf{mod}_{\mu'}\ \Lambda \overline{\alpha^K}.x = V\ \mathbf{in}\ M : B \mathbin{@} E'$$

Case T-TAbs, T-Abs, T-TApp, T-App, T-Do, T-Mask, T-Handler, T-ModAbs, other handlers and data types. Follow from IH. Similar to other cases we have shown.

□

As a corollary of Lemma B.11.3, the following sub-effecting rule is admissible.

Corollary B.12 (Sub-effecting). *The following rule is admissible.*

$$\frac{\Gamma \vdash M : A \mathbin{@} E \qquad \mathsf{locks}(\Gamma) : E \longrightarrow F \qquad F \leqslant F' \qquad \mathsf{locks}(\langle\!|\Gamma|\!\rangle_{F'}) : E' \longrightarrow F'}{\langle\!|\Gamma|\!\rangle_{F'} \vdash M : A \mathbin{@} E'}$$

Proof. Follow from Lemma B.11.3 by adding the lock $\blacksquare_{[F]}$ to the left of Γ in $\Gamma \vdash M : A \mathbin{@} E$, and weaken it to $\blacksquare_{[F']}$. Note that typing judgements still hold after adding a lock to or removing a lock from the left of the context, as long as the new contexts are still well-defined. □

The following lemma reflects the intuition that pure values can be used in any effect context.

Lemma B.13 (Pure Promotion). *The following promotion rule is admissible.*

$$\frac{\Gamma_1, \Gamma \vdash V : A \mathbin{@} E \qquad \Gamma_1 \vdash A : \mathsf{Abs} \qquad \mathsf{locks}(\Gamma) : E \to F \qquad \mathsf{locks}(\Gamma') : E' \to F \qquad \mathsf{fv}(V) \cap \mathsf{dom}(\Gamma') = \emptyset}{\Gamma_1, \Gamma' \vdash V : A \mathbin{@} E'}$$

Proof. By induction on the typing derivation of V.

Case T-Var. Trivial.

Case

$$\frac{\mu_E : F_1 \to E \qquad \Gamma_1, \Gamma, \blacksquare_{\mu_E} \vdash V : A \mathbin{@} F_1 \ (1)}{\Gamma_1, \Gamma \vdash \mathbf{mod}_\mu\, V : \mu A \mathbin{@} E} \ \text{T-Mod}$$

Case analysis on the shape of μ.

Case μ is relative. By kinding, A is also pure. By IH on (1), we have

$$\Gamma_1, \Gamma', \blacksquare_{\mu_{E'}} \vdash V : A \mathbin{@} F_1'$$

where $\mu_{E'} : F_1' \to E'$. Then by T-Mod we have

$$\Gamma_1, \Gamma' \vdash \mathbf{mod}_\mu\, V : \mu A \mathbin{@} E'$$

Case μ is absolute. We have $\mu = [F_1]$ and $\mathsf{locks}(\Gamma', \blacksquare_{\mu_{E'}}) = [F_1]_F = \mathsf{locks}(\Gamma, \blacksquare_{\mu_E})$. Thus, replacing the context $(\Gamma, \blacksquare_{\mu_E})$ with $(\Gamma', \blacksquare_{\mu_{E'}})$ in (1) does not influence all usages of T-Var in the derivation tree of (1). We have

$$\Gamma_1, \Gamma', \blacksquare_{\mu_{E'}} \vdash V : A \mathbin{@} F_1$$

Then by T-Mod we have

$$\Gamma_1, \Gamma' \vdash \mathbf{mod}_\mu\, V : \mu A \mathbin{@} E'$$

Case T-TAbs. Follow from IH and Lemma B.11.5.

Case T-Abs. Impossible since function types are impure.

Case Data types. Follow from IHs.

□

Lemma B.14 (Substitution). *The following substitution rules are admissible.*

1. *Preservation of kinds under type substitution.*

$$\frac{\Gamma \vdash A : K \qquad \Gamma, \alpha : K, \Gamma' \vdash B : K'}{\Gamma, \Gamma' \vdash B[A/\alpha] : K'}$$

2. *Preservation of types under type substitution.*

$$\frac{\Gamma \vdash A : K \qquad \Gamma, \alpha : K, \Gamma' \vdash M : B \mathbin{@} E}{\Gamma, \Gamma' \vdash M[A/\alpha] : B[A/\alpha] \mathbin{@} E}$$

3. *Preservation of types under value substitution.*

$$\frac{\Gamma, \blacksquare_{\mu_F} \vdash V : A \mathbin{@} F' \qquad \Gamma, x :_{\mu_F} A, \Gamma' \vdash M : B \mathbin{@} E}{\Gamma, \Gamma' \vdash M[V/x] : B \mathbin{@} E}$$

Proof.

1. By straightforward induction on the kinding derivation.
2. By straightforward induction on the typing derivation of M.
3. By induction on the typing derivation of M. Trivial when variable x is not used. In the following induction we always assume x is used.

Case

$$\text{T-Var}\ \frac{\nu_F = \mathsf{locks}(\Gamma') : E \to F \qquad \mu_F \Rightarrow \nu_F\ (1) \text{ or } \Gamma \vdash A : \mathsf{Abs}}{\Gamma, x :_{\mu_F} A, \Gamma' \vdash x : A \ @\ E}$$

Case analysis on the purity of A

Case Impure. By $\Gamma, \text{🔒}_{\mu_F} \vdash V : A \ @\ F'$, (1), and Lemma B.11.3, we have

$$\Gamma, \text{🔒}_{\nu_F} \vdash V : A \ @\ E.$$

Then, by context equivalence, Lemma B.11.1, and Lemma B.11.4, we have

$$\Gamma, \Gamma' \vdash V : A \ @\ E.$$

Case Pure. By $\Gamma, \text{🔒}_{\mu_F} \vdash V : A \ @\ F'$ and Lemma B.13, we have

$$\Gamma, \Gamma' \vdash V : A \ @\ E.$$

Case

$$\text{T-Mod}\ \frac{\mu'_E : F_1 \to E \qquad \Gamma, x :_{\mu_F} A, \Gamma', \text{🔒}_{\mu'_E} \vdash W : B \ @\ F_1\ (1)}{\Gamma, x :_{\mu_F} A, \Gamma' \vdash \mathbf{mod}_{\mu'}\ W : \mu' B \ @\ E}$$

By IH on (1) we have

$$\Gamma, \Gamma', \text{🔒}_{\mu'_E} \vdash W[V/x] : B \ @\ F_1.$$

Then by T-Mod we have

$$\Gamma, \Gamma' \vdash (\mathbf{mod}_{\mu'}\ W)[V/x] : \mu' B \ @\ E$$

Case

$$\text{T-Letmod}\ \frac{\nu_E : F_1 \to E \qquad \Gamma, x :_{\mu_F} A, \Gamma', \text{🔒}_{\nu_E} \vdash W : \mu' A' \ @\ F_1\ (1) \qquad \Gamma, x :_{\mu_F} A, \Gamma', y :_{\nu_E \circ \mu'_{F_1}} A' \vdash M : B \ @\ E\ (2)}{\Gamma, x :_{\mu_F} A, \Gamma' \vdash \mathbf{let}_\nu\ \mathbf{mod}_{\mu'}\ y = W\ \mathbf{in}\ M : B \ @\ E}$$

By IH on (1), we have

$$\Gamma, \Gamma', \text{🔒}_{\nu_E} \vdash W[V/x] : \mu' A' \ @\ F_1.$$

By IH on (2), we have

$$\Gamma, \Gamma', y :_{\nu_E \circ \mu'_{F_1}} A' \vdash M[V/x] : B \ @\ E.$$

Then by T-Letmod, we have

$$\Gamma, \Gamma' \vdash (\mathbf{let}_\nu\ \mathbf{mod}_{\mu'}\ y = W\ \mathbf{in}\ M)[V/x] : B \ @\ E$$

Case

$$\text{T-Letmod'}\ \frac{\nu_E : F_1 \to E \qquad \Gamma, x :_{\mu_F} A, \Gamma', \text{🔒}_{\nu_E}, \overline{\alpha : K} \vdash V : \mu' A' \ @\ F_1\ (1) \qquad \Gamma, x :_{\mu_F} A, \Gamma', y :_{\nu_E \circ \mu'_{F_1}} \forall \overline{\alpha^K}.A' \vdash M : B \ @\ E\ (2)}{\Gamma, x :_{\mu_F} A, \Gamma' \vdash \mathbf{let}_\nu\ \mathbf{mod}_{\mu'}\ \Lambda \overline{\alpha^K}.y = V\ \mathbf{in}\ M : B \ @\ E}$$

Similar to the case for T-Letmod. Our goal follows from IH on (1), IH on (2), and T-Letmod'.

Case

$$\text{T-Mask}\ \frac{\Gamma, x :_{\mu_F} A, \Gamma', \blacksquare_{\langle L \rangle_E} \vdash M : B \mathbin{@} E - L\ (1)}{\Gamma, x :_{\mu_F} A, \Gamma' \vdash \mathbf{mask}_L\ M : \langle L \rangle B \mathbin{@} E}$$

By IH on (1) we have

$$\Gamma, \Gamma', \blacksquare_{\langle L \rangle_E} \vdash M[V/x] : B \mathbin{@} E - L.$$

Then by T-Mask we have

$$\Gamma, \Gamma' \vdash (\mathbf{mask}_L\ M)[V/x] : \langle L \rangle B \mathbin{@} E$$

Case

$$\text{T-Handler}\ \frac{\begin{array}{c} D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, x :_{\mu_F} A, \Gamma', \blacksquare_{\langle D \rangle_E} \vdash M : A_0 \mathbin{@} D + E\ (1) \\ \Gamma, x :_{\mu_F} A, \Gamma', y : \langle\!\langle D \rangle A_0 \vdash N : B \mathbin{@} E\ (2) \\ [\Gamma, x :_{\mu_F} A, \Gamma', p_i : A_i, r_i : B_i \to B \vdash N_i : B \mathbin{@} E\ (3)]_i \end{array}}{\Gamma, x :_{\mu_F} A, \Gamma' \vdash \mathbf{handle}\ M\ \mathbf{with}\ \{\mathbf{return}\ y \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i : B \mathbin{@} E}$$

Follow from IH on (1),(2),(3), and reapplying T-Handler.

Case T-TAbs, T-TApp, T-Abs, T-App, T-Do. Follow from IH.

Case T-ModAbs, other handlers and data types. Follow from IH.

□

B.4 Progress

Theorem 4.3 (Progress). *If $\vdash M : A \mathbin{@} E$, then either there exists N such that $M \rightsquigarrow N$ or M is in a normal form with respect to E.*

Proof. By induction on the typing derivation $\vdash M : A \mathbin{@} E$. The most non-trivial cases are T-Mask and T-Handler. Other cases follow from IHs and reduction rules, using Lemma B.8.

Case M is in a value normal form U. Trivial. Base case.

Case T-Do. Trivial. Base case.

Case T-Mod. $\mathbf{mod}_\mu\ V$. By IH on V.

Case T-Letmod. $\mathbf{let}_\nu\ \mathbf{mod}_\mu\ x = V\ \mathbf{in}\ N$. By IH on V, if V is reducible then M is reducible; otherwise, V is in a value normal form, then by Lemma B.8 we have that M is reducible by E-Letmod.

Case T-Letmod'. Similar to the case for T-Letmod.

Case T-TApp. $M\ A$. Similarly by IH on M, Lemma B.8, and E-TApp.

Case T-App. $M\ N$. Similarly by IH on M and N, Lemma B.8, and E-App.

Case T-Mask. $\mathbf{mask}^E\ M$. By IH on M.

- Case M is reducible. Trivial.
- Case M is in a value normal form. By E-Mask.
- Case $M = \mathcal{E}[\mathbf{do}\ \ell\ U]$ with $n\text{–free}(\ell, \mathcal{E})$. The whole term is in a normal form.

Case Handlers. The general form is $\mathbf{handle}^\delta\ M\ \mathbf{with}\ H$. By IH on M.

- Case M is reducible. Trivial.
- Case M is in a value normal form. By E-Ret.
- Case $M = \mathcal{E}[\mathbf{do}\ \ell\ U]$ with $n\text{–free}(\ell, \mathcal{E})$. If $n = 0$ and $\ell \in H$, then reducible by E-Op. Otherwise, the whole term is in a normal form.

Case T-ModAbs. $\mathbf{mod}_{[]}\ M$. If $M \rightsquigarrow N$, follow by IH on M. Otherwise, M must be in a value normal form because the T-ModAbs requires M to have the empty effect. In this case, $\mathbf{mod}_{[]}\ M$ is also in a value normal form.

Case Other handlers and data types. Similar to other cases.

□

B.5 Subject Reduction

Theorem 4.4 (Subject Reduction). *If* $\Gamma \vdash M : A \mathbin{@} E$ *and* $M \rightsquigarrow N$, *then* $\Gamma \vdash N : A \mathbin{@} E$.

Proof. By induction on the typing derivation $\Gamma \vdash M : A \mathbin{@} E$.

Case T-Var. Impossible as there is no further reduction.

Case

$$\frac{\text{T-Mod}\quad \mu_F : E \longrightarrow F \qquad \Gamma, \lock_{\mu_F} \vdash V : A \mathbin{@} E\ (1)}{\Gamma \vdash \mathbf{mod}_{\mu}\ V : \mu A \mathbin{@} F}$$

The only way to reduce is by E-Lift and $V \rightsquigarrow W$. IH on (1) gives

$$\Gamma, \lock_{\mu_F} \vdash W : A \mathbin{@} E.$$

Then by T-Mod we have

$$\Gamma \vdash \mathbf{mod}_{\mu}\ W : \mu A \mathbin{@} F.$$

Case

$$\frac{\text{T-Letmod}\quad \nu_F : E \longrightarrow F \qquad \Gamma, \lock_{\nu_F} \vdash V : \mu A \mathbin{@} E\ (1) \qquad \Gamma, x :_{\nu_F \circ \mu_E} A \vdash M : B \mathbin{@} F\ (2)}{\Gamma \vdash \mathbf{let}_{\nu}\ \mathbf{mod}_{\mu}\ x = V\ \mathbf{in}\ M : B \mathbin{@} F}$$

By case analysis on the reduction.

Case E-Lift with $V \rightsquigarrow W$. By IH on (1) and reapplying T-Letmod.

Case E-Letmod. We have $V = \mathbf{mod}_{\mu}\ U$ and

$$\mathbf{let}_{\nu}\ \mathbf{mod}_{\mu}\ x = \mathbf{mod}_{\mu}\ U\ \mathbf{in}\ M \rightsquigarrow M[U/x].$$

Inversion on (1) gives

$$\Gamma, \lock_{\nu_F}, \lock_{\mu_E} \vdash U : A \mathbin{@} E'.$$

where $\mu_E : E' \longrightarrow E$. By context equivalence, we have

$$\Gamma, \lock_{\nu_F \circ \mu_E} \vdash U : A \mathbin{@} E'$$

where $\nu_F \circ \mu_E : E' \longrightarrow F$. By Lemma B.14.3 and (2), we have

$$\Gamma \vdash M[U/x] : B \mathbin{@} F.$$

Case

$$\frac{\text{T-Letmod'}\quad \nu_F : E \longrightarrow F \qquad \Gamma, \lock_{\nu_F}, \overline{\alpha : K} \vdash V : \mu A \mathbin{@} E\ (1) \qquad \Gamma, x :_{\nu_F \circ \mu_E} \forall \overline{\alpha^K}.A \vdash M : B \mathbin{@} F\ (2)}{\Gamma \vdash \mathbf{let}_{\nu}\ \mathbf{mod}_{\mu}\ \Lambda\overline{\alpha^K}.x = V\ \mathbf{in}\ M : B \mathbin{@} F}$$

Similar to the case for T-Letmod'. By case analysis on the reduction.

Case E-Lift with $V \rightsquigarrow W$. By IH on (1) and reapplying T-Letmod'.

Case E-LetMod'. We have $V = \mathbf{mod}_\mu\, U$ and

$$\mathbf{let}_\nu\ \mathbf{mod}_\mu\ \Lambda\overline{\alpha^K}.x = \mathbf{mod}_\mu\, U\ \mathbf{in}\ M \rightsquigarrow M[(\forall\overline{\alpha^K}.U)/x].$$

Inversion on (1) gives

$$\Gamma, \blacksquare_{\nu_F}, \overline{\alpha : K}, \blacksquare_{\mu_E} \vdash U : \mu A \mathbin{@} E'.$$

where $\mu_E : E' \to E$. By Lemma B.11.5 we have

$$\Gamma, \blacksquare_{\nu_F}, \blacksquare_{\mu_E}, \overline{\alpha : K} \vdash U : A \mathbin{@} E'.$$

By context equivalence, we have

$$\Gamma, \blacksquare_{\nu_F \circ \mu_E}, \overline{\alpha : K} \vdash U : A \mathbin{@} E'.$$

where $\nu_F \circ \mu_E : E' \to F$. By T-TAbs we have

$$\Gamma, \blacksquare_{\nu_F \circ \mu_E} \vdash \Lambda\overline{\alpha^K}.U : \forall\overline{\alpha^K}.A \mathbin{@} E'.$$

By Lemma B.14.3 and (2), we have

$$\Gamma \vdash M[U/x] : B \mathbin{@} F.$$

Case T-TAbs,T-Abs. Impossible as there is no further reduction.

Case

$$\text{T-TApp}\ \frac{\Gamma \vdash M : \forall\alpha^K.B \mathbin{@} E\ (1) \qquad \Gamma \vdash A : K\ (2)}{\Gamma \vdash M\,A : B[A/\alpha] \mathbin{@} E}$$

By case analysis on the reduction.

Case E-Lift with $M \rightsquigarrow N$. By IH on (1) and reapplying T-TApp.

Case E-TApp. We have $M = \Lambda\alpha^K.V$ and

$$(\Lambda\alpha^K.V)\,A \rightsquigarrow V[A/\alpha].$$

Inversion on (1) gives

$$\Gamma, \alpha : K \vdash V : B \mathbin{@} E.$$

Then by Lemma B.14.2 on (2), we have

$$\Gamma \vdash V[A/\alpha] : B[A/\alpha] \mathbin{@} E.$$

Case

$$\text{T-App}\ \frac{\Gamma \vdash M : A \to B \mathbin{@} E\ (1) \qquad \Gamma \vdash N : A \mathbin{@} E\ (2)}{\Gamma \vdash M\,N : B \mathbin{@} E}$$

By case analysis on the reduction.

Case E-Lift with $M \rightsquigarrow M'$. By IH on (1) and reapplying T-App.

Case E-Lift with $N \rightsquigarrow N'$. By IH on (2) and reapplying T-App.

Case E-App. We have $M = \lambda x^A.M'$, $N = U$, and

$$M\,N \rightsquigarrow M'[U/x].$$

Inversion on (1) gives

$$\Gamma, x : A \vdash M' : B \mathbin{@} E.$$

Then by Lemma B.14.3 we have

$$\Gamma \vdash M'[U/x] : B \mathbin{@} E.$$

Case T-Do. The only way to reduce is by E-Lift. Follow from IH and reapplying T-Do.

Case

$$\textsc{T-Mask}\ \frac{\Gamma, \blacksquare_{\langle L \rangle_F} \vdash M : A \mathbin{@} F - L \ (1)}{\Gamma \vdash \mathbf{mask}_L\ M : \langle L \rangle A \mathbin{@} F}$$

By case analysis on the reduction.

Case E-Lift with $M \rightsquigarrow N$. By IH on (1) and reapplying T-Mask.

Case E-Mask. We have $M = U$ and

$$\mathbf{mask}_L\ U \rightsquigarrow \mathbf{mod}_{\langle L \rangle}\ U.$$

By $\langle L \rangle_F : F - L \longrightarrow F$ and T-Mod, we have

$$\Gamma \vdash \mathbf{mod}_{\langle L \rangle}\ U : \langle L \rangle A \mathbin{@} F.$$

Case

$$\textsc{T-Handler}\ \frac{\begin{array}{c} H = \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i \\ D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \blacksquare_{\langle D \rangle_F} \vdash M : A \mathbin{@} D + F \ (1) \\ \Gamma, x : \langle D \rangle A \vdash N : B \mathbin{@} F \ (2) \qquad [\Gamma, p_i : A_i, r_i : B_i \longrightarrow B \vdash N_i : B \mathbin{@} F \ (3)]_i \end{array}}{\Gamma \vdash \mathbf{handle}\ M\ \mathbf{with}\ H : B \mathbin{@} F}$$

By case analysis on the reduction.

Case E-Lift with $M \rightsquigarrow M'$. By IHs and reapplying T-Handler.

Case E-Ret. We have $M = U$ and

$$\mathbf{handle}\ U\ \mathbf{with}\ H \rightsquigarrow N[(\mathbf{mod}_{\langle D \rangle}\ U)/x].$$

By (1), $\langle D \rangle_F : F \longrightarrow D + F$, and T-Mod, we have

$$\Gamma \vdash \mathbf{mod}_{\langle D \rangle}\ U : A \mathbin{@} F.$$

Then by (2) and Lemma B.14.3 we have

$$\Gamma \vdash N[(\mathbf{mod}_{\langle D \rangle}\ U)/x] : B \mathbin{@} F.$$

Case E-Op. We have $M = \mathcal{E}[\mathbf{do}\ \ell_j\ U]$, $0\text{–free}(\ell_j, \mathcal{E})$, $\ell_j\ p_j\ r_j \mapsto N_j$, and

$$\mathbf{handle}\ M\ \mathbf{with}\ H \rightsquigarrow N_j[U/p, (\lambda y.\mathbf{handle}\ \mathcal{E}[y]\ \mathbf{with}\ H)/r].$$

Since D is well-kinded, A_j and B_j are pure. By inversion on $\mathbf{do}\ \ell_j\ U$ we have

$$\Gamma, \blacksquare_{\langle D \rangle_F} \vdash U : A_j \mathbin{@} D + F.$$

By A_j is pure and Lemma B.13, we have

$$\Gamma, \blacksquare_{\langle D \rangle_F}, \blacksquare_{\langle L \rangle_{D+F}} \vdash U : A_j \mathbin{@} F$$

where $L = \mathrm{dom}(D)$. By context equivalence, we have

$$\Gamma \vdash U : A_j \mathbin{@} F \ (4)$$

Observe that B_j being pure allows $y : B_j$ to be accessed in any context. By (1) and a straightforward induction on $\mathcal{E}$ we have

$$\Gamma, y : B_j, \blacksquare_{\langle D \rangle_F} \vdash \mathcal{E}[y] : A \mathbin{@} D + F.$$

Then by T-Handler and T-Abs we have

$$\Gamma \vdash \lambda y.\mathbf{handle}\ \mathcal{E}[y]\ \mathbf{with}\ H : B_j \longrightarrow B \mathbin{@} F \ (5).$$

Finally, by (3), (4), (5), and Lemma B.14.3 we have

$$\Gamma \vdash N_j[U/p, (\lambda y.\mathbf{handle}\ \mathcal{E}[y]\ \mathbf{with}\ H)/r] : B \mathbin{@} F.$$

Case

$$\frac{\begin{array}{c}\text{T-Handler}^{\mathbb{A}} \\ D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \blacksquare_{[D+E]_F} \vdash M : A \mathbin{@} D + E\ (1) \\ \Gamma, \blacksquare_{[E]_F}, x : [D+E]A \vdash N : B \mathbin{@} F\ (2) \\ [\Gamma, \blacksquare_{[E]_F}, p_i : A_i, r_i : [E](B_i \to B) \vdash N_i : B \mathbin{@} E\ (3)]_i \qquad [E]_F \Rightarrow \mathbb{1}_F\end{array}}{\Gamma \vdash \mathbf{handle}^{\mathbb{A}}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i : B \mathbin{@} F}$$

By case analysis on the reduction.

Case E-Lift with $M \rightsquigarrow M'$. By IHs and reapplying T-Handle$^{\mathbb{A}}$.

Case E-Ret$^{\mathbb{A}}$. We have $M = U$ and

$$\mathbf{handle}\ M\ \mathbf{with}\ H \rightsquigarrow N[(\mathbf{mod}_{[D+E]}U)/x].$$

By (1) and $[D+E]_F = [E]_F \circ [D+E]_E$ we have

$$\Gamma, \blacksquare_{[E]_F}, \blacksquare_{[D+E]_E} \vdash U : A \mathbin{@} E.$$

By $[D+E]_E : D + E \to E$ and T-Mod, we have

$$\Gamma, \blacksquare_{[E]_F} \vdash \mathbf{mod}_{[D+E]}\ U : [D+E]A \mathbin{@} E.$$

Then by (2) and Lemma B.14.3 we have

$$\Gamma, \blacksquare_{[E]_F} \vdash N[(\mathbf{mod}_{[D+E]}\ U)/x] : B \mathbin{@} E.$$

By $[E]_F \Rightarrow \mathbb{1}_F$ and Lemma B.11.3 we have

$$\Gamma \vdash N[(\mathbf{mod}_{[D+E]}\ U)/x] : B \mathbin{@} F.$$

Case E-Op$^{\mathbb{A}}$. We have $M = \mathcal{E}[\mathbf{do}\ \ell_j\ U]$, $0\text{–free}(\ell_j, \mathcal{E})$, $\ell_j\ p_j\ r_j \mapsto N_j$, and

$$\mathbf{handle}^{\mathbb{A}}\ M\ \mathbf{with}\ H \rightsquigarrow N_j[U/p, (\mathbf{mod}_{[E]}\ (\lambda y.\mathbf{handle}^{\mathbb{A}}\ \mathcal{E}[y]\ \mathbf{with}\ H))/r].$$

Since D is well-kinded, A_j and B_j are pure. By inversion on $\mathbf{do}\ \ell_j\ U$, we have

$$\Gamma, \blacksquare_{[D+E]_F} \vdash U : A_j \mathbin{@} D + E.$$

By A_j is pure and Lemma B.13, we have

$$\Gamma \vdash U : A_j \mathbin{@} F\ (4).$$

Observe that B_j being pure allows y to be accessed in any context. By (1) and a straightforward induction on $\mathcal{E}$ we have

$$\Gamma, y : B_j, \blacksquare_{[D+E]_F} \vdash \mathcal{E}[y] : A \mathbin{@} D + E.$$

By $[E]_F \circ [E]_E \circ [D+E]_E = [D+E]_F$ and context equivalence, we have

$$\Gamma, y : B_j, \blacksquare_{[E]_F}, \blacksquare_{[E]_E}, \blacksquare_{[D+E]_E} \vdash \mathcal{E}[y] : A \mathbin{@} D + E.$$

Since B_j is pure, we can swap $y : B_j$ with locks and derive

$$\Gamma, \blacksquare_{[E]_F}, \blacksquare_{[E]_E}, y : B_j, \blacksquare_{[D+E]_E} \vdash \mathcal{E}[y] : A \mathbin{@} D + E.$$

By T-Handler$^{\mathbb{A}}$, we have

$$\Gamma, \blacksquare_{[E]_F}, \blacksquare_{[E]_E}, y : B_j \vdash \mathbf{handle}^{\mathbb{A}}\ \mathcal{E}[y]\ \mathbf{with}\ H : B \mathbin{@} E.$$

Notice that we can put H after absolute locks because all clauses in H have an absolute lock $🔒_{[E]_E}$ in their contexts. Then by T-Abs and T-Mod we have

$$\Gamma, 🔒_{[E]_F} \vdash \mathbf{mod}_{[E]}\ (\lambda y.\mathbf{handle}^{\mathbb{A}}\ \mathcal{E}[y]\ \mathbf{with}\ H) : [E](B_j \to B)\ @\ E\ (5).$$

By (3), (4), (5), and Lemma B.14.3 we have

$$\Gamma, 🔒_{[E]_F} \vdash N_j[U/p, (\mathbf{mod}_{[F]}\ (\lambda y.\mathbf{handle}\ \mathcal{E}[y]\ \mathbf{with}\ H))/r] : B\ @\ E.$$

Finally, by $[E]_F \Rightarrow \mathbb{1}_F$ and Lemma B.11.3 we have

$$\Gamma \vdash N_j[U/p, (\mathbf{mod}_{[F]}\ (\lambda y.\mathbf{handle}\ \mathcal{E}[y]\ \mathbf{with}\ H))/r] : B\ @\ F.$$

Case Shallow handlers. Similar to the cases of deep handlers.

Case Data types. Nothing more special than the cases we have already shown. Introduction rules follows from IHs and reapplying the same typing rules. Elimination rules require to additionally consider their corresponding reduction rules.

□

C Specification, Proof, and Discussion of the Encoding

In this section, we show the full encoding of $\mathrm{F}^1_{\mathrm{eff}}$ into Met, prove its type preservation, and discuss its extensibility. The definition of $\mathrm{F}^1_{\mathrm{eff}}$ is in Section 5.1 and the definition of Met is in Section 4.

C.1 Full Encoding

We repeat the encoding of types and contexts here for easy reference.

$$\begin{aligned} [\![1]\!]_E &= \mathbb{1}1 \\ [\![A \to^F B]\!]_E &= \langle E - F | F - E\rangle([\![A]\!]_F \to [\![B]\!]_F) \\ [\![\forall.A]\!]_E &= []\,[\![A]\!]. \end{aligned} \qquad \begin{aligned} [\![\cdot]\!]_E &= \cdot \\ [\![\Gamma, x : A]\!]_E &= [\![\Gamma]\!]_E, x :_{\mu_E} A' \text{ for } \mu A' = [\![A]\!]_E \\ [\![\Gamma, \blacklozenge_F]\!]_E &= [\![\Gamma]\!]_F, 🔒_{\langle F-E|E-F\rangle} \\ [\![\Gamma, \blacklozenge^{\wedge}_F]\!]. &= [\![\Gamma]\!]_F, 🔒_{[]} \end{aligned}$$

Figure 11 shows the translation from $\mathrm{F}^1_{\mathrm{eff}}$ terms with their types and effect contexts to Met terms. We use the following syntactic sugar to simply the encoding.

$$\begin{aligned} \mathbf{let}\ \mathbf{mod}_{\mu}\ x = M\ \mathbf{in}\ N &\doteq (\lambda x.\mathbf{let}\ \mathbf{mod}_{\mu}\ x = x\ \mathbf{in}\ N)\ M \\ \mathbf{let}\ \mathbf{mod}_{\mu;\nu}\ x = V\ \mathbf{in}\ M &\doteq \mathbf{let}\ \mathbf{mod}_{\mu}\ x = V\ \mathbf{in}\ \mathbf{let}_{\mu}\ \mathbf{mod}_{\nu}\ x = x\ \mathbf{in}\ M \end{aligned}$$

We use an auxiliary function $\mathrm{topmod}(-)$ to extract the top-level modality.

$$\mathrm{topmod}(\mu A) = \mu$$

In the term translation, all terms are translated to boxed terms with proper modalities consistent with those given by the type translation. Recall that Met uses let-style unboxing; we cannot immediately unbox values at the place we need. To get a systematic encoding, we *greedily unbox* top-level modalities for term variables when they are bound, and rebox them when they are used.

Greedy unboxing happens for variable bindings such as λ-abstractions and handlers. In the R-Abs case, we unbox the top-level modality of variable x. Additionally, we box the whole function with the relative modality $\langle E - F | F - E\rangle$, reflecting the effect context transition. In the R-Handler case, we similarly unbox the bound variables for return and operation clauses. In the operation clauses (N_i''), we need only unbox the operation argument p_i; the resumption function r_i is introduced under the current effect context E. In the return clause (N''), we unbox x with $\langle \overline{\ell_i}\rangle \circ \mu$ and then transform this modality to μ' given by $\mathrm{topmod}([\![A]\!]_E)$ in order to match the effect context E.

$\boxed{M : A\,!\,E \dashrightarrow M'}$

$$\text{R-Var}\ \frac{\mu := \mathrm{topmod}(\llbracket A \rrbracket_E)}{x : A\,!\,E \dashrightarrow \mathbf{mod}_\mu\, x} \qquad \text{R-App}\ \frac{M : A \to^E B\,!\,E \dashrightarrow M' \qquad N : A\,!\,E \dashrightarrow N' \qquad x\ \text{fresh}}{M\,N : B\,!\,E \dashrightarrow \mathbf{let\ mod}_{\mathbb{1}}\ x = M'\ \mathbf{in}\ x\,N'}$$

$$\text{R-Abs}\ \frac{M : B\,!\,F \dashrightarrow M' \qquad \nu := \langle E - F \mid F - E \rangle \qquad \mu := \mathrm{topmod}(\llbracket A \rrbracket_F)}{\lambda^F x^A.M : A \to^F B\,!\,E \dashrightarrow \mathbf{mod}_\nu\,(\lambda x^{\llbracket A \rrbracket_F}.\mathbf{let\ mod}_\mu\ x = x\ \mathbf{in}\ M')} \qquad \text{R-EAbs}\ \frac{V : A\,!\,\cdot \dashrightarrow V'}{\Lambda.V : \forall.A\,!\,E \dashrightarrow \mathbf{mod}_{[]}\, V'}$$

$$\text{R-EApp}\ \frac{M : \forall.A\,!\,E \dashrightarrow M' \qquad x\ \text{fresh}}{M\,@ : A[E/]\,!\,E \dashrightarrow \mathbf{let\ mod}_{[]}\ x = M'\ \mathbf{in}\ x} \qquad \text{R-Do}\ \frac{M : A\,!\,\ell, E \dashrightarrow M'}{\mathbf{do}\ \ell\ M : B\,!\,\ell, E \dashrightarrow \mathbf{do}\ \ell\ M'}$$

$$\text{R-Mask}\ \frac{M : A\,!\,E \dashrightarrow M' \qquad \mu_1 := \mathrm{topmod}(\llbracket A \rrbracket_E) \qquad \mu_2 := \mathrm{topmod}(\llbracket A \rrbracket_{L+E})}{\mathbf{mask}_L\ M : A\,!\,L + E \dashrightarrow \mathbf{let\ mod}_{\langle L \rangle ; \mu_1}\ x = \mathbf{mask}_L\ M'\ \mathbf{in}\ \mathbf{mod}_{\mu_2}\, x}$$

$$\text{R-Handler}\ \frac{\begin{array}{c} M : A\,!\,\overline{\ell_i}, E \dashrightarrow M' \qquad N : B\,!\,E \dashrightarrow N' \qquad [N_i : B\,!\,E \dashrightarrow N_i']_i \\ \mu := \mathrm{topmod}(\llbracket A \rrbracket_{\overline{\ell_i}, E}) \qquad \mu' := \mathrm{topmod}(\llbracket A \rrbracket_E) \\ N'' := \mathbf{let\ mod}_{\langle \overline{\ell_i} \rangle ; \mu}\ x = x\ \mathbf{in\ let}_{\mu'}\ \mathbf{mod}_{\mathbb{1}}\ x = \mathbf{mod}_{\mathbb{1}}\ x\ \mathbf{in}\ N' \\ [\mu_i := \mathrm{topmod}(\llbracket A_i \rrbracket_\cdot) \qquad N_i'' := \mathbf{let\ mod}_{\mu_i}\ p_i = p_i\ \mathbf{in}\ N_i']_i \\ H = \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i \qquad H' := \{\mathbf{return}\ x \mapsto N''\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i''\}_i \end{array}}{\mathbf{handle}\ M\ \mathbf{with}\ H : B\,!\,E \dashrightarrow \mathbf{handle}\ M'\ \mathbf{with}\ H'}$$

Fig. 11. Encoding of $\mathrm{F}^1_{\mathrm{eff}}$ in Met.

Similar to the R-Abs case, the R-EAbs case boxes the translated value with the empty absolute modality. Similar to the return clauses of the R-Handler case, the R-Mask case transforms the modality $\langle L \rangle \circ \mu_1$ to μ_2 in order to match the current effect context $L + E$.

In R-Var, we rebox the variable with the appropriate modality given by the type translation.

As a result of translating all terms to boxed terms, we must insert unboxing for elimination rules such as R-App and R-EApp. Nothing special happens for the R-Do case.

C.2 Proof of Encoding

We prove the encoding from $\mathrm{F}^1_{\mathrm{eff}}$ into Met in Section 5.

Definition 5.1 (Well-scoped). A typing judgement $\Gamma_1, x :_\varepsilon A, \Gamma_2 \vdash M : B\,!\,E$ is *well-scoped for* x if either $x \notin \mathrm{fv}(M)$ or $\blacklozenge_F^\Lambda \notin \Gamma_2$ or $A = \forall.A'$. A typing judgement $\Gamma \vdash M : A\,!\,E$ is *well-scoped* if it is well-scoped for all $x \in \Gamma$.

Lemma C.1 (Well-scopedness of Derivation Trees). *If the judgement at the bottom of a derivation tree is well-scoped, then every judgement in the derivation tree is well-scoped.*

Proof. Assume the contrary. Let $\Gamma_1, x :_\varepsilon A, \Gamma_2 \vdash M : B\,!\,E$ be the top-most judgement in the derivation tree with $x \in \mathrm{fv}(M)$ and $\blacklozenge_F^\Lambda \in \Gamma_2$ and $A \neq \forall.A'$. By case analysis on whether $\blacklozenge_F^\Lambda \in \Gamma_2$ was introduced in the derivation tree.

Case not introduced in the derivation tree: Then the judgement at the bottom of the derivation tree must contain both the marker and x and is not well-scoped for x. Contradiction.

Case introduced in the derivation tree: since we chose the top-most judgement, the judgement must have introduced the marker by an application of the R-EAbs rule. Let ε' be the effect variable introduced at this judgement. Then $\varepsilon \neq \varepsilon'$ by the side-condition of the R-EAbs rule. We have that ε is the ambient effect at the R-Var rule where x is used as a free variable, since we chose the top-most judgement. By the side-condition of the R-Var rule, then $\varepsilon = \varepsilon'$ or $A = \forall.A'$. Contradiction.

□

In the special case we consider there are no absent signatures. This implies that submoding on effects can only add labels to the end. Furthermore, all labels are drawn from a global environment and thus have the same signatures. This allows us to freely permute them in the effect row. In this case, we can strengthen the statement to the following:

Corollary C.2 (Transformation from Index). *If* $\langle L_1|D_1\rangle(F) \leqslant \langle L_2|D_2\rangle(F)$ *and* $L_1 \leqslant F$ *and* $L_2 \leqslant F$ *and* $L_1 \bowtie D_1 = L_2 \bowtie D_2$, *then* $\langle L_1|D_1\rangle_F \Rightarrow \langle L_2|D_2\rangle_F$.

Proof. We show that for all F' with $F \leqslant F'$, we have $\langle L_1|D_1\rangle(F') \leqslant \langle L_2|D_2\rangle(F')$. Since all signatures are present in F, we have that $F' = F + \bar{l}$ for some collection of labels with signatures $\bar{l}$. Then we use that $L_1 \leqslant F$:

$$\begin{aligned}\langle L_1|D_1\rangle(F') &= \langle L_1|D_1\rangle(F + \bar{l}) \\ &= D_1 + ((F + \bar{l}) - L_1) \\ &= D_1 + ((F - L_1) + \bar{l}) \\ &= \langle L_1|D_1\rangle(F) + \bar{l}\end{aligned}$$

and the same for $\langle L_2|D_2\rangle(F')$. Since $\langle L_1|D_1\rangle(F) \leqslant \langle L_2|D_2\rangle(F)$ and we can freely permute labels, we have that $(\langle L_1|D_1\rangle(F) + \bar{l}) \leqslant (\langle L_2|D_2\rangle(F) + \bar{l})$. □

The condition that $L_1 \bowtie D_1 = L_2 \bowtie D_2$ can be checked easily, where for the composition of modalities we use the fact that for $\langle L|D\rangle = \langle L_1|D_1\rangle \circ \langle L_2|D_2\rangle$, we have $L \bowtie D = (L_1, L_2) \bowtie (D_1, D_2)$.

Lemma C.3 (First Modality Transformation). *For all* E_1, E_2, E_3:

$$(\langle E_1 - E_2|E_2 - E_1\rangle \circ \langle E_2 - E_3|E_3 - E_2\rangle)_{E_1} \Leftrightarrow \langle E_1 - E_3|E_3 - E_1\rangle_{E_1}$$

Proof. We can use Corollary C.2 since $(E_1 - E_3) \leqslant E_1$ and $(E_1 - E_2) + L \leqslant E_1$ where $(L, D) = (E_2 - E_3) \bowtie (E_2 - E_1)$. We have:

$$\begin{aligned}\langle E_1 - E_3|E_3 - E_1\rangle(E_1) &= (E_3 - E_1) + (E_1 - (E_1 - E_3)) \\ &= (E_3 - E_1) + (E_1 \cap E_3) \\ &= E_3\end{aligned}$$

and using this calculation:

$$\begin{aligned}\langle E_1 - E_2|E_2 - E_1\rangle \circ \langle E_2 - E_3|E_3 - E_2\rangle(E_1) &= \langle E_2 - E_3|E_3 - E_2\rangle(\langle E_1 - E_2|E_2 - E_1\rangle(E_1)) \\ &= \langle E_2 - E_3|E_3 - E_2\rangle(E_2) \\ &= E_3\end{aligned}$$

□

Lemma C.4 (Second Modality Transformation). *For all L, E, F:*

$$\langle L + (E - F) | F - E \rangle_{L+E} \Rightarrow \langle (L + E) - F | F - (L + E) \rangle_{L+E}$$

Proof. We can use Corollary C.2 since $(L + E) - F \leqslant L + E$ and $L + (E - F) \leqslant L + E$. We have:

$$\begin{aligned}\langle (L + E) - F | F - (L + E) \rangle (L + E) &= (F - (L + E)) + ((L + E) - (L + E - F)) \\ &= (F - (L + E)) + ((L + E) \cap F) \\ &= F\end{aligned}$$

and:

$$\begin{aligned}\langle L + (E - F) | F - E \rangle (L + E) &= (F - E) + ((L + E) - (L + (E - F))) \\ &= (F - E) + (E - (E - F)) \\ &= (F - E) + (E \cap F) \\ &= F\end{aligned}$$

□

Lemma C.5 (Third Modality Transformation). *For all $\overline{\ell_i}, E, F$:*

$$(\langle\!| \overline{\ell_i} \rangle \circ \langle \overline{\ell_i}, E - F | F - \overline{\ell_i}, E \rangle)_E \Rightarrow \langle E - F | F - E \rangle_E$$

Proof. We can use Corollary C.2 since $(\langle\!| \overline{\ell_i} \rangle \circ \langle \overline{\ell_i}, E - F | F - \overline{\ell_i}, E \rangle) = \langle \overline{\ell_i}, E - F | F - \overline{\ell_i}, E \rangle (\overline{\ell_i}, E)$ and $\overline{\ell_i}, E - F \leqslant \overline{\ell_i}, E$ and $E - F \leqslant E$. We have $\langle E - F | F - E \rangle (E) = F$ and:

$$\begin{aligned}\langle\!| \overline{\ell_i} \rangle \circ \langle \overline{\ell_i}, E - F | F - \overline{\ell_i}, E \rangle (E) &= \langle \overline{\ell_i}, E - F | F - \overline{\ell_i}, E \rangle (\langle\!| \overline{\ell_i} \rangle (E)) \\ &= \langle \overline{\ell_i}, E - F | F - \overline{\ell_i}, E \rangle (\overline{\ell_i}, E) \\ &= F\end{aligned}$$

□

Lemma C.6 (Translating Instantiated Types). *For all $\mathrm{F}^1_{\mathrm{eff}}$ types A: $[\![A]\!]_E = [\![A[E'/]]\!]_{E,E'}$.*

Proof. By induction on the type A.

Case $A = \mathtt{Int}$. Trivial.

Case $A = \forall.A'$. Trivial.

Case $A = A' \to^F B'$. Then:

$$[\![A]\!]_E = \langle E - F | F - E \rangle ([\![A']\!]_F \to [\![B']\!]_F)$$

$$[\![A[E'/]]\!]_{E,E'} = \langle E, E' - F, E' | F, E' - E, E' \rangle ([\![A'[E'/]]\!]_{F,E'} \to [\![B'[E'/]]\!]_{F,E'})$$

By the induction hypothesis we have:

$$[\![A']\!]_F = [\![A'[E'/]]\!]_{F,E'}$$

$$[\![B']\!]_F = [\![B'[E'/]]\!]_{F,E'}$$

Since we can freely permute labels:

$$\begin{aligned}\langle E, E' - F, E' | F, E' - E, E' \rangle &= \langle E', E - E', F | E', F - E', E \rangle \\ &= \langle E - F | F - E \rangle\end{aligned}$$

□

Lemma 5.2 (Type preservation of encoding). *If $\Gamma \vdash M : A \,!\, \{E | \varepsilon\}$ is well-scoped, then $M : A \,!\, E \dashrightarrow M'$ and $[\![\Gamma]\!]_E \vdash M' : [\![A]\!]_E \,@\, E$.*

Proof. By induction on the typing derivation $\Gamma \vdash M : A \,!\, E$. We prove this for each rule of the translation. As a visual aid, we repeat each rule where we replace the translation premises by the Met judgement implied by the induction hypothesis and the translation in the conclusion by the Met judgement we need to prove.

Case

$$\text{R-Var}\ \frac{}{[\![\Gamma_1, x : A, \Gamma_2]\!]_E \vdash \mathsf{rebox}(x; A; E) : [\![A]\!]_E \ @\ E}$$

We use the $\mathsf{rebox}(x; A; E)$ function defined as follows:

$$\mathsf{rebox}(x; A; E) \quad = \quad \begin{cases} \mathbf{mod}_{\langle|\rangle}\, x, & \text{if } A = \mathtt{Int} \\ \mathbf{mod}_{\langle E-F|F-E\rangle}\, x, & \text{if } A = A' \to^F B' \\ \mathbf{mod}_{[]}\, x, & \text{if } A = \forall.A' \end{cases}$$

This function is exactly equivalent to $\mathbf{mod}_\mu\, x$ where $\mu = \mathsf{topmod}([\![A]\!]_E)$ We use the T-Mod rule to introduce the box. By cases on the type A:

Case $A = \mathtt{Int}$. We can use the T-Var rule since $\cdot \vdash \mathtt{Int} : \mathsf{Abs}$.

Case $A = \forall.A'$. Then $[\![A]\!]_F = []\,[\![A']\!]$. for all F. By rule MT-Abs, the pure modality transforms into any other modality and so we can use the T-Var rule.

Case $A = A' \to^F B'$. Since the $\mathrm{F}^1_{\mathrm{eff}}$ judgement is well-scoped, we have that $\mathsf{locks}(\Gamma_2)$ is the composition of transition modalities. Furthermore, $\mathsf{locks}(\Gamma') \circ \langle E - F|F - E\rangle : F \to F'$ for the context F' where x as introduced and x is annotated by the modality $\langle F' - F|F - F'\rangle_{F'} : F \to F'$. By Lemma C.3, we can use the T-Var rule.

Case

$$\text{R-App}\ \frac{[\![\Gamma]\!]_E \vdash M' : [\![A \to^E B]\!]_E \ @\ E \qquad [\![\Gamma]\!]_E \vdash N' : [\![A]\!]_E \ @\ E \qquad x \text{ fresh}}{[\![\Gamma]\!]_E \vdash \mathbf{let}\ \mathbf{mod}_{\langle|\rangle}\ x = M'\ \mathbf{in}\ x\, N' : [\![B]\!]_E \ @\ E}$$

We have $[\![A \to^E B]\!]_E = \langle|\rangle([\![A]\!]_E \to [\![B]\!]_E)$. The claim follows by the T-Letmod and T-App rules.

Case

$$\text{R-Abs}\ \frac{[\![\Gamma, \blacklozenge_E, x : A]\!]_F \vdash M' : [\![B]\!]_F \ @\ F \qquad v \coloneqq \langle E - F|F - E\rangle \qquad \mu \coloneqq \mathsf{topmod}([\![A]\!]_F)}{[\![\Gamma]\!]_E \vdash \mathbf{mod}_\nu\, (\lambda x^{[\![A]\!]_F}.\mathbf{let}\ \mathbf{mod}_\mu\, x = x\ \mathbf{in}\ M') : [\![A \to^F B]\!]_E \ @\ E}$$

We have $[\![\Gamma, \blacklozenge_E, x : A]\!]_F = [\![\Gamma]\!]_E, \blacklozenge_{\langle E-F|F-E\rangle}, x :_{\mu_F} A'$ where $\mu A' = [\![A]\!]_F$. Further $[\![A \to^F B]\!]_E = \langle E - F|F - E\rangle([\![A]\!]_F \to [\![B]\!]_F)$. The claim follows from the T-Letmod, T-Abs and T-Mod rules.

Case

$$\text{R-EAbs}\ \frac{[\![\Gamma, \blacklozenge^{\Lambda}_E]\!]. \vdash V' : [\![A]\!]. \ @\ \cdot}{[\![\Gamma]\!]_E \vdash \mathbf{mod}_{[]}\, V' : [\![\forall.A]\!]_E \ @\ E}$$

We have $[\![\Gamma, \blacklozenge^{\Lambda}_E]\!]. = [\![\Gamma]\!]_E, \blacklozenge_{[]}$. Further, $[\![\forall.A]\!]_E = []\,[\![A]\!].$. The claim follows from the T-Mod rule.

Case

$$\textsc{R-EApp}\quad \frac{[\![\Gamma]\!]_E \vdash M' : [\![\forall.A]\!]_E \ @\ E \qquad x \text{ fresh}}{[\![\Gamma]\!]_E \vdash \mathbf{let\ mod}_{[]}\ x = M'\ \mathbf{in}\ x : [\![A[E/]]\!]_E \ @\ E}$$

We have $[\![\forall.A]\!]_E = [][\![A]\!].$. By Lemma C.6, $[\![A]\!]. = [\![A[E/]]\!]_E$. The claim follows by the T-Letmod rule.

Case

$$\textsc{R-Do}\quad \frac{\ell : A \twoheadrightarrow B \in \Sigma \qquad [\![\Gamma]\!]_{\ell,E} \vdash M' : [\![A]\!]_{\ell,E} \ @\ \ell, E}{[\![\Gamma]\!]_{\ell,E} \vdash \mathbf{do}\ \ell\ M' : [\![B]\!]_{\ell,E} \ @\ \ell, E}$$

Because we only allow pure values in the effect signatures of $\mathrm{F}^1_{\mathrm{eff}}$, we have that $[\![A]\!]_{\ell,E} = [\![A]\!].$ and $[\![B]\!]_{\ell,E} = [\![B]\!].$, where $\ell : [\![A]\!]. \twoheadrightarrow [\![B]\!].$ in Met. The claim follows directly by the T-Do rule.

Case

$$\textsc{R-Mask}\quad \frac{[\![\Gamma, \blacklozenge_{L+E}]\!]_E \vdash M' : [\![A]\!]_E \ @\ E \qquad \mu_1 \coloneqq \mathrm{topmod}([\![A]\!]_E) \qquad \mu_2 \coloneqq \mathrm{topmod}([\![A]\!]_{L+E})}{[\![\Gamma]\!]_{L+E} \vdash \mathbf{let\ mod}_{\langle L|\rangle;\mu_1}\ x = \mathbf{mask}_L\ M'\ \mathbf{in}\ \mathbf{mod}_{\mu_2}\ x : [\![A]\!]_{L+E} \ @\ L+E}$$

We have $[\![\Gamma, \blacklozenge_{L+E}]\!]_E = [\![\Gamma]\!]_{L+E}, \blacklozenge_{\langle (L+E)-E|E-(L+E)\rangle}$. By permuting labels, we have $\langle (L+E)-E|E-(L+E)\rangle = \langle L|\rangle$. The goal follows by the T-Letmod, T-Mask and T-Mod rules if we can show that x can be used under the box. This is clear for integers, since they are pure and otherwise we need to show that $(\langle L|\rangle \circ \mu_1)_{L+E} \Rightarrow (\mu_2)_{L+E}$. For $A = \forall.A'$ this is clear since $\mu_1 = \mu_2 = []$ and $\langle L|\rangle \circ [] = []$. For functions, this follows from Lemma C.4.

Case

$$\textsc{R-Handler}\quad \frac{\begin{array}{c}[\![\Gamma, \blacklozenge_E]\!]_{\overline{\ell_i},E} \vdash M' : [\![A]\!]_{\overline{\ell_i},E} \ @\ \overline{\ell_i}, E \\ [\![\Gamma, x : A]\!]_E \vdash N' : [\![B]\!]_E \ @\ E \qquad [[\![\Gamma, p_i : A_i, r_i : B_i \to^E B]\!]_E \vdash N_i' : [\![B]\!]_E \ @\ E]_i \\ \mu \coloneqq \mathrm{topmod}([\![A]\!]_{\overline{\ell_i},E}) \qquad \mu' \coloneqq \mathrm{topmod}([\![A]\!]_E) \\ N'' \coloneqq \mathbf{let\ mod}_{\langle|\overline{\ell_i}\rangle;\mu}\ x = x\ \mathbf{in}\ \mathbf{let}_{\mu'}\ \mathbf{mod}_{\langle|\rangle}\ x = \mathbf{mod}_{\langle|\rangle}\ x\ \mathbf{in}\ N' \\ [\mu_i \coloneqq \mathrm{topmod}([\![A_i]\!].) \qquad N_i'' \coloneqq \mathbf{let\ mod}_{\mu_i}\ p_i = p_i\ \mathbf{in}\ N_i']_i \\ H = \{\mathbf{return}\ x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i \\ H' \coloneqq \{\mathbf{return}\ x \mapsto N''\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i'\}_i\end{array}}{[\![\Gamma]\!]_E \vdash \mathbf{handle}\ M'\ \mathbf{with}\ H' : [\![B]\!]_E \ @\ E}$$

We have $[\![\Gamma, \blacklozenge_E]\!]_{\overline{\ell_i},E} = [\![\Gamma]\!]_E, \blacklozenge_{\langle E-\overline{\ell_i},E|\overline{\ell_i},E-E\rangle}$. By permuting labels, we have $\langle E-\overline{\ell_i}, E|\overline{\ell_i}, E-E\rangle = \langle|\overline{\ell_i}\rangle$. In the operation clauses, we have that $[\![B_i \to^E B]\!]_E = \langle|\rangle([\![B_i]\!]_E \to [\![B]\!]_E)$. Because the argument and return of effects are pure, we have that $[\![B_i]\!]_E = [\![B_i]\!].$ and $[\![A_i]\!]_E = [\![A]\!].$. We need to unbox the argument p_i though. In the return clause, Met gives us $x : \langle|\overline{\ell_i}\rangle[\![A]\!]_{\overline{\ell_i},E}$, but we need $x : [\![A]\!]_E$. We achieve this by unboxing x fully and then re-boxing it with the modality μ'. This is possible for integers because they are pure, for $\forall$s because of the MT-Abs rule and for functions due to the modality transformation in Lemma C.5.

□

C.3 Extensibility of the Encoding

Our encoding in Section 5 does not consider any extensions of Met. With the extension of effect polymorphism, the encoding certainly becomes trivial. Thus, we only consider extensions of data types and polymorphism for value types and discuss how to extend the encoding.

Recall that in Section 5.2 we always translate types to modal types and perform greedy unboxing and lazy boxing for variables. For variables of data types such as a pair (A, B), we just need to further destruct the pair before greedy unboxing, and reconstruct the pair when lazy boxing. This is because the translation on its components might give terms of type $\mu A'$ and $\nu B'$ with different modalities which require separate unboxing. For variables of recursive data types, we need to destruct only to the extent that the data type is unfolded in the function body (where we may treat recursive invocations as opaque). While this requires a somewhat global translation, it does not require destructing and unboxing the recursive data type more than a small number of times.

The essential reason for the translation being global comes from the fact that we use let-style unboxing following MTT. For modalities with certain structure (right adjoints), it is possible to use Fitch-style unboxing [12] which allows terms to be directly unboxed without binding [19, 49]. We are interested in exploring whether we could extend Met to use Fitch-style unboxing and thus give a compositional local encoding for recursive data types. Fortunately, these issues appear to only cause problems for encoding but not in practice. Functional programs typically use pattern-matching in a structured way that plays nicely with automatic unboxing of bidirectional typing.

Extending the encoding with polymorphism for value types is tricky as the source calculus $\mathrm{F}^1_{\mathrm{eff}}$ is not stable under value type substitution. For instance, the following substitution breaks the condition that function arrows only refer to the lexically closest effect variable: $(\forall \varepsilon'.\alpha)[(1 \to^{\{|\varepsilon\}} 1)/\alpha] = \forall \varepsilon'.1 \to^{\{|\varepsilon\}} 1$. This exemplifies the fragility of the syntactic approach of Frank. It is possible to still define the encoding by forcing the substituted type to satisfy the lexical restriction. We leave the full development of such an encoding as future work.

D Specification and Implementation of Metl

In this section, we provide the syntax and typing rules for Metl introduced in Section 7 and the elaboration from it to Met. We also briefly discuss our prototype implementation.

D.1 Syntax and Typing Rules

The syntax of Metl is shown in Figure 12. We include extensions of data types and polymorphism in Section 6. For the latter we require explicit type application $M\ A$ (explicit type abstraction is optional since the checking mode tells us when to introduce type abstraction).

The bidirectional typing rules for Metl are shown in Figure 13.

We repeat the definition of across used in B-Var here for easy reference.

$$\mathrm{across}(\Gamma, A, \nu, F) \quad = \quad \begin{cases} A, & \text{if } \Gamma \vdash A : \mathrm{Abs} \\ \zeta G, & \text{otherwise, where } A = \overline{\mu} G \text{ and } \nu_F \backslash \overline{\mu}_F = \zeta_E \end{cases}$$

For $\mu_F : E \to F$ and $\nu_F : F' \to F$, the right residual $\nu_F \backslash \mu_F$ is a partial operation defined as follows.

$$\begin{aligned} \nu_F \setminus [E]_F &= [E]_{F'} \\ \langle L' | D' \rangle_F \setminus \langle L | D \rangle_F &= \begin{cases} \langle \lfloor D' \rfloor + (L - L') | D + F|_{L'-L} \rangle_{D'+(F-L')}, & \text{if present}(F|_{L'-L}) \\ \mathrm{fail}, & \text{otherwise} \end{cases} \\ [E]_F \setminus \langle L | D \rangle_F &= \mathrm{fail} \end{aligned}$$

Types	$A, B ::= A \to B \mid \mu A \mid \alpha \mid A * B \mid A + B$
Guarded Types	$G ::= A \to B \mid \alpha \mid A * B \mid A + B$
Masks	$L ::= \cdot \mid \ell, L$
Extensions	$D ::= \cdot \mid \ell : P, D$
Effect Contexts	$E, F ::= \cdot \mid \ell : P, E$
Modalities	$\mu ::= [E] \mid \langle L \mid D \rangle$
Kinds	$K ::=$ Abs $\mid$ Any $\mid$ Effect
Contexts	$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \alpha : K \mid \Gamma, \blacksquare_{\mu_F}$
Terms	$M, N ::= x \mid \lambda x.M \mid M\ N \mid M : A \mid M\ A \mid \mathbf{do}\ \ell\ M$
	$\mid\ \mathbf{mask}_L\ M \mid \mathbf{handle}\ M\ \mathbf{with}\ H$
	$\mid\ (M, N) \mid \mathbf{case}\ M\ \mathbf{of}\ (x, y) \mapsto N$
	$\mid\ \mathbf{inl}\ M \mid \mathbf{inr}\ M \mid \mathbf{case}\ M\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1, \mathbf{inr}\ y \mapsto M_2\}$
Values	$V, W ::= x \mid \lambda x.M \mid V : A \mid V\ A \mid (V, W) \mid \mathbf{inl}\ V \mid \mathbf{inr}\ V$
Handlers	$H ::= \{\mathbf{return}\ x \mapsto M\} \mid \{(\ell : A \twoheadrightarrow B)\ p\ r \mapsto M\} \uplus H$

Fig. 12. Syntax of Metl.

We define $\lfloor D \rfloor$ as converting an extension to a mask by taking the multiset of all its labels. We define $E|_L$ as the extension derived by extracting the entries in E with labels in L from the left. The predicate present(D) checks if all labels in D are present.

$$\begin{array}{rcl} \lfloor \cdot \rfloor & = & \cdot \\ \lfloor \ell : P, D \rfloor & = & \ell, \lfloor D \rfloor \end{array} \qquad \begin{array}{rcl} (\ell : P, E)|_{\ell, L} & = & \ell : P, E|_L \\ (\ell' : P, E)|_{\ell, L} & = & E|_{\ell, L} \\ \varepsilon|_{\ell, L} & = & \text{fail} \end{array}$$

B-Mod introduces a lock and B-Forall introduces a type variable into the context, respectively. B-Annotation is standard for bidirectional typing. B-Switch not only switches the direction from checking to inference, but also transforms the top-level modalities when there is a mismatch. B-Abs is standard. Both B-App and B-TApp unbox the eliminated term M when it has top-level modalities. B-Do is standard. For masks and handlers, we have typing rules in both checking and inference modes. For B-HandlerInfer, we use a partial join operation $A \vee_{\Gamma, F} B$ to join the potentially different types of different branches. The join operation fails when A and B are different types modulo top-level modalities; otherwise, it tries to transform the top-level modalities of A and B to the same one. As a special case, if A and B give some absolute guarded type G after removing top-level modalities, the join operation succeeds and directly returns G, which is a general enough solution because an absolute type has no restriction on its accessibility.

We define join on types as follows.

$$\overline{\mu} G \vee_{\Gamma, F} \overline{\nu} G \quad = \quad \begin{cases} G, & \text{if } \Gamma \vdash G : \text{Abs} \\ (\overline{\mu} \vee_F \overline{\nu}) G, & \text{otherwise} \end{cases}$$

In order to define $\mu \vee_F \nu$, we first define some auxiliary join operations.

We define the join of operation signatures $P \vee P'$ as follows, in order to obtain the minimal signature P'' such that $P \leqslant P''$ and $P' \leqslant P''$.

$$\begin{array}{rcl} - \vee P & = & P \\ P \vee - & = & P \\ P \vee P' & = & \begin{cases} P, & \text{if } P \equiv P' \\ \text{fail}, & \text{otherwise} \end{cases} \end{array}$$

We define the join of effect contexts $E \vee E'$ as follows, in order to obtain the minimal effect context F such that $E \leqslant F$ and $E' \leqslant F$.

$$\begin{array}{rcl} \cdot \vee \cdot & = & \cdot \\ \ell : P, E \vee \ell : P', E' & = & \ell : (P \vee P'), (E \vee E') \end{array}$$

We define the join of an extension and an effect context $D \vee E$ as follows, creating an extension based on D by joining the signatures with those of corresponding labels in E.

$$\begin{array}{rcl} \cdot \vee E & = & \cdot \\ \ell : P, D \vee \ell : P', E' & = & \ell : (P \vee P'), (D \vee E') \end{array}$$

We define the meet of extensions $D \wedge D'$ as follows.

$$\begin{array}{rcl} \cdot \wedge D & = & \cdot \\ \ell : P, D \wedge \ell : P', D' & = & \ell : (P \vee P'), (D \wedge D') \\ \ell : P, D \wedge D' & = & D \wedge D', \quad \text{where } \ell \notin \mathrm{dom}(D') \end{array}$$

Note that we take the join of the signatures when the labels are present in both extensions.

We write $L \cup L'$ and $L \cap L'$ for the standard union and intersection of multisets.

We define the join of modalities $\mu \vee_F \nu$ at mode F as follows, where if the join operation succeeds, we have $\mu_F \Rightarrow (\mu \vee_F \nu)_F$ and $\nu_F \Rightarrow (\mu \vee_F \nu)_F$, and for any other ζ such that $\mu_F \Rightarrow \zeta_F$ and $\nu_F \Rightarrow \zeta_F$, we have $(\mu \vee_F \nu)_F \Rightarrow \zeta_F$.

$$\begin{array}{rcl} [E] \vee_F [E'] & = & [E \vee E'] \\ [E] \vee_F \langle L | D \rangle & = & \begin{cases} \langle L | D \vee E \rangle, & \text{if } E \leqslant D \vee E + (F - L) \\ \text{fail}, & \text{otherwise} \end{cases} \\ \langle L | D \rangle \vee_F \langle L' | D' \rangle & = & \begin{cases} \langle L \cap L' | D \wedge D' \rangle, & \text{if } \langle L | D \rangle_F \Rightarrow \langle L \cap L' | D \wedge D' \rangle_F \\ & & \text{and } \langle L' | D' \rangle_F \Rightarrow \langle L \cap L' | D \wedge D' \rangle_F \\ \text{fail}, & & \text{otherwise} \end{cases} \end{array}$$

The third case may appear surprising since it takes the meet of masks and extensions. It works because the MT-Shrink rule in Section 4.3 allows us to remove corresponding elements from masks and extensions simultaneously. The side conditions guarantee that both modalities can be transformed to their join.

The introduction rules of data types are standard. For their elimination, we have both checking and inference version. Both B-CrispPair and B-CrispSum extract the top-level modalities of the data values and distribute them to the types of the variables for their components. We also use the join operation to unify the different branches of B-CrispSumInfer.

D.2 Elaboration and Implementation

It is easy to elaborate MetL terms into Met terms in a type-directed manner. We formally define the elaboration alongside the typing rules in Figures 14 and 15. We have $\Gamma \vdash M \Rightarrow A \mathbin{@} E \dashrightarrow N$ for the inference mode and $\Gamma \vdash M \Leftarrow A \mathbin{@} E \dashrightarrow N$ for the checking mode, both of which elaborates M in MetL to N in Met.

$\boxed{\Gamma \vdash M \Rightarrow A \mathbin{@} E}$ $\boxed{\Gamma \vdash M \Leftarrow A \mathbin{@} E}$

B-Var
$$\frac{\nu_F = \mathsf{locks}(\Gamma') : E \to F \qquad B = \mathsf{across}(\Gamma, A, \nu, F)}{\Gamma, x : A, \Gamma' \vdash x \Rightarrow B \mathbin{@} E}$$

B-Mod
$$\frac{\Gamma, \blacksquare_{\mu_F} \vdash V \Leftarrow A \mathbin{@} E \qquad \mu_F : E \to F}{\Gamma \vdash V \Leftarrow \mu A \mathbin{@} F}$$

B-Forall
$$\frac{\Gamma, \alpha : K \vdash V \Leftarrow A \mathbin{@} E}{\Gamma \vdash V \Leftarrow \forall \alpha : K.A \mathbin{@} E}$$

B-Annotation
$$\frac{\Gamma \vdash V \Leftarrow A \mathbin{@} E}{\Gamma \vdash V : A \Rightarrow A \mathbin{@} E}$$

B-Switch
$$\frac{\Gamma \vdash M \Rightarrow \overline{\mu} G \mathbin{@} E \qquad \Gamma \vdash (\overline{\mu}, G) \Rightarrow \overline{\nu} \mathbin{@} E}{\Gamma \vdash M \Leftarrow \overline{\nu} G \mathbin{@} E}$$

B-Abs
$$\frac{\Gamma, x : A \vdash M \Leftarrow B \mathbin{@} E}{\Gamma \vdash \lambda x.M \Leftarrow A \to B \mathbin{@} E}$$

B-App
$$\frac{\Gamma \vdash M \Rightarrow \overline{\mu}(A \to B) \mathbin{@} E \qquad \overline{\mu}_E \Rightarrow \mathbb{1}_E \qquad \Gamma \vdash N \Leftarrow A \mathbin{@} E}{\Gamma \vdash M\ N \Rightarrow B \mathbin{@} E}$$

B-TApp
$$\frac{\Gamma \vdash M \Rightarrow \overline{\mu}(\forall \alpha : K.B) \mathbin{@} E \qquad \overline{\mu}_E \Rightarrow \mathbb{1}_E \qquad \Gamma \vdash A : K}{\Gamma \vdash M\ A \Rightarrow B[A/\alpha] \mathbin{@} E}$$

B-Do
$$\frac{E = \ell : A \twoheadrightarrow B, F \qquad \Gamma \vdash M \Leftarrow A \mathbin{@} E}{\Gamma \vdash \mathbf{do}\ \ell\ M \Rightarrow B \mathbin{@} E}$$

B-MaskCheck
$$\frac{\Gamma, \blacksquare_{\langle L \rangle_F} \vdash M \Leftarrow A \mathbin{@} F - L}{\Gamma \vdash \mathbf{mask}_L\ M \Leftarrow \langle L \rangle A \mathbin{@} F}$$

B-MaskInfer
$$\frac{\Gamma, \blacksquare_{\langle L \rangle_F} \vdash M \Rightarrow A \mathbin{@} F - L}{\Gamma \vdash \mathbf{mask}_L\ M \Rightarrow \langle L \rangle A \mathbin{@} F}$$

B-HandlerInfer
$$\frac{D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \blacksquare_{\langle D \rangle_E} \vdash M \Rightarrow A \mathbin{@} D + E \qquad \Gamma, x : \langle D \rangle A \vdash N \Rightarrow B' \mathbin{@} E \qquad [\Gamma, p_i : A_i, r_i : B_i \to B' \vdash N_i \Rightarrow B_i \mathbin{@} E]_i \qquad B = B'(\vee_{\Gamma,E} B_i)_i}{\Gamma \vdash \mathbf{handle}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{(\ell_i : A_i \twoheadrightarrow B_i)\ p_i\ r_i \mapsto N_i\}_i \Rightarrow B \mathbin{@} E}$$

B-HandlerCheck
$$\frac{D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \blacksquare_{\langle D \rangle_E} \vdash M \Rightarrow A \mathbin{@} D + E \qquad \Gamma, x : \langle D \rangle A \vdash N \Leftarrow B \mathbin{@} E \qquad [\Gamma, p_i : A_i, r_i : B_i \to B \vdash N_i \Leftarrow B \mathbin{@} E]_i}{\Gamma \vdash \mathbf{handle}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{(\ell_i : A_i \twoheadrightarrow B_i)\ p_i\ r_i \mapsto N_i\}_i \Leftarrow B \mathbin{@} E}$$

B-Pair
$$\frac{\Gamma \vdash M \Leftarrow A \mathbin{@} E \qquad \Gamma \vdash N \Leftarrow B \mathbin{@} E}{\Gamma \vdash (M, N) \Leftarrow A * B \mathbin{@} E}$$

B-Inl
$$\frac{\Gamma \vdash M \Leftarrow A \mathbin{@} E}{\Gamma \vdash \mathbf{inl}\ M \Leftarrow A + B \mathbin{@} E}$$

B-Inr
$$\frac{\Gamma \vdash M \Leftarrow B \mathbin{@} E}{\Gamma \vdash \mathbf{inr}\ M \Leftarrow A + B \mathbin{@} E}$$

B-CrispPairInfer
$$\frac{\Gamma \vdash V \Rightarrow \overline{\mu}(A * B) \mathbin{@} E \qquad \Gamma, x : \overline{\mu}A, y : \overline{\mu}B \vdash M \Rightarrow A' \mathbin{@} E}{\Gamma \vdash \mathbf{case}\ V\ \mathbf{of}\ (x, y) \mapsto M \Rightarrow A' \mathbin{@} E}$$

B-CrispPairCheck
$$\frac{\Gamma \vdash V \Rightarrow \overline{\mu}(A * B) \mathbin{@} E \qquad \Gamma, x : \overline{\mu}A, y : \overline{\mu}B \vdash M \Leftarrow A' \mathbin{@} E}{\Gamma \vdash \mathbf{case}\ V\ \mathbf{of}\ (x, y) \mapsto M \Leftarrow A' \mathbin{@} E}$$

B-CrispSumInfer
$$\frac{\Gamma \vdash V \Rightarrow \overline{\mu}(A + B) \mathbin{@} E \qquad \Gamma, x : \overline{\mu}A \vdash M_1 \Rightarrow A_1 \mathbin{@} E \qquad \Gamma, y : \overline{\mu}B \vdash M_2 \Rightarrow A_2 \mathbin{@} E}{\Gamma \vdash \mathbf{case}\ V\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1, \mathbf{inr}\ y \mapsto M_2\} \Rightarrow A_1 \vee_{\Gamma,E} A_2 \mathbin{@} E}$$

B-CrispSumCheck
$$\frac{\Gamma \vdash V \Rightarrow \overline{\mu}(A + B) \mathbin{@} E \qquad \Gamma, x : \overline{\mu}A \vdash M_1 \Leftarrow A' \mathbin{@} E \qquad \Gamma, y : \overline{\mu}B \vdash M_2 \Leftarrow A' \mathbin{@} E}{\Gamma \vdash \mathbf{case}\ V\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1, \mathbf{inr}\ y \mapsto M_2\} \Leftarrow A' \mathbin{@} E}$$

Fig. 13. Bidirectional typing rules for Metl.

We use the following auxiliary functions in the elaboration.

$$\begin{aligned}
\mathsf{unmod}(M; \overline{\mu}) &= (\lambda y.\mathbf{let}\ \mathbf{mod}_{\overline{\mu}}\ y = y\ \mathbf{in}\ y)\ M \\
\mathsf{unvar}(x; A; M) &= \mathbf{let}\ \mathbf{mod}_{\overline{\mu}}\ x = x\ \mathbf{in}\ M \quad \text{where } A = \overline{\mu} G \\
\\
\mathsf{join}_{\Gamma,E}(M_1 : \overline{\mu_1} G_1, M_2 : \overline{\mu_2} G_2, \cdots) &= (\mathbf{let}\ \mathbf{mod}_{\overline{\mu_1}}\ x = M_1\ \mathbf{in}\ \mathbf{mod}_{\overline{\zeta}}\ x, \\
&\qquad \mathbf{let}\ \mathbf{mod}_{\overline{\mu_2}}\ x = M_2\ \mathbf{in}\ \mathbf{mod}_{\overline{\zeta}}\ x, \cdots) \\
&\qquad \text{where } \overline{\mu_1} G_1 \vee_{\Gamma,E} \overline{\mu_2} G_2 \vee_{\Gamma,E} \cdots = \overline{\zeta} G
\end{aligned}$$

The core idea of the elaboration is similar to the greedy unboxing strategy of the encoding we have in Appendix C.1. For B-Abs (and also handler rules), we immediately fully unbox the bound variables. For B-Mod, we insert explicit boxing. For B-Var, we re-box them with the appropriate modality ζ. For B-Switch, we insert an unboxing followed by boxing. For B-App and B-TApp, we unbox $\overline{\mu}$. For B-HandlerCheck and B-HandlerInfer, we unbox the bound variables in handler clauses. We do not need to unbox the continuation functions since they have no top-level modality. For B-CrispPairCheck/Infer and B-CrispSumCheck/Infer, we unbox V before case splitting. Also, when the $\wedge_E$ operation is used (such as in B-CrispSumInfer and B-HandlerInfer), we use join to unbox the terms correspondingly.

We implement a prototype of Metl with all features mentioned above as well as algebraic data types and pattern matching. We do not encounter any challenges in generalising the pairs and sums to algebraic data types. In our implementation, we do not strictly follow the conventional bidirectional typing approach, which distinguishes between the checking and inference mode as in the above rules. Instead, we use the form $\Gamma \vdash M \Leftarrow S \Rightarrow A \ @\ E$ where each rule has an input shape S and output type A, similar to contextual typing [56]. A shape S is a type with some holes. When S is empty, we are in inference mode; when S is a complete type, we are in checking mode; otherwise, we can still use S to pass in partial type information which could allow us to type check more programs.

Our implementation supports polymorphism with explicit type instantiation. As we have discussed in Section 7, it is natural to extend it with inference for polymorphism, following the literature on bidirectional typing [14, 17, 60]. We plan to explore this extension in the future.

$\boxed{\Gamma \vdash M \Rightarrow A \mathbin{@} E \dashrightarrow N}$ $\boxed{\Gamma \vdash M \Leftarrow A \mathbin{@} E \dashrightarrow N}$

B-Var
$$\frac{\nu_F = \mathsf{locks}(\Gamma') : E \to F \qquad \overline{\zeta} G = \mathsf{across}(\Gamma, A, \nu, F)}{\Gamma, x : A, \Gamma' \vdash x \Rightarrow \overline{\zeta} G \mathbin{@} E \dashrightarrow \mathbf{mod}_{\overline{\zeta}}\, x}$$

B-Mod
$$\frac{\mu_F : E \to F \qquad \Gamma, \text{🔒}_{\mu_F} \vdash V \Leftarrow A \mathbin{@} E \dashrightarrow V'}{\Gamma \vdash V \Leftarrow \mu A \mathbin{@} F \dashrightarrow \mathbf{mod}_{\mu}\, V'}$$

B-Forall
$$\frac{\Gamma, \alpha : K \vdash V \Leftarrow A \mathbin{@} E \dashrightarrow V'}{\Gamma \vdash V \Leftarrow \forall \alpha : K.A \mathbin{@} E \dashrightarrow \Lambda \alpha^K.V'}$$

B-Annotation
$$\frac{\Gamma \vdash V \Leftarrow A \mathbin{@} E \dashrightarrow V'}{\Gamma \vdash V : A \Rightarrow A \mathbin{@} E \dashrightarrow V'}$$

B-Switch
$$\frac{\Gamma \vdash M \Rightarrow \overline{\mu} G \mathbin{@} E \dashrightarrow M' \qquad (G, \overline{\mu}) \Rightarrow \overline{\nu} \mathbin{@} E \qquad N = \mathbf{let}\ \mathbf{mod}_{\overline{\mu}}\, x = M'\ \mathbf{in}\ \mathbf{mod}_{\overline{\nu}}\, x}{\Gamma \vdash M \Leftarrow \overline{\nu} G \mathbin{@} E \dashrightarrow N}$$

B-Abs
$$\frac{\Gamma, x : A \vdash M \Leftarrow B \mathbin{@} E \dashrightarrow M' \qquad M'' = \lambda x.\mathsf{unvar}(x; A; M')}{\Gamma \vdash \lambda x.M \Leftarrow A \to B \mathbin{@} E \dashrightarrow M''}$$

B-App
$$\frac{\Gamma \vdash M \Rightarrow \overline{\mu}(A \to B) \mathbin{@} E \dashrightarrow M' \qquad \overline{\mu}_E \Rightarrow \mathbb{1}_E \qquad M'' = \mathsf{unmod}(M'; \overline{\mu}) \qquad \Gamma \vdash N \Leftarrow A \mathbin{@} E \dashrightarrow N'}{\Gamma \vdash M\ N \Rightarrow B \mathbin{@} E \dashrightarrow M''\ N'}$$

B-TApp
$$\frac{\Gamma \vdash M \Rightarrow \overline{\mu}(\forall \alpha : K.B) \mathbin{@} E \dashrightarrow M' \qquad \overline{\mu}_E \Rightarrow \mathbb{1}_E \qquad M'' = \mathsf{unmod}(M'; \overline{\mu}) \qquad \Gamma \vdash A : K}{\Gamma \vdash M\ A \Rightarrow B[A/\alpha] \mathbin{@} E \dashrightarrow M''\ A}$$

B-Do
$$\frac{E = \ell : A \twoheadrightarrow B, F \qquad \Gamma \vdash M \Leftarrow A \mathbin{@} E \dashrightarrow M'}{\Gamma \vdash \mathbf{do}\ \ell\ M \Rightarrow B \mathbin{@} E \dashrightarrow \mathbf{do}\ \ell\ M'}$$

B-MaskCheck
$$\frac{\Gamma, \text{🔒}_{\langle L \rangle_F} \vdash M \Leftarrow A \mathbin{@} F - L \dashrightarrow M'}{\Gamma \vdash \mathbf{mask}_L\ M \Leftarrow \langle L \rangle A \mathbin{@} F \dashrightarrow \mathbf{mask}_L\ M'}$$

B-MaskInfer
$$\frac{\Gamma, \text{🔒}_{\langle L \rangle_F} \vdash M \Rightarrow A \mathbin{@} F - L \dashrightarrow M'}{\Gamma \vdash \mathbf{mask}_L\ M \Rightarrow \langle L \rangle A \mathbin{@} F \dashrightarrow \mathbf{mask}_L\ M'}$$

B-HandlerCheck
$$\frac{\begin{array}{c} D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \text{🔒}_{\langle D \rangle_E} \vdash M \Rightarrow A \mathbin{@} D + E \dashrightarrow M' \\ \Gamma, x : \langle D \rangle A \vdash N \Leftarrow B \mathbin{@} E \dashrightarrow N' \qquad [\Gamma, p_i : A_i, r_i : B_i \to B \vdash N_i \Leftarrow B \mathbin{@} E \dashrightarrow N_i']_i \end{array}}{\begin{array}{c} \Gamma \vdash \mathbf{handle}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{(\ell_i : A_i \twoheadrightarrow B_i)\ p_i\ r_i \mapsto N_i\}_i \Leftarrow B \mathbin{@} E \\ \dashrightarrow \mathbf{handle}\ M'\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto \mathsf{unvar}(x; \langle D \rangle A; N')\} \uplus \{(\ell_i : A_i \twoheadrightarrow B_i)\ p_i\ r_i \mapsto \mathsf{unvar}(p_i; A_i; N_i')\}_i \end{array}}$$

B-HandlerInfer
$$\frac{\begin{array}{c} D = \{\ell_i : A_i \twoheadrightarrow B_i\}_i \qquad \Gamma, \text{🔒}_{\langle D \rangle_E} \vdash M \Rightarrow A \mathbin{@} D + E \dashrightarrow M' \\ \Gamma, x : \langle D \rangle A \vdash N \Rightarrow B' \mathbin{@} E \dashrightarrow N' \qquad [\Gamma, p_i : A_i, r_i : B_i \to B' \vdash N_i \Rightarrow B_i \mathbin{@} E \dashrightarrow N_i']_i \\ B = B'(\vee_{\Gamma,E} B_i)_i \qquad N'', (N_i'')_i = \mathsf{join}_{\Gamma,E}(N' : B', (N_i' : B_i')_i) \end{array}}{\begin{array}{c} \Gamma \vdash \mathbf{handle}\ M\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto N\} \uplus \{(\ell_i : A_i \twoheadrightarrow B_i)\ p_i\ r_i \mapsto N_i\}_i \Rightarrow B \mathbin{@} E \\ \dashrightarrow \mathbf{handle}\ M'\ \mathbf{with}\ \{\mathbf{return}\ x \mapsto \mathsf{unvar}(x; \langle D \rangle A; N'')\} \uplus \{(\ell_i : A_i \twoheadrightarrow B_i)\ p_i\ r_i \mapsto \mathsf{unvar}(p_i; A_i; N_i'')\}_i \end{array}}$$

Fig. 14. Elaboration from MetL to Met (part I).

$\boxed{\Gamma \vdash M \Rightarrow A \mathbin{@} E \dashrightarrow N}$ $\boxed{\Gamma \vdash M \Leftarrow A \mathbin{@} E \dashrightarrow N}$

B-Pair
$$\frac{\Gamma \vdash M \Leftarrow A \mathbin{@} E \dashrightarrow M' \qquad \Gamma \vdash N \Leftarrow B \mathbin{@} E \dashrightarrow N'}{\Gamma \vdash (M, N) \Leftarrow A * B \mathbin{@} E \dashrightarrow (M', N')}$$

B-Inl
$$\frac{\Gamma \vdash M \Leftarrow A \mathbin{@} E \dashrightarrow M'}{\Gamma \vdash \mathbf{inl}\ M \Leftarrow A + B \mathbin{@} E \dashrightarrow \mathbf{inl}\ M'}$$

B-Inr
$$\frac{\Gamma \vdash M \Leftarrow B \mathbin{@} E \dashrightarrow M'}{\Gamma \vdash \mathbf{inr}\ M \Leftarrow A + B \mathbin{@} E \dashrightarrow \mathbf{inr}\ M'}$$

B-CrispPairInfer
$$\frac{\Gamma \vdash V \Rightarrow \overline{\mu}(A * B) \mathbin{@} E \dashrightarrow V' \qquad \Gamma, x : \overline{\mu}A, y : \overline{\mu}B \vdash M \Rightarrow A' \mathbin{@} E \dashrightarrow M'}{\Gamma \vdash \mathbf{case}\ V\ \mathbf{of}\ (x, y) \mapsto M \Rightarrow A' \mathbin{@} E \dashrightarrow \mathbf{let\ mod}_{\overline{\mu}}\ x = V'\ \mathbf{in\ case}_{\overline{\mu}}\ x\ \mathbf{of}\ (x, y) \mapsto M'}$$

B-CrispPairCheck
$$\frac{\Gamma \vdash V \Rightarrow \overline{\mu}(A * B) \mathbin{@} E \dashrightarrow V' \qquad \Gamma, x : \overline{\mu}A, y : \overline{\mu}B \vdash M \Leftarrow A' \mathbin{@} E \dashrightarrow M'}{\Gamma \vdash \mathbf{case}\ V\ \mathbf{of}\ (x, y) \mapsto M \Leftarrow A' \mathbin{@} E \dashrightarrow \mathbf{let\ mod}_{\overline{\mu}}\ x = V'\ \mathbf{in\ case}_{\overline{\mu}}\ x\ \mathbf{of}\ (x, y) \mapsto M'}$$

B-CrispSumInfer
$$\frac{\Gamma \vdash V \Rightarrow \overline{\mu}(A + B) \mathbin{@} E \dashrightarrow V' \qquad \Gamma, x : \overline{\mu}A \vdash M_1 \Rightarrow A_1 \mathbin{@} E \dashrightarrow M_1' \qquad \Gamma, y : \overline{\mu}B \vdash M_2 \Rightarrow A_2 \mathbin{@} E \dashrightarrow M_2' \qquad M_1'', M_2'' = \mathsf{join}_{\Gamma,E}(M_1 : A_1, M_2 : A_2)}{\Gamma \vdash \mathbf{case}\ V\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1, \mathbf{inr}\ y \mapsto M_2\} \Rightarrow A_1 \vee_{\Gamma,E} A_2 \mathbin{@} E \dashrightarrow \mathbf{let\ mod}_{\overline{\mu}}\ x = V'\ \mathbf{in\ case}_{\overline{\mu}}\ x\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1'', \mathbf{inr}\ y \mapsto M_2''\}}$$

B-CrispSumCheck
$$\frac{\Gamma \vdash V \Rightarrow \overline{\mu}(A + B) \mathbin{@} E \dashrightarrow V' \qquad \Gamma, x : \overline{\mu}A \vdash M_1 \Leftarrow A' \mathbin{@} E \dashrightarrow M_1' \qquad \Gamma, y : \overline{\mu}B \vdash M_2 \Leftarrow A' \mathbin{@} E \dashrightarrow M_2'}{\Gamma \vdash \mathbf{case}\ V\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1, \mathbf{inr}\ y \mapsto M_2\} \Leftarrow A' \mathbin{@} E \dashrightarrow \mathbf{let\ mod}_{\overline{\mu}}\ x = V'\ \mathbf{in\ case}_{\overline{\mu}}\ x\ \mathbf{of}\ \{\mathbf{inl}\ x \mapsto M_1', \mathbf{inr}\ y \mapsto M_2'\}}$$

Fig. 15. Elaboration from MetL to Met (part II).