

Modal Effect Types

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Effect handlers are a powerful abstraction for defining, customising, and composing computational effects. Statically ensuring that all effect operations are handled requires some form of effect system, but using a traditional effect system would require adding extensive effect annotations to the millions of lines of existing code in these languages. Recent proposals seek to address this problem by removing the need for explicit effect polymorphism. However, they typically rely on fragile syntactic mechanisms or on introducing a separate notion of second-class function. We introduce a novel semantic approach based on modal effect types.

1 Introduction

Effect handlers [44] allow programmers to define, customise, and compose a range of computational effects including concurrency, exceptions, state, backtracking, and probability, in direct-style inside the programming language. Following their pioneering use in languages such as Eff [4], Effekt [8, 9], Frank [13, 34], Koka [32], and Links [22], they are now increasingly being adopted in production languages and systems such as OCaml [50], Scala [7], and WebAssembly [42].

In a statically typed programming language with effect handlers some form of effect system to track effectful operations is necessary in order to ensure that a given program handles all of its effects. However, traditional effect systems require extensive effect annotations even for code that does not use effects. Consider the standard map function:

```
map : ∀ a b . (a → b) → List a → List b
```

This type is a statement about the values that map accepts and returns, but is silent about which effects may occur during its evaluation. In the effect system of Koka, for instance, this map function is thus presumed to be a total function that takes a function which cannot perform any effects and itself does not perform any effects.

However, this would prevent programmers from passing any effectful function to map. To use map in effectful code, in Koka we must give it a more permissive type such as:

```
map' : ∀ a b e . (a  $\overset{e}{\rightarrow}$  b)  $\overset{e}{\rightarrow}$  List a  $\overset{e}{\rightarrow}$  List b
```

This type uses *effect polymorphism*, quantifying over an *effect variable* e , which occurs on every arrow. Such effect annotations pollute the type signature of map to convey the obvious: map' is polymorphic in its effects, that is, the effects of $\text{map}' \ f \ xs$ depend on the effects of the function argument f . Effect annotations impose a mild burden to authors of new code, but pose a significant problem when extending an existing language with effectful features.

Signatures of existing library code must be rewritten to support effect polymorphism [7, 39], even in legacy libraries that do not use effects, making it challenging to retrofit such an effect system onto an existing language in a backwards-compatible way without causing friction for existing codebases. However, if we can eliminate the need to annotate effect polymorphism, then

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50 retrofitting an effect system ought to become a tractable problem. Our goal is to design a principled
51 effect system, where effect polymorphism silence is a virtue.

52 An important step towards that goal was taken by the language Frank [13, 34]. Frank gives `map`
53 its original unannotated type, whilst still allowing it to be passed effectful functions. The key idea
54 is that expressions are typed assuming an unknown set of possible effects—the *ambient effects*—will
55 be provided by the context in which the expression occurs. Rather than assuming unannotated
56 function types perform no effects, they are assumed to perform the ambient effects.

57 Frank still uses effect variables behind the scenes, implicitly inserting effect variables for passing
58 the ambient effects around. For instance, Frank simply treats the type signature of `map` above
59 as syntactic sugar for `map'`. (Frank has certain other syntactic idiosyncrasies, so in order to ease
60 readability, we render Frank code using the syntax of the rest of the paper.) This syntactic mechanism
61 is fragile. For instance, effect variables can appear in error messages as in the following example in
62 which we use a `yield` effect to write a function that yields all values in a list.

```
63 gen : List Int  $\xrightarrow{\text{yield}}$  1  
64 gen xs = map (fun x → do yield x) xs; ()
```

65 If the user forgets the `yield` annotation, Frank complains:

```
66 cannot unify effects e and yield, f
```

67 Here `f` and `e` are the underlying effect variables inserted by the Frank compiler. They do not appear
68 in the source program and in larger programs it can be unclear how to fix such errors.

69 Effekt [9] and Scala [7] also make use of ambient effects to avoid effect polymorphism by tracking
70 effects as capabilities. However, they either restrict functions to be second-class or require advanced
71 features like path-dependent types, as we discuss further in Section 7.2.

72 We build on the insight that ambient effect contexts can substantially reduce the annotation
73 burden. Instead of relying on desugaring to traditional effect polymorphism like Frank, we develop
74 MET (Modal Effect Types), a novel effect system with a theoretical foundation based on modal
75 types. We follow multimodal type theory (MTT) [20, 21] in tracking *modes* for types and terms
76 and consider *modalities* as the transitions between such modes. We treat each possible ambient
77 effect context as a mode, and each possible transition between effect contexts as a *modality*. We
78 support *absolute* modalities, which override the ambient effect context. We also support *relative*
79 modalities, which describe a local change to the ambient effect context, as exemplified by effect
80 handlers which handles certain effects and let all others pass through unchanged.

81 MET precludes hidden effect variables in error messages as there are no hidden effect variables.
82 Moreover, MET works smoothly with pure first-class higher-order functions, which require neither
83 hidden effect variables nor extra annotations, and can be applied to effectful arguments. Both Frank
84 and MET strive to capture the essence of modular programming with effects. Frank relies on a loose
85 *syntactic* characterisation based on polymorphic types. In contrast, MET provides a tight *semantic*
86 characterisation based on simple types.

87 The main contributions of this paper are as follows.

- 88 • We give a high-level overview of the key ideas of modal effect types: effect contexts, absolute
89 and relative modalities. We provide a series of practical examples to show how modal effect
90 types enable us to write modular effectful programs without effect polymorphism (Section 2).
- 91 • We introduce MET, a simply-typed core calculus with effect handlers and modal effect types
92 (Section 3). We prove its type soundness and effect safety.
- 93 • Intuitively, MET can type check all functions that can be written in traditional effect systems
94 using a single effect variable, which is the most common case in practice. We formally prove
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this intuition by presenting a calculus for row-based effect systems with a single effect variable and encoding it in MET (Section 4).

- We extend MET with data types and value polymorphism. To recover the full power of traditional effect systems, we also extend MET with effect polymorphism which can be seamlessly used alongside modal effect types to express effectful programs that use higher-order effects (Section 5).
- We outline and prototype a surface language METL which uses bidirectional type checking to infer modality introduction and elimination (Section 6).

Section 7 discusses related and future work.

2 Overview

In this section we illustrate the main ideas of modal effect types through a series of examples. We demonstrate how modal effect types support modular composition of higher-order functions and effect handlers without effect polymorphism. The examples are written in METL, whose core calculus we introduce in Sections 3 and 5, and whose design we discuss further in Section 6. METL adopts simple bidirectional typing to infer introduction and elimination of modalities. In order to elucidate the core idea that modal effect types support modular effectful programming without polymorphism we begin with examples in the simply-typed fragment of METL.

2.1 From Function Arrows to Effect Contexts

Traditional effect systems annotate a function type with the effects that the function may perform when invoked. For instance, consider the following typing judgement for the `app` function specialised to take a pair of a function from integers to the unit type and an integer.

$$\vdash \text{fun } (f, x) \rightarrow f\ x : (\text{Int} \xrightarrow{E} \mathbf{1}, \text{Int}) \xrightarrow{E} \mathbf{1}$$

The effect annotation E is a row of typed operations that f may perform. For instance, if E is `get : $\mathbf{1} \Rightarrow \text{Int}$, put : $\text{Int} \Rightarrow \mathbf{1}$` then f may perform a `get` operation which takes a unit value and returns an integer and a `put` operation which takes an integer and returns a unit value. As in Frank [13] and Koka [32], rows are scoped [31] meaning that they allow duplicate operations with the same name (but possibly different signatures). The order of duplicates matters, but the relative order of distinct operations does not.

Since `app` invokes its argument, E also denotes the operations that invoking `app` might perform. As we saw in the introduction, the standard way to support modularity is to be polymorphic in E . But this introduces an annotation burden for all higher-order functions, including those (like `app`) which do not themselves perform effects.

In the spirit of Frank, METL decouples effects from function types and tracks *effect contexts* in typing judgements. All components of the term and type share the same effect context (unless manipulated by modalities as we will see in Sections 2.2 and 2.3). For instance, we have the following typing judgement for the same `app` function as above.

$$\vdash \text{fun } \underbrace{(f)}_{@E}, x) \rightarrow \underbrace{f\ x}_{@E} : \underbrace{(\text{Int} \rightarrow \mathbf{1}, \text{Int})}_{@E} \rightarrow \underbrace{\mathbf{1}}_{@E} @ E$$

As a visual aid, we use braces to explicitly annotate the effect contexts for the argument f and the whole function in the term and type. The $@ E$ annotation belongs to the judgement and indicates the effect context E . This is the *ambient effect context* for the whole term and type of this typing judgement. An effect context specifies which operations may be performed. In this example, the effect contexts are all the same as the ambient effect context. We know that `app` can perform the same effects as its argument f as they share the same effect context.

2.2 Overriding the Ambient Effect Context with Absolute Modalities

An *absolute modality* $[E]$ defines a new effect context E that overrides the ambient effect context. For instance, the following function invokes the operation `yield` via the `do` keyword. The `yield` operation takes an integer and returns a unit value.

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$$\vdash \text{fun } x \rightarrow \underbrace{\text{do } \text{yield } (x + 42)}_{@ \text{yield} : \text{Int} \Rightarrow 1} : [\text{yield} : \text{Int} \Rightarrow 1](\underbrace{\text{Int} \rightarrow 1}_{@ \text{yield} : \text{Int} \Rightarrow 1}) @ .$$

The absolute modality $[\text{yield} : \text{Int} \Rightarrow 1]$ specifies a singleton effect context in which the `yield` operation with signature $\text{Int} \Rightarrow 1$ may be performed. Here, it enables the function to override the empty ambient effect context $(.)$, allowing `yield` to be performed in the function body. In MET we would need to write the explicit introduction form for the modality, but in METL the bidirectional type system manages the introduction implicitly. Effect contexts specified by absolute modalities percolate through the structure of a type. A function of type $[E](A \rightarrow B)$ may perform effects E when invoked. A list of type $[E](\text{List } (A \rightarrow B))$ may perform effects E when its components are invoked. For brevity, we define an effect context abbreviation.

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$$\text{eff Gen } a = \text{yield} : a \Rightarrow 1$$

Such abbreviations are merely macros, such that, for instance, $[\text{Gen Int}]$ denotes the modality $[\text{yield} : \text{Int} \Rightarrow 1]$ and $[\text{Gen Int}, E]$ denotes the modality $[\text{yield} : \text{Int} \Rightarrow 1, E]$.

For higher-order functions like `map` and `app` which do not directly perform any effects, we use the empty absolute modality $[\]$. For instance, in METL, the curried first-class higher-order `iter` function, specialised to iterate over a list of integers, is defined as follows.

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```

$$\begin{aligned} \text{iter} &: []((\text{Int} \rightarrow 1) \rightarrow \text{List Int} \rightarrow 1) \\ \text{iter } f \ \text{nil} &= () \\ \text{iter } f \ (\text{cons } x \ xs) &= f \ x; \text{iter } f \ xs \end{aligned}$$

The empty absolute modality $[\]$ specifies an empty effect context in which the function is defined. However, due to subeffecting, `iter` is not limited to only the empty effect context. For instance, we can apply `iter` to the previous function which uses `yield`.

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```

$$\vdash \text{iter } (\text{fun } x \rightarrow \text{do } \text{yield } (x + 42)) : 1 @ \text{Gen Int}$$

What happened? Bidirectional typing eliminates the $[\]$ modality of `iter` and then upcasts its empty effect context to the singleton effect context Gen Int . Following the literature on modal types, we refer to introduction of modalities as *boxing* and elimination as *unboxing*.

To achieve the same flexibility of applying `iter` to any effectful arguments in a traditional row-based effect system, we would need effect polymorphism:

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```

$$\text{iter} : \forall e . (\text{Int} \xrightarrow{e} 1) \xrightarrow{e} \text{List Int} \xrightarrow{e} 1$$

In the METL implementation we permit the $[\]$ annotation to be omitted from global definitions, but for clarity we choose to keep it in the remaining examples in this section.

2.3 Transforming the Ambient Effect Context with Relative Modalities

So far we have only seen examples that are either pure or just perform effects. Absolute modalities suffice for modular programming with such examples without requiring any use of effect polymorphism. However, the situation becomes more interesting when we introduce constructs that manipulate effect contexts non-trivially, such as effect handlers. Effect handlers provide a way of interpreting such effects inside the object language. For instance, we can use an effect handler to interpret Gen Int thanks by simply generating a list of integers.

```

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```

$$\text{asList } f = \text{handle } f \ () \ \text{with}$$

```

197     return ()                ⇒ nil
198     (yield : Int ⇒ 1) x r ⇒ cons x (r ())

```

199 The body of `asList` invokes the function `f` inside a handler. The handler must account for two cases:
200 1) what happens when `f` returns; and 2) what happens when `f` performs `yield`. In the first case,
201 it directly returns the empty list `nil`. In the second case, it conses the integer `x` onto the head of
202 the list returned by the application of `r`. Here `r` is bound to the continuation of performing `yield`
203 inside `f`. The argument type of `r` is determined by the return type of the operation being handled
204 (unit in the case of `yield`) and its return type is determined by the return type of the handler. Thus
205 $r : 1 \rightarrow \text{List Int}$. The continuation `r` reinstalls the handler around its body such that if `yield` is
206 performed again then it will be handled by the same handler. (This kind of handler is known as
207 *deep* in the literature [27].)

208 What type should `asList` have? Naively, we might simply expect to ignore the handler.

```

209 asList : []((1 → 1) → List Int)

```

210 This would be unsound as it would allow us to write:

```

211 crash : [Gen String](String → List Int)
212 crash s = asList (fun () → do yield s)

```

213 The function passed to `asList` yields a string. This is then accidentally handled by the handler in
214 `asList`, which expects an integer.

215 One fix is to box the argument of `asList` with an absolute modality `[Gen Int]`.

```

216 asList : []([Gen Int](1 → 1) → List a)

```

217 To see what happens here, consider the following typing judgement for the inlined function body
218 of `asList` under some effect context `E`.

```

219 ⊢ fun f → handle f () with ... : [Gen Int](1 → 1) → List Int @ E
220                                     @ Gen Int, E

```

221 The effect handler extends the ambient effect context `E` with a `yield` operation to give an effect
222 context of `Gen Int, E`. Meanwhile, the argument `f` has the effect context `Gen Int` specified by the
223 absolute modality `[Gen Int]`. This is sound, because it is safe to invoke a function which can only
224 use `Gen Int` under the effect context `Gen Int, E`.

225 However, the restriction that the argument can only use `Gen Int` severely hinders reusability. We
226 would like to apply `asList` to arguments that may perform other operations in addition to `yield`.
227 To this end, we introduce *relative modalities* which enable us to describe the relative change that a
228 handler makes to the effect context. For instance, consider:

```

229 asList : [](<Gen Int>(1 → 1) → List Int)

```

230 The relative modality `<Gen Int>` is part of the argument type and extends the ambient effect context
231 with `Gen Int` for the inner function `1 → 1`. The typing judgement becomes:

```

232 ⊢ fun f → handle f () with ... : <Gen Int>( 1 → 1 ) → List Int @ E
233                                     @ Gen Int, E

```

234 Now, the effect context for the function of argument `f` is also `Gen Int, E`, matching the effect
235 context at its invocation. This allows the argument `f` to perform other effects from the ambient
236 effect context `E` (which will be forwarded to outer handlers).

237 In practice relative modalities often appear in an argument position and specify which effects of
238 an argument will be handled in the function body. A function that handles effects `D` of its argument
239 typically has a type of the form `<D>(A → B) → C`.

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In a traditional row-based effect system, in order to be able to use `asList` across different effect contexts, we would typically require effect polymorphism.

$$\text{asList} : \forall e . (1 \xrightarrow{\text{Gen Int}, e} 1) \xrightarrow{e} \text{List Int}$$

2.4 Coercions Between Modalities

The automatic unboxing and boxing performed by the bidirectional type system of METL allows values to be coerced between different modalities. For instance, we can extend an absolute modality.

$$\vdash \text{fun } f \rightarrow f : [\text{Gen Int}](1 \rightarrow 1) \rightarrow [\text{Gen Int}, \text{Gen String}](1 \rightarrow 1) @ E$$

In contrast, a relative modality cannot be similarly extended

$$\not\vdash \text{fun } f \rightarrow f : \langle (1 \rightarrow 1) \rangle @ E \rightarrow \langle \text{Gen Int} \rangle (1 \rightarrow 1) @ E \quad \# \text{ Ill-typed}$$

as doing so would insert a fresh `yield : Int ⇒ 1` operation which may shadow other `yield` operations in E , consequently permitting bad programs like `crash` in Section 2.3.

An absolute modality can be coerced into the corresponding relative modality.

$$\vdash \text{fun } f \rightarrow f : [\text{Gen Int}](1 \rightarrow 1) \rightarrow \langle \text{Gen Int} \rangle (1 \rightarrow 1) @ E$$

But the converse is not permitted

$$\not\vdash \text{fun } f \rightarrow f : \langle \text{Gen Int} \rangle (1 \rightarrow 1) \rightarrow [\text{Gen Int}](1 \rightarrow 1) @ E \quad \# \text{ Ill-typed}$$

because the argument may also use effects from the ambient effect context E .

Similarly, the following typing judgement is invalid

$$\not\vdash \text{fun } \underbrace{f}_{@ \text{Gen Int}, E} \rightarrow \underbrace{f ()}_{@ E} : \langle \text{Gen Int} \rangle (1 \rightarrow 1) \rightarrow 1 @ E \quad \# \text{ Ill-typed}$$

because the argument may use `Gen Int` in addition to the ambient effect context E .

2.5 Composing Handlers

We can compose handlers modularly in METL. For example, consider state operations `get` and `put`.

```
eff State s = get : 1 ⇒ s, put : s ⇒ 1
```

We can implement a standard state handler, specialised to integer state, by interpreting a computation over state operations as a state-passing function.

```
state : [](<State Int>(1 → 1) → Int → 1)
```

```
state m = handle m () with
```

```
  return x           ⇒ fun s → x
```

```
  (get : 1 ⇒ Int) () r ⇒ fun s → r s s
```

```
  (put : Int ⇒ 1) s' r ⇒ fun s → r () s'
```

Using integer state we can write a generator which yields the prefix sum of a list.

```
prefixSum : [Gen Int, State Int](List Int → 1)
```

```
prefixSum xs = iter (fun x → do put (do get () + x); do yield (do get ())) xs
```

The absolute modality `[Gen Int, State Int]` aggregates all effects performed in `prefixSum`.

We can now handle `prefixSum` by composing two handlers in sequence.

```
> asList (fun () → state (fun () → prefixSum [3,1,4,1,5,9]) 0)
```

```
# [3,4,8,9,14,23] : List Int
```

The type signature of `state` mentions only `State Int` even though it is applied to a computation which invokes `prefixSum`, which also uses `Gen Int`. In contrast, to achieve the same modularity, conventional row-based effect systems would ascribe the following type to `state`.

```
state : ∀ e . (1  $\xrightarrow{\text{State Int, } e}$  1)  $\xrightarrow{e}$  Int  $\xrightarrow{e}$  1
```

2.6 Storing Effectful Functions in Data Types

We show how modal effect types allow us to smoothly store effectful functions into data types. We consider a richer effect handler example that implements cooperative concurrency using a UNIX-style fork operation [25, 47]. A `Coop` effect context includes two operations.

```
eff Coop = ufork : 1 ⇒ Bool, suspend : 1 ⇒ 1
```

The `ufork` operation returns a boolean. As we shall see, concurrency can be implemented by a handler that invokes the continuation twice. The idea is that passing true to the continuation defines the behaviour of the parent, whereas passing false defines the behaviour of the child. The `suspend` operation suspends the current process allowing another process to run.

We model a process as a data type that embeds a continuation function which takes a list of suspended processes and returns unit. In addition, we define auxiliary functions `push` to append a process onto the end of the list and `next` to remove and then run the process at the head of the list.

```
data Proc = proc (List Proc → 1)                next : [](List Proc → 1)
                                                next q = case q of
push : [](Proc → List Proc → List Proc)        nil           → ()
push x xs = xs ++ cons x nil                    cons (proc p) ps → p ps
```

The following handler implements a scheduler parameterised by a list of suspended processes.

```
schedule : [](<Coop>(1 → 1) → List Proc → 1)
schedule m = handle m () with
  return ()           ⇒ fun q → next q
  (suspend : 1 ⇒ 1) () r ⇒ fun q → next (push (proc (r ())) q)
  (ufork : 1 ⇒ Bool) () r ⇒ fun q → r true (push (proc (r false)) q)
```

The `return`-case is triggered when a process finishes, and runs the next available process. The `suspend`-case pushes the continuation onto the end of the list, before running the next available process. The `ufork`-case implements the process duplication behaviour of UNIX fork by first pushing one application of the continuation onto the end of the list, and then immediately applying the other. Observe that the above code seamlessly stores continuation functions in `Proc` and then puts `Proc` in `List` without even mentioning any effects. These functions are not restricted to be pure; they may use any effects from the ambient effect context.

The `schedule` function allows processes to use any other effects. To achieve this flexibility, a traditional row-based effect system requires effect polymorphism and a parameterised data type.

```
data Proc e = proc (List Proc  $\xrightarrow{e}$  1)
schedule : ∀ e . (1  $\xrightarrow{\text{Coop, } e}$  1)  $\xrightarrow{e}$  List (Proc e)  $\xrightarrow{e}$  1
```

2.7 Masking

Whereas handlers extend the effect context, masking restricts the effect context [5]. Masking is a useful device to conceal private implementation details [36]. We illustrate masking by using a generator to implement a function to find an integer satisfying a predicate.

```
findWrong : []((Int → Bool) → List Int → Maybe Int) # ill-typed
```

```

344 findWrong p xs = handle (iter (fun x → if p x then do yield x) xs) with
345     return _           ⇒ nothing
346     (yield : Int ⇒ 1) x _ ⇒ just x

```

The `findWrong` program is ill-typed because it is unsound to invoke predicate `p` inside the handler, as this would accidentally handle any `yield` operations performed by `p`.

```

349 ⊢ ... handle (iter (fun x → if (p x) then do yield x) xs) with ... : _ @ E
350
351                               @ Gen Int, E

```

Changing the type of `p` from `Int → Bool` to `<Gen Int>(Int → Bool)` would fix the type error but leak the implementation detail that `findWrong` uses `yield`. A better solution is to mask `yield` for the argument `p` and rewrite the handled expression as follows.

```

355 ⊢ ... handle (iter (fun x → if mask<yield>(p x) ... ) with ... : _ @ E
356
357                               @ E

```

The term `mask<yield>(M)` masks the effect `yield` from the ambient effect context for `M`. Now the effect context for `p` is equivalent to the ambient one, since the transformations of extending with `yield` (performed by the handler) followed by masking with `yield` (performed by the mask) cancel each other out. Like a handler, a mask wraps its return value in a relative modality. The term `mask<yield>(p x)` initially returns a value of type `<yield|>Bool` instead of `Bool`, where `<yield|>` is a relative modality masking `yield` from the ambient effect context. In this case METL automatically unboxes the relative modality, as booleans are pure and do not rely on effect contexts.

In general, relative modalities have the form `<L|D>` which specifies a local transformation on the effect context: `L` is a row of effect labels that are removed from the effect context and `D` is a row of effects that are added to the effect context. We write `<D>` as a shorthand for `<|D>`.

2.8 Kinds

A handler extends the effect context with those effects it handles. When a value leaves the scope of a handler, its effect context changes, and we must keep track of this change.

Let us now consider `state'`, a variation of the `state` function defined in Section 2.5 in which the return type of the handled computation is changed from `1` to `1 → 1`. The body of `state'` is exactly the same as that of `state`. We might naively expect its type signature to be the following.

```

375 state' : [](<State Int>(1 → (1 → 1)) → Int → (1 → 1))

```

However, this type is unsound. Suppose we apply `state'` as follows.

```

377 state' (fun () → fun () → do put (do get () + 42)) 0 : 1 → 1
378

```

The function `fun () → do put (do get () + 42)` is returned by the return clause of `state'`, escaping the scope of their handler. To guarantee effect safety, we must capture the fact that the returned function might perform `get` and `put` when invoked. The following type signature is sound.

```

382 state' : [](<State Int>(1 → (1 → 1)) → Int → <State Int>(1 → 1))

```

Let us contrast the types of `state` and `state'`:

```

384 state : [](<State Int>(1 → 1) → Int → 1)
385 state' : [](<State Int>(1 → (1 → 1)) → Int → <State Int>(1 → 1))
386

```

The crucial difference is that the former cannot leak the state effect as the handled computation has unit type, whereas the latter can as the handled computation is a function.

In practice, it is useful to allow a value of base type or an algebraic data type that contains only base types or a type boxed with absolute modalities to appear anywhere, including escaping the scope of a handler. Such values can never depend on the effect context in which they are used. MET

and METL include a kind system in which the **Abs** kind classifies such *absolute types*, whereas the **Any** kind classifies unrestricted types. Subkinding allows absolute types to be treated as unrestricted.

2.9 Value Polymorphism

Now that we have explored the expressive power of the simply-typed fragment of METL, we briefly outline its extension with value polymorphism. For simplicity, METL supports value polymorphism but requires explicit type abstraction and type application. We write explicit type abstractions and applications using braces. For instance, we can define the polymorphic iterate function as follows.

```
iter : ∀ a . [ ]((a → 1) → List a → 1)
iter {a} f nil      = ()
iter {a} f (cons x xs) = f x; iter {a} f xs
```

The extension is mostly routine, however, we must respect kinds. The `state` and `state'` examples in Section 2.8 illustrate a non-uniformity that we must account for. We may generalise them such that the former allows any absolute return type and the latter allows any return type at all.

```
state  : ∀ [a] . [ ](<State Int>(1 → a) → Int → a)
state' : ∀ a . [ ](<State Int>(1 → a) → Int → <State Int>a)
```

The syntax $\forall [a]$ ascribes kind **Abs** to a , allowing values of type a to escape the handler. The syntax $\forall a$ ascribes kind **Any** to a , not allowing values of type a to escape the handler. Though in practice it is usually desirable for return types of computations inside handler scopes to be absolute, the latter type signature is the more general in that simply by η -expanding we can coerce it to the former.

```
⊢ fun {a} m s → state' {a} m s : ∀ [a] . [ ](<State Int>(1 → a) → Int → a) @ .
```

2.10 Effect Polymorphism

Though modal effect types alone suffice for writing a remarkably rich class of modular effectful programs, occasionally effect variables are still useful. In particular, they are required for the implementation of higher-order operations [53, 54, 57], which take closures as arguments.

METL restricts operation arguments and results to be absolute. This is because effect handlers provide non-trivial manipulation of control-flow, which allows operation arguments and results to jump between different effect contexts. For example, if we were to allow an operation `leak` : $(1 \rightarrow 1) \Rightarrow 1$, then we could write the following unsafe program.

```
handle asList (fun () → do leak (fun () → do yield 42)) with
return      _ ⇒ fun () ⇒ 37
(leak : (1 → 1) ⇒ 1) p _ ⇒ p
```

The `asList` handler extends the ambient effect context with `yield`. However, the `leak` handler binds the closure $(\text{fun } () \rightarrow \text{do yield } 42)$ to `p` and returns this closure, leaking the `yield` operation.

Consider a higher-order fork operation which takes a thunk as an argument. (One reason to prefer such an operation over the UNIX fork operation of Section 2.6 is that it can be implemented using an affine handler that invokes each captured continuation at most once.) We may define a recursive effect context for cooperative processes as follows.

```
eff Coop = fork : [Coop](1 → 1) ⇒ 1, suspend : 1 ⇒ 1
```

In order to allow processes to use additional operations as well as `fork` and `suspend` we must extend our modal type system with effect variables. With an effect variable e , we can define the following higher-order `Coop` parameterised over effect context e .

```
eff Coop e = fork : [Coop e, e](1 → 1) ⇒ 1, suspend : 1 ⇒ 1
```

As we show in Section 5.2, modal effect types are compatible with effect variables. Nonetheless, effect variables are only necessary for use-cases such as higher-order effects in which a computation needs to be stored for use in an effect context different from the ambient one.

3 A Multimodal Core Calculus with Effect Handlers

In this section we introduce MET, a simply-typed call-by-value calculus with effect handlers and modal effect types. We present its static and dynamic semantics as well as its meta theory. We aim at a minimal core calculus here and defer extensions such as data types, alternative forms of handlers, and polymorphism (including both value and effect polymorphism) to Section 5.

3.1 Syntax

The syntax of MET is as follows.

Types	$A, B ::= \cdot \mid A \rightarrow B \mid \mu A$	Contexts	$\Gamma ::= \cdot \mid \Gamma, x : \mu_F A \mid \Gamma, \mathbf{\mu}_{\mu_F}$
Masks	$L ::= \cdot \mid \ell, L$	Terms	$M, N ::= () \mid x \mid \lambda x^A. M \mid MN \mid \mathbf{mod}_{\mu} V$
Extensions	$D ::= \cdot \mid \ell : P, D$		$\mid \mathbf{let}_V \mathbf{mod}_{\mu} x = V \mathbf{in} M$
Effect Contexts	$E, F ::= \cdot \mid \ell : P, E$		$\mid \mathbf{do} \ell M \mid \mathbf{mask}_L M$
Signatures	$P ::= A \rightarrow B \mid -$		$\mid \mathbf{handle} M \mathbf{with} H$
Modalities	$\mu ::= [E] \mid \langle L \mid D \rangle$	Values	$V, W ::= () \mid x \mid \lambda x^A. M \mid \mathbf{mod}_{\mu} V$
Kinds	$K ::= \mathbf{Abs} \mid \mathbf{Any}$	Handlers	$H ::= \{\mathbf{return} x \mapsto M\} \mid \{\ell p r \mapsto M\} \uplus H$

MET extends a simply-typed λ -calculus with standard constructs for effects and handlers as well as the main novelty of this work: modal effect types. We highlight the novel parts in grey.

MET is closely related to multimodal type theory (MTT) [20, 21], especially its simply-typed fragment [30]. MTT is a type theory parameterised over a mode theory, which specifies the structures of modes and modalities. It has both multiple modalities and multiple modes. Each type and term is at some mode. Modalities transform types and terms from one mode to another. We present MET without assuming deep familiarity with MTT and discuss further in Section 7.3.

MTT provides us with the flexibility to define our own mode theory. In the following, we first illustrate the structures of modes and modalities for MET before presenting the typing rules.

3.2 Effect Contexts as Modes

The *modes* of MET are effect contexts E . Each type and term is at some mode E , specifying the available effects from the context. As we have seen in Section 2.1, unlike type qualifiers [19], modes (effect contexts) are not decorations on types; instead, they appear in judgements.

Effect contexts E are defined as scoped rows of effect labels [31]. Each label denotes an effectful operation. An effect context may contain the same label multiple times. Each label has a signature. A signature can be an arrow of the form $A \rightarrow B$, which indicates that the operation takes an argument of type A and returns a value of type B , or absent $-$ (similar to presence types [46]), which indicates that the operation of this label cannot be invoked.

Following Rémy [46] and Leijen [31], we identify effects up to reordering of distinct labels, and allow absent labels to be freely added to or removed from the right of effect contexts. For instance, $\ell : P, \ell' : -$ is equivalent to $\ell : P$. We can think of an effect context as denoting a map from labels to infinite sequences of signatures where a cofinite tail of each sequence contains only $-$.

Extensions D and masks L are used respectively to extend effect contexts with more labels or removes some labels from them. Extensions are like effect contexts except that we do not ignore labels with absent signatures in their equivalence relation, so $\ell : P, \ell' : -$ and $\ell : P$ are distinct.

We define a sub-effecting relation on effect contexts: $E \leq E'$ if we can replace the absent signatures in E with proper signatures to obtain E' . We also have a subtyping relation on extensions $D \leq D'$. Different from sub-effecting, it requires D and D' to contain the same row of labels, but allows absent signatures in D to be replaced by other signatures in D' . We give the full rules for type equivalence and sub-effecting in Appendix A.3.

Masks L are simply multisets of labels without signatures; we do not require signatures when removing labels from effect contexts. We define extending $D + E$ and masking $E - L$ as follows.

$$\begin{array}{l}
 D + E = D, E \\
 \cdot - L = \cdot \\
 (\ell : P, E) - L = \begin{cases} E - L & \text{if } L \equiv \ell, L' \\ \ell : P, (E - L) & \text{otherwise} \end{cases} \\
 \cdot \bowtie D = (\cdot, D) \\
 (\ell, L) \bowtie D = \begin{cases} (L', D'') & \text{if } D' \equiv \ell : P, D'' \\ ((\ell, L'), D') & \text{otherwise} \end{cases} \\
 \text{where } (L', D') = L \bowtie D
 \end{array}$$

We write $L \bowtie D = (L', D')$ for the difference between L and D . The L' are those labels in L not appearing in the domain of D , and the D' are those entries in D with labels not in L .

3.3 Modalities Manipulating Effect Contexts

Components of types and terms may have different effect contexts from the ambient one. We use *modalities* to manipulate effect contexts. For the modal type μA , the effect context for A is derived from the ambient effect context manipulated by the modality μ as follows.

$$[E](F) = E \qquad \langle L|D \rangle(F) = D + (F - L)$$

The absolute modality $[E]$ completely replaces the effect context F with E , similar to effect annotations on function types in traditional effect systems. The relative modality $\langle L|D \rangle$ is the key novelty of MET. It specifies a transformation on the input effect context. It masks the labels L in F before extending the resulting context with D . We call $\langle | \rangle$ the identity modality and write $\mathbb{1}$ for it. Modalities are monotone total functions on effect contexts. If $E \leq F$, we have $\mu(E) \leq \mu(F)$.

We write μ_F for the pair of μ and F where F is the effect context that μ acts on. We refer to such a pair as an indexed modality. We write $\mu_F : E \rightarrow F$ if $\mu(F) = E$. (The arrow goes from E to F instead of the other direction to keep closer to MTT [20, 21]. For readers familiar with MTT, indexed modalities μ_F correspond to the notion of modalities in MTT as they are concrete morphisms between modes and our modalities μ actually correspond to indexed families of modalities in MTT.)

Modality Composition. We can compose the actions of modalities in the intuitive way.

$$\begin{array}{l}
 \mu \circ [E] = [E] \\
 [E] \circ \langle L|D \rangle = [D + (E - L)] \\
 \langle L_1|D_1 \rangle \circ \langle L_2|D_2 \rangle = \langle L_1 + L_2|D_2 + D_1 \rangle \quad \text{where } (L, D) = L_2 \bowtie D_1
 \end{array}$$

To keep close to MTT, our composition reads from left to right. First, an absolute modality completely specifies the new effect context, thus shadowing any other modality μ . Second, replacing the effect context with E and then masking L and extending with D is equivalent to just replacing with $D + (E - L)$. Third, sequential masking and extending can be combined into one by using $L_2 \bowtie D_1$ to cancel the overlapping part of L_2 and D_1 . For instance, we have $\langle \text{yield} : \text{Int} \rightarrow 1 \rangle \circ \langle \text{yield} \rangle = \mathbb{1}$.

Composition is well-defined since composing followed by applying is equivalent to sequentially applying $(\mu \circ \nu)(E) = \nu(\mu(E))$. We also have associativity $(\mu \circ \nu) \circ \xi = \mu \circ (\nu \circ \xi)$ and identity $\mathbb{1}$. The definition of composition naturally generalises to indexed modalities μ_F . We can compose $\mu_F : E \rightarrow F$ and $\nu_E : E' \rightarrow E$ to get $\mu_F \circ \nu_E : E' \rightarrow F$ which is defined as $(\mu \circ \nu)_F$.

540 *Modality Transformations.* Just as modalities allow us to manipulate effect contexts, we need a
 541 transformation relation that tells us when we can change modalities.

542 We write $\mu_F \Rightarrow \nu_F$ for a transformation between indexed modalities $\mu_F : E \rightarrow F$ and $\nu_F : E' \rightarrow F$.
 543 Intuitively, such a transformation indicates that under ambient effect context F , the action of μ can
 544 be replaced by the action of ν . This relation is used in the typing rule for variables in Section 3.5. It
 545 indicates that for a variable boxed by μ under the effect context F , whether we can unbox and use
 546 it inside another modality ν . For instance, supposing we have a variable of type $\mu(1 \rightarrow 1)$ under
 547 ambient effect context F , we can rewrap it to a function of type $\nu(1 \rightarrow 1)$ if $\mu_F \Rightarrow \nu_F$.

548 Intuitively, $\mu_F \Rightarrow \nu_F$ is safe when $\nu(F)$ is larger than $\mu(F)$ so that we have not lost any operations.
 549 Moreover, subeffecting should not break the safety guarantee of transformations. That is, $\mu(F') \leq$
 550 $\nu(F')$ should hold for any effect context F' with $F \leq F'$. We formally define $\mu_F \Rightarrow \nu_F$ by the
 551 transitive closure of the following four rules.

553 MT-ABS

$$\begin{array}{c}
 554 \mu_F : E' \rightarrow F \quad \text{MT-UPCAST} \quad \text{MT-EXPAND} \quad \text{MT-SHRINK} \\
 555 \frac{E \leq E'}{[E]_F \Rightarrow \mu_F} \quad \frac{D \leq D'}{\langle L|D \rangle_F \Rightarrow \langle L|D' \rangle_F} \quad \frac{(F-L) \equiv \ell : A \rightarrow B, E}{\langle L|D \rangle_F \Rightarrow \langle \ell, L|D, \ell : A \rightarrow B \rangle_F} \quad \frac{(F-L) \equiv \ell : P, E}{\langle \ell, L|D, \ell : P \rangle_F \Rightarrow \langle L|D \rangle_F}
 \end{array}$$

556 MT-ABS allows us to transform an absolute modality to any other modality as long as no effect
 557 leaks. MT-UPCAST allow us to upcast a label with an absent signature in D to an arbitrary signature,
 558 since the corresponding operation is unused. Recall that the subtyping relation between extensions
 559 only upcasts signatures. MT-EXPAND allows us to simultaneously mask and extend some present
 560 operations given that these operations exist in the ambient effect context F . MT-SHRINK allows us
 561 to do the reverse for any operations regardless of their presence.

562 The following lemma shows that the syntactic definition of transformation matches our intuition.
 563 The proof is in Appendix B.2.

564 LEMMA 3.1 (SEMANTICS OF MODALITY TRANSFORMATION). *We have $\mu_F \Rightarrow \nu_F$ if and only if*
 565 *$\mu(F') \leq \nu(F')$ for all F' with $F \leq F'$.*

566 Let us give some examples here. First, $[\]_E \Rightarrow \mu_E$ always holds, consistent with the intuition that
 567 pure values can be used anywhere safely. Second, $\langle \ell : - \rangle_E \Rightarrow \langle \ell : P \rangle_E$ always holds. Third, we
 568 have $\langle \ell | \ell : P \rangle_{\ell, P, E} \Leftrightarrow \langle \ell \rangle_{\ell, P, E}$ in both directions. Last, $\mathbb{1}_E \Rightarrow \langle \ell : P \rangle_E$ does not hold for any E .

572 3.4 Kinds and Contexts

573 As illustrated in Section 2.8, we have two kinds Abs and Any. The Abs kind is a sub-kind of the kind
 574 of all types Any, and denotes types of values that are guaranteed not to use operations from the
 575 ambient effect context. We show the kinding and well-formedness rules for types and signatures in
 576 Figure 1, relying on the well-formedness of modalities and effect contexts, which is standard and
 577 defined in Appendix A.3. Function arrows have kind Any due to the possibility of using operations
 578 from the ambient effect context. A modal type $[E]A$ is absolute as it cannot depend on the ambient
 579 effect context. We restrict the kind of the argument and return value of effects to be Abs in order to
 580 prevent effect leakage as discussed in Section 2.10.

581 Contexts are ordered. Each term variable binding $x :_{\mu_F} A$ in contexts is tagged with an indexed
 582 modality μ_F . Intuitively, this annotation means that the term bound to x is defined inside modality
 583 μ under the effect context F . We omit this annotation when μ is identity. Contexts contain locks μ_F
 584 carrying indexed modalities μ_F which track the transformations on the effect context. Whenever
 585 we go inside a modality we put a corresponding lock into the context. They are used to control the
 586 accessibility of variables as we will see in Section 3.5.

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 \end{array}$$

$$\begin{array}{c}
 \boxed{\Gamma \vdash A : K} \quad \boxed{\Gamma \vdash P} \quad \boxed{\Gamma \vdash (\mu, A) \Rightarrow v @ F} \\
 \\
 \frac{}{\Gamma \vdash 1 : \text{Abs}} \quad \frac{\Gamma \vdash A : \text{Abs}}{\Gamma \vdash A : \text{Any}} \quad \frac{\Gamma \vdash [E] \quad \Gamma \vdash A : \text{Any}}{\Gamma \vdash [E]A : \text{Abs}} \quad \frac{\Gamma \vdash \langle L|D \rangle \quad \Gamma \vdash A : K}{\Gamma \vdash \langle L|D \rangle A : K} \\
 \\
 \frac{\Gamma \vdash A : \text{Any} \quad \Gamma \vdash B : \text{Any}}{\Gamma \vdash A \rightarrow B : \text{Any}} \quad \frac{\Gamma \vdash A : \text{Abs} \quad \Gamma \vdash B : \text{Abs}}{\Gamma \vdash A \rightarrow B} \quad \frac{\Gamma \vdash A : \text{Abs}}{\Gamma \vdash (\mu, A) \Rightarrow v @ F} \quad \frac{\mu_F \Rightarrow v_F}{\Gamma \vdash (\mu, A) \Rightarrow v @ F} \\
 \\
 \boxed{\Gamma @ E} \\
 \\
 \frac{}{\cdot @ E} \quad \frac{\Gamma @ F \quad \mu_F : E \rightarrow F \quad \Gamma \vdash A : K}{\Gamma, x :_{\mu_F} A @ F} \quad \frac{\Gamma @ F \quad \mu_F : E \rightarrow F}{\Gamma, \blacksquare_{\mu_F} @ E}
 \end{array}$$

Fig. 1. Representative kinding, well-formedness, and auxiliary rules for MET.

We define the relation $\Gamma @ E$ that context Γ is well-formed at effect context E in Figure 1. For instance, the following context is well-formed at effect context E . Reading from left to right, the lock $\blacksquare_{[E]_F}$ switches the effect context from F to E as $[E](F) = E$.

$$x :_{\mu_F} A_1, y :_{v_F} A_2, \blacksquare_{[E]_F}, z :_{\xi_E} A_3 @ E$$

Following MTT, we define locks(\cdot) to compose all the modalities on the locks in a context.

$$\text{locks}(\cdot) = \mathbb{1} \quad \text{locks}(\Gamma, \blacksquare_{\mu_F}) = \text{locks}(\Gamma) \circ \mu_F \quad \text{locks}(\Gamma, x :_{\mu_F} A) = \text{locks}(\Gamma)$$

Following MTT, we identify contexts up to the following two equations.

$$\Gamma, \blacksquare_{\mathbb{1}_E} @ E = \Gamma @ E \quad \Gamma, \blacksquare_{\mu_F}, \blacksquare_{v_F'} @ E = \Gamma, \blacksquare_{\mu_F \circ v_F'} @ E$$

3.5 Typing

The typing rules of MET are shown in Figure 2. The typing judgement $\Gamma \vdash M : A @ E$ means that the term M has type A under context Γ and effect context E . As usual, we require $\Gamma @ E$, $\Gamma \vdash E$, $\Gamma \vdash A : K$ for some K , and well-formedness for type annotations as well-formedness conditions. We explain the interesting rules, which are highlighted in grey; the other rules are standard.

Modality Introduction and Elimination. Modalities are introduced by T-MOD and eliminated by T-LETMOD. The term $\mathbf{mod}_{\mu} V$ introduces modality μ to the type of the conclusion and lock \blacksquare_{μ_F} into the context of the premise, and requires the value V to be well-typed under the new effect context E manipulated by μ . The lock \blacksquare_{μ_F} tracks the change to the effect context. Specialising the modality μ to concrete absolute or relative modalities, we get the following two rules.

$$\frac{\Gamma, \blacksquare_{[E]_F} \vdash V : A @ E}{\Gamma \vdash \mathbf{mod}_{[E]} V : [E]A @ F} \quad \frac{\Gamma, \blacksquare_{\langle L|D \rangle_F} \vdash V : A @ D + (F - L)}{\Gamma \vdash \mathbf{mod}_{\langle L|D \rangle} V : \langle L|D \rangle A @ F}$$

Note that we use modality μ instead of indexed modality μ_F in types and terms, because the index can always be inferred from the effect context. We restrict \mathbf{mod} to values to avoid effect leakage [2, 35]. Otherwise, a term such as $\mathbf{mod}_{\langle \ell:P \rangle} (\mathbf{do} \ell V)$ would type check under the empty effect context but get stuck due to the unhandled operation ℓ .

The term $\mathbf{let}_v \mathbf{mod}_{\mu} x = V \mathbf{in} M$ moves the modality μ from the type of V to the binding of x . As with boxing, unboxing is restricted to values. Following MTT, we use let-style modality elimination which takes another modality v in addition to the modality μ that is eliminated from V . This is

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$$\boxed{\Gamma \vdash M : A @ E}$$

$ \begin{array}{c} \text{T-VAR} \\ v_F = \text{locks}(\Gamma') : E \rightarrow F \\ \Gamma \vdash (\mu, A) \Rightarrow v @ F \\ \hline \Gamma, x : \mu_F A, \Gamma' \vdash x : A @ E \end{array} $	$ \begin{array}{c} \text{T-MOD} \\ \mu_F : E \rightarrow F \\ \Gamma, \mu_F \vdash V : A @ E \\ \hline \Gamma \vdash \mathbf{mod}_\mu V : \mu A @ F \end{array} $	$ \begin{array}{c} \text{T-LETMOD} \\ v_F : E \rightarrow F \quad \Gamma, \mu_{v_F} \vdash V : \mu A @ E \\ \Gamma, x : v_F \circ \mu_E A \vdash M : B @ F \\ \hline \Gamma \vdash \mathbf{let}_v \mathbf{mod}_\mu x = V \mathbf{in} M : B @ F \end{array} $
$ \begin{array}{c} \text{T-ABS} \\ \Gamma, x : A \vdash M : B @ E \\ \hline \Gamma \vdash \lambda x^A. M : A \rightarrow B @ E \end{array} $	$ \begin{array}{c} \text{T-APP} \\ \Gamma \vdash M : A \rightarrow B @ E \\ \Gamma \vdash N : A @ E \\ \hline \Gamma \vdash M N : B @ E \end{array} $	$ \begin{array}{c} \text{T-DO} \\ E = \ell : A \rightarrow B, F \\ \Gamma \vdash N : A @ E \\ \hline \Gamma \vdash \mathbf{do} \ell N : B @ E \end{array} $
$ \begin{array}{c} \text{T-MASK} \\ \Gamma, \mu_{\langle L \rangle_F} \vdash M : A @ F - L \\ \hline \Gamma \vdash \mathbf{mask}_L M : \langle L \rangle A @ F \end{array} $	$ \begin{array}{c} \text{T-HANDLER} \\ H = \{\mathbf{return} \ x \mapsto N\} \uplus \{\ell_i \ p_i \ r_i \mapsto N_i\}_i \\ \Gamma, \mu_{\langle D \rangle_F} \vdash M : A @ D + F \quad \Gamma, x : \langle D \rangle A \vdash N : B @ F \\ D = \{\ell_i : A_i \rightarrow B_i\}_i \quad [\Gamma, p_i : A_i, r_i : B_i \rightarrow B \vdash N_i : B @ F]_i \\ \hline \Gamma \vdash \mathbf{handle} \ M \ \mathbf{with} \ H : B @ F \end{array} $	

Fig. 2. Typing rules for MET.

crucial for sequential unboxing. For instance, the following term sequentially unboxes $x : v\mu A$. The variables y and z are bound as $y : v\mu A$ and $z : v\circ\mu A$, respectively.

$$\mathbf{let} \ \mathbf{mod}_v \ y = x \ \mathbf{in} \ \mathbf{let}_v \ \mathbf{mod}_\mu \ z = y \ \mathbf{in} \ M$$

Masking and Handling. Masking and handling also introduce relative modalities. Unlike **mod**, these constructs can apply to computations as they perform masking and handling semantically. In T-MASK, the mask $\mathbf{mask}_L M$ removes effects L from the ambient effect context for M . For the return value of M , we need to box it with $\langle L \rangle$ to reconcile the mismatch between $F - L$ and F . In T-HANDLER, the handler $\mathbf{handle} \ M \ \mathbf{with} \ H$ extends the ambient effect context with effects D for M . For the return value of M which is bound as x in the return clause, we need to box it with $\langle D \rangle$ to reconcile the mismatch between $D + F$ and F . The other parts of the handler rule are standard.

Accessing Variables. The T-VAR rule uses the auxiliary judgement $\Gamma \vdash (\mu, A) \Rightarrow v @ F$ defined in Figure 1. Variables of absolute types can always be used as they do not depend on the effect context. For a non-absolute term variable binding $x : \mu_F A$ from context $\Gamma, x : \mu_F A, \Gamma'$, we must guarantee that it is safe to use x in the current effect context. The term bound to x is defined inside μ under the effect context F . As we track all transformations on effect contexts up to the binding of x as locks in Γ' , the current effect context E is obtained by applying $\text{locks}(\Gamma')$ to F . Thus, we need the transformation $\mu_F \Rightarrow \text{locks}(\Gamma')_F$ to hold for effect safety.

Subeffecting. Subeffecting is incorporated into the T-VAR rule within the transformation relation $\mu_F \Rightarrow v_F$. We have seen how subeffecting works in Section 2.4. We give another example here which upcasts the empty effect context to E . It is well-typed because $[\] \Rightarrow [E]$ holds.

$$\lambda x^{[\](\text{Int} \rightarrow \text{Int})}. \mathbf{let} \ \mathbf{mod}_{[\]} \ y = x \ \mathbf{in} \ \mathbf{mod}_{[E]} \ y : [\](\text{Int} \rightarrow \text{Int}) \rightarrow [E](\text{Int} \rightarrow \text{Int})$$

3.6 Operational Semantics

The operational semantics for MET is quite standard [24]. We first define evaluation contexts \mathcal{E} :

Evaluation contexts $\mathcal{E} ::= [] \mid \mathcal{E} N \mid V \mathcal{E} \mid \mathbf{do} \ell \mathcal{E} \mid \mathbf{mask}_L \mathcal{E} \mid \mathbf{handle} \mathcal{E} \mathbf{with} H$

The reduction rules are as follows.

E-APP	$(\lambda x^A.M) V \rightsquigarrow M[V/x]$
E-LETMOD	$\mathbf{let}_V \mathbf{mod}_\mu x = \mathbf{mod}_\mu V \mathbf{in} M \rightsquigarrow M[V/x]$
E-MASK	$\mathbf{mask}_L V \rightsquigarrow \mathbf{mod}_{\langle L \rangle} V$
E-RET	$\mathbf{handle} V \mathbf{with} H \rightsquigarrow N[(\mathbf{mod}_{\langle D \rangle} V)/x]$, where $(\mathbf{return} x \mapsto N) \in H$
E-OP	$\mathbf{handle} \mathcal{E}[\mathbf{do} \ell V] \mathbf{with} H \rightsquigarrow N[V/p, (\lambda y.\mathbf{handle} \mathcal{E}[y] \mathbf{with} H)/r]$, where $0\text{-free}(\ell, \mathcal{E})$ and $(\ell p r \mapsto N) \in H$
E-LIFT	$\mathcal{E}[M] \rightsquigarrow \mathcal{E}[N]$, if $M \rightsquigarrow N$

The only slightly non-standard aspect of the rules is the boxing of values escaping masks and handlers. They coincide with the typing rules for masks and handlers. In E-RET, we assume handlers are decorated with the operations D that they handle as in Section 2.

Following Biernacki et al. [5], the predicate $n\text{-free}(\ell, \mathcal{E})$ is defined inductively on evaluation contexts as follows. The meta function $\text{count}(\ell; L)$ yields the number of ℓ labels in L . We omit the inductive cases that do not change n . Notice that the cases for introduction and elimination of modalities fall into this category as they require values which cannot be of the form $\mathbf{do} \ell V$.

$0\text{-free}(\ell, [])$	$\frac{n\text{-free}(\ell, \mathcal{E})}{(n)\text{-free}(\ell, \mathbf{do} \ell' \mathcal{E})}$	$\frac{n\text{-free}(\ell, \mathcal{E}) \quad \text{count}(\ell; L) = m}{(n+m)\text{-free}(\ell, \mathbf{mask}_L \mathcal{E})}$
$\frac{(n+1)\text{-free}(\ell, \mathcal{E}) \quad \ell \in \text{dom}(H)}{n\text{-free}(\ell, \mathbf{handle} \mathcal{E} \mathbf{with} H)}$	$\frac{n\text{-free}(\ell, \mathcal{E}) \quad \ell \notin \text{dom}(H)}{n\text{-free}(\ell, \mathbf{handle} \mathcal{E} \mathbf{with} H)}$	

3.7 Type Soundness and Effect Safety

We prove type soundness and effect safety for MET. Our proofs cover the extensions in Section 5.

MET enjoys relatively standard substitution properties along the lines of Kavvos and Gratzner [30]. For example, we have the following rule for substituting values with modalities into terms.

$$\frac{\Gamma, \mathbf{mod}_{\mu_F} \vdash V : A @ F' \quad \Gamma, x :_{\mu_F} A, \Gamma' \vdash M : B @ E}{\Gamma, \Gamma' \vdash M[V/x] : B @ E}$$

We state and prove the relevant properties in Appendix B.3.

To state syntactic type soundness, we first define normal forms.

Definition 3.2 (Normal Forms). We say a term M is in a normal form with respect to effect type E , if it is either in value normal form $M = U$ or of form $M = \mathcal{E}[\mathbf{do} \ell U]$ for $\ell \in E$ and $n\text{-free}(\ell, \mathcal{E})$.

The following together give type soundness and effect safety (proofs in Appendices B.4 and B.5).

THEOREM 3.3 (PROGRESS). *If $\vdash M : A @ E$, then either there exists N such that $M \rightsquigarrow N$ or M is in a normal form with respect to E .*

THEOREM 3.4 (SUBJECT REDUCTION). *If $\Gamma \vdash M : A @ E$ and $M \rightsquigarrow N$, then $\Gamma \vdash N : A @ E$.*

4 Encoding Effect Polymorphism in MET

Even without effect variables, MET is sufficiently expressive to encode programs from conventional row-based effect systems providing effect variables on function arrows always refer to the lexically closest one. This is an important special case, since most functions in practice use at most one effect variable. For example, as of July 2024, the Koka repository contains 520 effectful functions across 112 files but only 86 functions across 5 files use more than one effect variable, almost all of them internal primitives for handlers not exposed to programmers. Moreover, almost all programs in the Frank repository make no mention of effect variables, relying on syntactic sugar to hide the single effect variable. We formally characterise and prove this intuition on the expressiveness of MET.

4.1 Row Effect Types with a Single Effect Variable

We first define F_{eff}^1 , a core calculus with row-based effect types in the style of Koka [32], but where each scope can only refer to the lexically closest effect variable.

Types	$A, B ::= 1 \mid A \rightarrow^{\{E \varepsilon\}} B \mid \forall \varepsilon. A$	Terms	$M, N ::= () \mid x \mid \lambda^{\{E \varepsilon\}} x^A. M \mid MN$
Effects	$L, D, E, F ::= \cdot \mid \ell, E$	Contexts	$\Gamma ::= \cdot \mid \Gamma, x :_{\varepsilon} A \mid \Gamma, \blacklozenge_E \mid \Gamma, \blacklozenge_E^{\Delta} \mid \mathbf{mask}_L M \mid \mathbf{handle} M \mathbf{with} H$
Values	$V, W ::= x \mid \lambda^{\{E \varepsilon\}} x^A. M \mid \Lambda \varepsilon. V$	Handlers	$H ::= \{\mathbf{return} x \mapsto M\} \mid \{\ell pr \mapsto M\} \uplus H$

In types we include units, effectful functions, and effect abstraction $\forall \varepsilon. A$. As we consider only one effect variable at a time, we need not track effect variables on function types and effect abstraction. Nonetheless, we include them in grey font for easier comparison with existing calculi. In Γ , each term variable is annotated with the effect variable ε that was referred to at the time of its introduction. Further, we add markers \blacklozenge_E and \blacklozenge_E^{Δ} to the context, which track the change of effects due to functions, masks, handlers, and effect abstraction. These markers are not needed by the typing rules but help with the encoding. As with MET, we require contexts to be ordered. For simplicity we assume operation signatures come from a global context $\Sigma = \{\ell : A \rightarrow B\}$, thus unifying extensions D , masks L , and effects (effect contexts) E into one syntactic category. Mirroring our kind restriction for operation signatures in MET, we assume that these A and B are not function arrows, but they can be effect abstractions (which may themselves contain function arrows).

Figure 3 gives the typing rules of F_{eff}^1 . The judgement $\Gamma \vdash M : A \mid \{E|\varepsilon\}$ states that in context Γ , the term M has type A and might use concrete effects E extended with effect variable ε . The typing rules are mostly standard for row-based effect systems. The R-VAR rule ensures that either the current effect variable matches the effect variable at which the variable was introduced or that the value is an effect abstraction or unit. These constraints guarantee that programs can only refer to one effect variable in one scope. The R-APP, R-DO, R-MASK, and R-HANDLER rules are standard. The R-ABS rule is standard except for requiring the effect variable to remain unchanged. The R-EABS rule introduces a new effect variable ε' and the R-EAPP rule instantiates an effect abstraction. While conventional systems allow instantiating with any effect row, this rule only allows instantiation with the ambient effects E . The instantiation operator $[\{E|\varepsilon\}/]$ implements standard type substitution for the single effect variable as follows.

$$\begin{aligned} 1[\{E|\varepsilon\}/] &= 1 & (\forall \varepsilon'. A)[\{E|\varepsilon\}/] &= \forall \varepsilon'. A \\ (A \rightarrow^{\{F|\varepsilon'\}} B)[\{E|\varepsilon\}/] &= A[\{E|\varepsilon\}/] \rightarrow^{\{F, E|\varepsilon\}} B[\{E|\varepsilon\}/] \end{aligned}$$

The following F_{eff}^1 function (grey parts omitted) sums up all integers yielded by its argument.

```
asSum :  $\forall. (1 \rightarrow^{\text{yield}} 1) \rightarrow \text{Int}$ 
asSum =  $\Lambda. \lambda f. \mathbf{handle} f () \mathbf{with} \{\mathbf{return} x \mapsto 0, \text{yield } x \ r \mapsto x + r\}$ 
```


<div style="border: 1px solid black; padding: 2px; margin-bottom: 10px;">$\Gamma \vdash M : A! \{E \varepsilon\}$</div> <div style="margin-bottom: 10px;"> $\frac{\varepsilon = \varepsilon' \text{ or } A = \forall \varepsilon'' . A' \text{ or } A = 1}{\Gamma_1, x :_{\varepsilon'} A, \Gamma_2 \vdash x : A! \{E \varepsilon\}}$ </div> <div> $\frac{\varepsilon' \notin \text{ftv}(\Gamma) \quad \Gamma, \blacklozenge_E^\Delta \vdash V : A! \{ \cdot \varepsilon' \}}{\Gamma \vdash \Lambda \varepsilon' . V : \forall \varepsilon' . A! \{E \varepsilon\}}$ </div>	<div style="margin-bottom: 10px;"> $\frac{\Gamma, \blacklozenge_E, x :_{\varepsilon} A \vdash M : B! \{F \varepsilon\}}{\Gamma \vdash \lambda^{(F \varepsilon)} x^A . M : A \rightarrow^{(F \varepsilon)} B! \{E \varepsilon\}}$ </div> <div> $\frac{\Gamma \vdash M : \forall \varepsilon' . A! \{E \varepsilon\}}{\Gamma \vdash M \blacklozenge \{E \varepsilon\} : A[\{E \varepsilon\}/\cdot]! \{E \varepsilon\}}$ </div>	<div style="margin-bottom: 10px;"> $\frac{\Gamma \vdash M : A \rightarrow^{\{E \varepsilon\}} B! \{E \varepsilon\}}{\Gamma \vdash MN : B! \{E \varepsilon\}}$ </div> <div> $\frac{\Gamma, \blacklozenge_{L+E} \vdash M : A! \{E \varepsilon\}}{\Gamma \vdash \mathbf{mask}_L M : A! \{L + E \varepsilon\}}$ </div>
<div style="margin-bottom: 10px;"> $\frac{(\ell : A \rightarrow B) \in \Sigma \quad \Gamma \vdash M : A! \{\ell, E \varepsilon\}}{\Gamma \vdash \mathbf{do} \ell M : B! \{\ell, E \varepsilon\}}$ </div>	<div style="margin-bottom: 10px;"> $\frac{\Gamma, \blacklozenge_E \vdash M : A! \{\bar{\ell}_i, E \varepsilon\} \quad \Gamma, x :_{\varepsilon} A \vdash N : B! \{E \varepsilon\} \quad \{\ell_i : A_i \rightarrow B_i\} \subseteq \Sigma \quad [\Gamma, p_i :_{\varepsilon} A_i, r_i :_{\varepsilon} B_i \rightarrow^E B \vdash N_i : B! \{E \varepsilon\}]_i}{\Gamma \vdash \mathbf{handle} M \mathbf{with} \{\mathbf{return} x \mapsto N\} \uplus \{\ell_i p_i r_i \mapsto N_i\}_i : B! \{E \varepsilon\}}$ </div>	

 Fig. 3. Typing rules of F_{eff}^1 .

4.2 Encoding

We now give compositional translations for types and contexts of F_{eff}^1 into MET. We transform F_{eff}^1 types at effect context E to modal types in MET by the translation $\llbracket - \rrbracket_E$. For the unit type, we insert the identity modality. For a function arrow $A \rightarrow^F B$, the relative modality $\langle E - F | F - E \rangle$ heralds the transition from effect context E to effect context F as we enter the function. For effect abstraction, the empty absolute modality simulates entering a new effect context with different effect variables. We translate contexts by translating each type and moving top-level modalities to their bindings. For each marker, we insert a corresponding lock to reflect the changes of effect context. The auxiliary function $\text{topmod}(-)$ extracts the top-level modality.

$$\begin{array}{ll}
 \llbracket \mathbb{1} \rrbracket_E = \mathbb{1} & \llbracket \cdot \rrbracket_E = \cdot \\
 \llbracket A \rightarrow^F B \rrbracket_E = \langle E - F | F - E \rangle (\llbracket A \rrbracket_F \rightarrow \llbracket B \rrbracket_F) & \llbracket \Gamma, x : A \rrbracket_E = \llbracket \Gamma \rrbracket_E, x :_{\mu_E} A' \text{ for } \mu A' = \llbracket A \rrbracket_E \\
 \llbracket \forall . A \rrbracket_E = \llbracket \square \rrbracket \llbracket A \rrbracket . & \llbracket \Gamma, \blacklozenge_F \rrbracket_E = \llbracket \Gamma \rrbracket_F, \blacklozenge_{\langle F-E | E-F \rangle} \\
 \text{topmod}(\mu A) = \mu & \llbracket \Gamma, \blacklozenge_F^\Delta \rrbracket = \llbracket \Gamma \rrbracket_F, \blacklozenge_{\square}
 \end{array}$$

Observe that not every valid typing judgement in F_{eff}^1 can be transformed to valid typing judgement in MET, because the translation depends on markers in contexts, while the typing of F_{eff}^1 does not. We define well-scoped typing judgements, which characterise the typing judgements for which our encoding is well-defined, as follows.

Definition 4.1 (Well-scoped). A typing judgement $\Gamma_1, x :_{\varepsilon} A, \Gamma_2 \vdash M : B! E$ is *well-scoped* for x if either $x \notin \text{fv}(M)$ or $\blacklozenge_F^\Delta \notin \Gamma_2$ or $A = \forall . A'$. A typing judgement $\Gamma \vdash M : A! E$ is *well-scoped* if it is well-scoped for all $x \in \Gamma$.

In particular, if the judgement at the bottom of a derivation tree is well-scoped, then every judgement in the derivation tree is well-scoped.

Figure 4 shows the translation from F_{eff}^1 terms with their types and effect contexts to MET terms. We use the following syntactic sugar to simply the encoding.

$$\begin{aligned} \text{let mod}_\mu = M \text{ in } N &\doteq (\lambda x. \text{let mod}_\mu x = x \text{ in } N) M \\ \text{let mod}_{\mu, \nu} x = V \text{ in } M &\doteq \text{let mod}_\mu x = V \text{ in let}_\mu \text{ mod}_\nu x = x \text{ in } M \end{aligned}$$

$$\boxed{M : A ! E \dashrightarrow M'}$$

$$\begin{array}{c} \text{R-VAR} \\ \frac{\mu := \text{topmod}(\llbracket A \rrbracket_E)}{x : A ! E \dashrightarrow \text{mod}_\mu x} \end{array} \quad \begin{array}{c} \text{R-APP} \\ \frac{M : A \xrightarrow{E} B ! E \dashrightarrow M' \quad N : A ! E \dashrightarrow N' \quad x \text{ fresh}}{MN : B ! E \dashrightarrow \text{let mod}_\perp x = M' \text{ in } x N'} \end{array}$$

$$\begin{array}{c} \text{R-ABS} \\ \frac{M : B ! F \dashrightarrow M' \quad \nu := \langle E - F | F - E \rangle \quad \mu := \text{topmod}(\llbracket A \rrbracket_F)}{\lambda^F x^A. M : A \xrightarrow{F} B ! E \dashrightarrow \text{mod}_\nu (\lambda x^{\llbracket A \rrbracket_F}. \text{let mod}_\mu x = x \text{ in } M')} \end{array} \quad \begin{array}{c} \text{R-EABS} \\ \frac{V : A ! \cdot \dashrightarrow V'}{\Lambda. V : \forall. A ! E \dashrightarrow \text{mod}_\square V'} \end{array}$$

$$\begin{array}{c} \text{R-EAPP} \\ \frac{M : \forall. A ! E \dashrightarrow M' \quad x \text{ fresh}}{M @ : A[E/] ! E \dashrightarrow \text{let mod}_\square x = M' \text{ in } x} \end{array} \quad \begin{array}{c} \text{R-DO} \\ \frac{M : A ! \ell, E \dashrightarrow M'}{\text{do } \ell M : B ! \ell, E \dashrightarrow \text{do } \ell M'} \end{array}$$

$$\begin{array}{c} \text{R-MASK} \\ \frac{M : A ! E \dashrightarrow M' \quad \mu_1 := \text{topmod}(\llbracket A \rrbracket_E) \quad \mu_2 := \text{topmod}(\llbracket A \rrbracket_{L+E})}{\text{mask}_L M : A ! L + E \dashrightarrow \text{let mod}_{\langle L \rangle, \mu_1} x = \text{mask}_L M' \text{ in mod}_{\mu_2} x} \end{array}$$

$$\begin{array}{c} \text{R-HANDLER} \\ \frac{M : A ! \bar{\ell}_i, E \dashrightarrow M' \quad N : B ! E \dashrightarrow N' \quad [N_i : B ! E \dashrightarrow N'_i]_i \\ \mu := \text{topmod}(\llbracket A \rrbracket_{\bar{\ell}_i, E}) \quad \mu' := \text{topmod}(\llbracket A \rrbracket_E) \\ N'' := \text{let mod}_{\langle \bar{\ell}_i \rangle, \mu} x = x \text{ in let}_{\mu'} \text{ mod}_\perp x = \text{mod}_\perp x \text{ in } N' \\ [\mu_i := \text{topmod}(\llbracket A_i \rrbracket \cdot)] \quad N''_i := \text{let mod}_{\mu_i} p_i = p_i \text{ in } N''_i]_i \\ H = \{\text{return } x \mapsto N\} \uplus \{\ell_i p_i r_i \mapsto N_i\}_i \quad H' := \{\text{return } x \mapsto N''\} \uplus \{\ell_i p_i r_i \mapsto N''_i\}_i}{\text{handle } M \text{ with } H : B ! E \dashrightarrow \text{handle } M' \text{ with } H'} \end{array}$$

Fig. 4. Encoding of F_{eff}^1 in MET.

In the term translation, all terms are translated to boxed terms with proper modalities consistent with those given by the type translation. Recall that MET uses let-style unboxing; we cannot immediately unbox values at the place we need. To get a systematic encoding, we *greedily unbox* top-level modalities for term variables when they are bound, and rebox them when they are used.

Greedy unboxing happens for variable bindings such as λ -abstractions and handlers. In the R-ABS case, we unbox the top-level modality of variable x . Additionally, we box the whole function with the relative modality $\langle E - F | F - E \rangle$, reflecting the effect context transition. In the R-HANDLER case, we similarly unbox the bound variables for return and operation clauses. In the operation clauses (N'_i), we need only unbox the operation argument p_i ; the resumption function r_i is introduced under the current effect context E . In the return clause (N''), we unbox x with $\langle \bar{\ell}_i \rangle \circ \mu$ and then transform this modality to μ' given by $\text{topmod}(\llbracket A \rrbracket_E)$ in order to match the effect context E .

Similar to the R-ABS case, the R-EABS case boxes the translated value with the empty absolute modality. Similar to the return clauses of the R-HANDLER case, the R-MASK case transforms the modality $\langle L \rangle \circ \mu_1$ to μ_2 in order to match the current effect context $L + E$.

In R-VAR, we rebox the variable with the appropriate modality given by the type translation.

As a result of translating all terms to boxed terms, we must insert unboxing for elimination rules such as R-APP and R-EAPP. Nothing special happens for the R-DO case.

We have the following type preservation theorem. The proof is given in Appendix B.6. We have proved the existence of all modality transformations used in the encoding.

LEMMA 4.2 (TYPE PRESERVATION OF ENCODING). *If $\Gamma \vdash M : A! \{E|\varepsilon\}$ is well-scoped, then $M : A! E \dashrightarrow M'$ and $[\Gamma]_E \vdash M' : [A]_E @ E$.*

As an example, we can translate the asSum handler in F_{eff}^1 defined in Section 4.1 as follows into MET, omitting term-level boxing and unboxing of the identity modality $\mathbb{1}$.

$$\begin{aligned} \text{asSum} &: [](\langle \text{yield} \rangle(\mathbb{1} \rightarrow \mathbb{1}) \rightarrow \mathbb{1}\text{Int}) \\ \text{asSum} &= \mathbf{mod}_{[]}(\lambda f. \mathbf{let} \mathbf{mod}_{\langle \text{yield} \rangle} f = f \mathbf{in} \mathbf{handle} f () \mathbf{with} \\ &\quad \{\mathbf{return} \ x \mapsto \mathbf{let} \mathbf{mod}_{\langle \text{yield} \rangle} x = x \mathbf{in} \ 0, \text{yield } x \ r \mapsto x + r \ ()\}) \end{aligned}$$

The explicit boxing and unboxing as well as identity modalities here are only generated to keep the encoding systematic. We do not need them in practice as we have seen in Section 2.

Our encoding focuses on presenting the core idea and does not consider advanced language features including data types and value polymorphism. In Section 5 we will show how to extend MET with these features and briefly discuss how to extend the encoding to cover them.

5 Extensions to MET

In this section we demonstrate that MET scales to support data types and polymorphism including both value and effect polymorphism. Effect polymorphism helps deal with situations in which it is useful to refer to one or more effect contexts that differ from the ambient one (such as the higher-order fork operation in Section 2.10), recovering the full expressive power of row-based effect systems. We only discuss the key ideas of extensions here; their full specification as well as more extensions including shallow handlers [23, 27] are given in Appendix A. We prove type soundness and effect safety for all extensions in Appendix B.

5.1 Making Data Types Crisp

We demonstrate the extensibility of MET with data types by extending it with pair and sum types. We expect no extra challenge to extend MET with algebraic data types. The syntax and typing rules are shown as follows.

$$\begin{array}{c} \text{T-PAIR} \\ \frac{\Gamma \vdash M : A @ E \quad \Gamma \vdash N : B @ E}{\Gamma \vdash (M, N) : (A, B) @ E} \end{array}$$

$$\begin{array}{c} \text{T-INL} \\ \frac{\Gamma \vdash M : A @ E}{\Gamma \vdash \mathbf{inl} \ M : A + B @ E} \end{array}$$

$$\begin{array}{c} \text{T-INR} \\ \frac{\Gamma \vdash M : B @ E}{\Gamma \vdash \mathbf{inr} \ M : A + B @ E} \end{array}$$

$$\begin{array}{c} \text{T-CRISPPAIR} \\ \frac{v_F : E \rightarrow F \quad \Gamma, \mathbf{mod}_{v_F} \vdash V : (A, B) @ E \quad \Gamma, x : v_F A, y : v_F B \vdash M : A' @ F}{\Gamma \vdash \mathbf{case}_v \ V \ \mathbf{of} \ (x, y) \mapsto M : A' @ F} \end{array}$$

$$\begin{array}{c} \text{T-CRISPSUM} \\ \frac{v_F : E \rightarrow F \quad \Gamma, \mathbf{mod}_{v_F} \vdash V : A + B @ E \quad \Gamma, x : v_F A \vdash M_1 : A' @ F \quad \Gamma, y : v_F B \vdash M_2 : A' @ F}{\Gamma \vdash \mathbf{case}_v \ V \ \mathbf{of} \ \{\mathbf{inl} \ x \mapsto M_1, \mathbf{inr} \ y \mapsto M_2\} : A' @ F} \end{array}$$

The T-PAIR, T-INL, and T-INR are standard introduction rules. The elimination rules T-CRISPPAIR and T-CRISPSUM are more interesting. In addition to normal pattern matching, they interpret the value V under the effect context transformed by certain modalities v , which can then be tagged to the variable bindings in case clauses. They follow the crisp induction principles of multimodal

type theory [21, 48]. These crisp elimination rules provide extra expressiveness. For example, we can write the following function which transforms a sum of type $\mu(A + B)$ to another sum of type $(\mu A + \mu B)$. This function is not expressible without crisp elimination rules.

$$\lambda x^{\mu(A+B)}. \text{let mod}_\mu y = x \text{ in case}_\mu y \text{ of } \{\text{inl } x_1 \mapsto \text{inl } (\text{mod}_\mu x_1), \text{inr } x_2 \mapsto \text{inr } (\text{mod}_\mu x_2)\}$$

5.2 Value and Effect Polymorphism

The extensions to syntax and typing rules with polymorphism are as follows.

Types	$A, B ::= \dots \mid \forall \alpha^K. A$	Contexts	$\Gamma ::= \dots \mid \Gamma, \alpha : K$
Effects	$E ::= \dots \mid \varepsilon \mid E \setminus L$	Terms	$M, N ::= \dots \mid \Lambda \alpha^K. V \mid M A$
Kinds	$K ::= \dots \mid \text{Effect}$	Values	$V, W ::= \dots \mid \Lambda \alpha^K. V \mid V A$

$\frac{\text{T-TABS}}{\Gamma, \alpha : K \vdash V : A @ E}$ $\Gamma \vdash \Lambda \alpha^K. V : \forall \alpha^K. A @ E$	$\frac{\text{T-TAPP}}{\Gamma \vdash M : \forall \alpha^K. B @ E \quad \Gamma \vdash A : K}$ $\Gamma \vdash M A : B[A/\alpha] @ E$
---	---

It may appear surprising that we treat type application $V A$ as values. This is useful in practice to allow instantiation inside boxes. We also extend the semantics to allow reduction in values.

To support effect polymorphism, we extend the syntax of effect contexts E with effect variables ε and introduce a new kind `Effect` for them. As is typical for row polymorphism, we restrict each effect type to contain at most one effect variable. We also extend the syntax with effect masking $E \setminus L$, which means the effect types given by masking L from E . The latter is needed to keep the syntax of effect contexts closed under the masking operation $E - L$; otherwise we cannot define $\varepsilon - L$. In other words, the syntax of effects is the free algebra generated from extending D, E and masking $E \setminus L$ with base elements \cdot and ε .

The effect equivalence and subtyping rules are extended in a relatively standard way.

$\overline{E \setminus \cdot \equiv E}$	$\overline{\cdot \setminus L \equiv \cdot}$	$\overline{(\ell : P, E) \setminus (\ell, L) \equiv E \setminus L}$	$\frac{\ell \notin L}{(\ell : P, E) \setminus L \equiv \ell : P, E \setminus L}$
$\overline{(\varepsilon \setminus L) \setminus L' \equiv \varepsilon \setminus (L, L')}$	$\overline{\varepsilon \setminus L \equiv \varepsilon \setminus L}$	$\overline{\cdot \leq \varepsilon \setminus L}$	$\overline{\varepsilon \setminus L \leq \varepsilon \setminus L}$

We do not allow non-trivial equivalence or subtyping between different effect variables. We always identify effects up to the equivalence relation. That is, we can directly treat syntax of effects as the free algebra quotiented by the equivalence relation $E \equiv F$. Observe that using the equivalence relation, all open effect types with effect variable ε can be simplified to an equivalent normal form $D, \varepsilon \setminus L$. We assume the operation $E - L$ is defined for effects E in normal form and extend it with one case for effect variables as $\varepsilon \setminus L - L' = \varepsilon \setminus (L, L')$.

5.3 Extensibility of the Encoding

Our encoding in Section 4 does not consider any extensions of `MET`. With the extension of effect polymorphism, the encoding certainly becomes trivial. Thus, we only consider extensions of data types and value polymorphism and discuss how to extend the encoding.

Recall that in Section 4.2 we always translate types to modal types and perform greedy unboxing and lazy boxing for variables. For variables of data types such as a pair (A, B) , we just need to further destruct the pair before greedy unboxing, and reconstruct the pair when lazy boxing. This is because the translation on its components might give terms of type $\mu A'$ and $\nu B'$ with different modalities which require separate unboxing. For variables of recursive data types, we need to destruct only to the extent that the data type is unfolded in the function body (where we may treat

recursive invocations as opaque). While this requires a somewhat global translation, it does not require destructuring and unboxing the recursive data type more than a small number of times.

The essential reason for the translation being global comes from the fact that we use let-style unboxing following MTT. For modalities with certain structure (right adjoints), it is possible to use Fitch-style unboxing [12] which allows terms to be directly unboxed without binding [20, 49]. We are interested in exploring whether we could extend MET to use Fitch-style unboxing and thus give a compositional local encoding for recursive data types. Fortunately, these issues appear to only cause problems for encoding but not in practice. Functional programs typically use pattern-matching in a structured way that plays nicely with automatic unboxing of bidirectional typing.

Extending the encoding with value polymorphism is tricky as the source calculus F_{eff}^1 is not stable under value type substitution. For instance, the following substitution breaks the condition that function arrows only refer to the lexically closest effect variable: $(\forall \varepsilon'. \alpha)[(1 \rightarrow^{\{\varepsilon\}} 1)/\alpha] = \forall \varepsilon'. 1 \rightarrow^{\{\varepsilon\}} 1$. This exemplifies the fragility of the syntactic approach of Frank. It is possible to still define the encoding by forcing the substituted type to satisfy the lexical restriction. We leave the full development of such an encoding as future work.

6 Simple Bidirectional Type Checking for MET

In this section we outline the design and implementation of METL, a basic surface language on top of MET, which uses a simple bidirectional typing strategy to infer all boxing and unboxing [16, 43].

The bidirectional typing rules for simply-typed λ -calculus and modalities of METL are shown in Figure 5. Rules for effect handlers and extensions are shown in Appendix C.1.

$$\begin{array}{c}
 \boxed{\Gamma \vdash M \Rightarrow A @ E} \quad \boxed{\Gamma \vdash M \Leftarrow A @ E} \\
 \\
 \begin{array}{ccc}
 \text{B-VAR} & & \text{B-SWITCH} \\
 \frac{v_F = \text{locks}(\Gamma') : E \rightarrow F \quad B = \text{across}(\Gamma, A, v, F)}{\Gamma, x : A, \Gamma' \vdash x \Rightarrow B @ E} & \frac{\text{B-ANNOTATION} \quad \Gamma \vdash V \Leftarrow A @ E}{\Gamma \vdash V : A \Rightarrow A @ E} & \frac{\Gamma \vdash M \Rightarrow \bar{\mu}A @ E \quad \Gamma \vdash (\bar{\mu}, A) \Rightarrow \bar{v} @ E}{\Gamma \vdash M \Leftarrow \bar{v}A @ E} \\
 \\
 \text{B-ABS} & \text{B-APP} & \text{B-MOD} \\
 \frac{\Gamma, x : A \vdash M \Leftarrow B @ E}{\Gamma \vdash \lambda x. M \Leftarrow A \rightarrow B @ E} & \frac{\Gamma \vdash M \Rightarrow \bar{\mu}(A \rightarrow B) @ E \quad \bar{\mu}_E \Rightarrow \perp_E \quad \Gamma \vdash N \Leftarrow A @ E}{\Gamma \vdash M N \Rightarrow B @ E} & \frac{\Gamma, \mu_F \vdash V \Leftarrow A @ E \quad \mu_F : E \rightarrow F}{\Gamma \vdash V \Leftarrow \mu A @ F}
 \end{array}
 \end{array}$$

Fig. 5. Representative bidirectional typing rules for METL.

As with usual bidirectional typing, we have inference mode $\Gamma \vdash M \Rightarrow A @ E$ and checking mode $\Gamma \vdash M \Leftarrow A @ E$. They both additionally take the effect context E as an input.

The B-VAR rule uses the auxiliary function `across` defined as follows.

$$\text{across}(\Gamma, A, v, F) = \begin{cases} A, & \text{if } \Gamma \vdash A : \text{Abs} \\ \zeta G, & \text{otherwise, where } A = \bar{\mu}G \text{ and } v_F \setminus \bar{\mu}_F = \zeta_E \end{cases}$$

We write G for guarded types which do not have top-level modalities, and $\bar{\mu}$ for a sequence of modalities. When A is absolute, we can always access the variable. Otherwise, in order to know how far we should unbox the modalities $\bar{\mu}$ of the variable, we define a right residual operation $v_F \setminus \mu_F$ for the modality transformation relation. Given $\mu_F : E \rightarrow F$ and $v_F : F' \rightarrow F$, the partial operation $v_F \setminus \mu_F$ fails if there does not exist $\zeta_{F'}$ such that $\mu_F \Rightarrow v_F \circ \zeta_{F'}$. Otherwise, it gives an indexed modality such that $\mu_F \Rightarrow v_F \circ (\zeta_{F'} \setminus \mu_F)$ and for any $\zeta_{F'}$ with $\mu_F \Rightarrow v_F \circ \zeta_{F'}$, we have

1030 $v_F \setminus \mu_F \Rightarrow \zeta_{F'}$. Intuitively, $v_F \setminus \mu_F$ gives the best solution $\zeta_{F'}$ for the transformation $\mu_F \Rightarrow v_F \circ \zeta_{F'}$
 1031 to hold. The concrete definition of $v_F \setminus \mu_F$ is given in Appendix C.1. B-MOD introduces a lock into
 1032 the context. B-ANNOTATION is standard for bidirectional typing. B-SWITCH not only switches the
 1033 direction from checking to inference, but also transforms the top-level modalities when there is
 1034 a mismatch. It uses the judgement $\Gamma \vdash (\mu, A) \Rightarrow v @ E$ defined in Section 3.4. B-ABS is standard.
 1035 B-APP unboxes M when it has top-level modalities.

1036 Our bidirectional typing rules are sufficiently syntax-directed to yield an algorithm. We specify
 1037 the elaboration of METL into MET and discuss our prototype implementation in Appendix C.2.

1038 Though incorporating polymorphic type inference is beyond the scope of this paper, we are
 1039 confident that modal effect types are compatible with it. The key observation here is that in the
 1040 presence of polymorphism, the problem of automatically boxing and unboxing modal effect types
 1041 is closely related to that of inferring first-class polymorphism. Box introduction is analogous to
 1042 type abstraction (which type inference algorithms realise through generalisation). Box elimination
 1043 is analogous to type application (which type inference algorithms realise through instantiation).
 1044 As such, one can adapt any of the myriad techniques for combining first-class polymorphism with
 1045 Hindley-Milner type inference. As METL adopts a bidirectional type system, we could simply follow
 1046 the literature on extending bidirectional typing with sound and complete inference for higher-rank
 1047 polymorphism [17], first-class polymorphism [60] and bounded quantification [14].

1048 In the future, we plan to further explore type inference for modal effect types and in particular
 1049 design an extension to OCaml, building on and complementing recent work on modal types for
 1050 OCaml [35] and making use of existing techniques for supporting first-class polymorphism.

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7 Related and Future Work

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7.1 Frank

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7.2 Capability-based Effect Systems

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Capability-based effect systems such as Effekt [8, 9] and $CC_{<:\square}$ [7] interpret effects as capabilities and offer a form of implicit effect polymorphism through capability passing.

For example, in Effekt the `asList` for `Yield` has the following type:

```
def asList{ f: 1 ⇒ List[Int] / { Yield } }: List[Int] / {}
```

Here the block parameter `f` is allowed to use the capability `Yield` in addition to those from the context. The capability annotation `{Yield}` on its type is similar to our relative modalities.

A key difference between Effekt and MET is that Effekt requires blocks to be second-class, while MET supports first-class functions smoothly. Brachthäuser et al. [8] recovers first-class functions by boxing blocks. However, such boxed blocks cannot use capabilities from the context any more, because the boxes on types fully specifies the required capabilities, similar to our absolute modalities. For example, we can obtain a curried version of map in Effekt by boxing the result.

```
map1[A, B]{ f: A ⇒ B }: List[A] ⇒ List[B] at {f} / {}
```

The return value has type `List[A] ⇒ List[B] at {f}`. The decoration `{f}` indicates that the return function captures the capability `f`. This sort of annotation is reminiscent of an effect variable. This illustrates why MET without effect polymorphism is not expressive enough to encode all of Effekt. To encode captured capability variables, as the `at {f}` in `map1`, we need the expressiveness provided by effect variables in Section 5.2.

Another key difference is that Effekt uses named handlers [6, 55, 59] where operations are dispatched to a specific named handler, whereas MET uses Plotkin and Pretnar [44]-style handlers where operations are dispatched to the first matching handler in the evaluation context. Named handlers provide a form of effect generativity. In the future it would be interesting to explore variants of modal effect types with capabilities and generative effects [15].

`CC<·□` [7], the basis for capture tracking in Scala 3, also provides succinct types for uncurried higher-order functions like `map`. As in Effekt, the curried version requires the result function to be explicitly annotated with its capture set `{f}`. Although `CC<·□` works well in the setting of Scala 3, it does rely on existing advanced features like path-dependent types and implicit parameters. Modal effect types do not require the language to support such advanced features.

7.3 Relationship Between MET and Multimodal Type Theory

The literature on multimodal type theory organises the structure of modes (objects), modalities (morphisms between objects), and their transformations (2-cells between morphisms) in a 2-*category* [20, 21, 30] (or, in the case of a single mode, a semiring [1, 10, 40, 41]). In MET, modes are effect contexts E , modalities are $\mu_F : E \rightarrow F$, and transformations are $\mu_F \Rightarrow \nu_F$. However, we have found that 2-categories are not sufficient in a system that also includes submoding. To deal with this extra structure, we extend the 2-category to a *double category* with an additional kind of vertical morphisms between objects (in MET, vertical morphisms are the subeffecting relation $E \leq F$), as also proposed by Katsumata [29]. As a result, the transformations do not strictly require the two modalities to have the same sources and targets, enabling us to have $[\]_F \Rightarrow [E]_F$ in MET. The relationship between MET and MTT is explained in detail in Appendix B.1.

7.4 Other Related Work

We discuss other related work on effect systems and modal types.

Row-based Effect Systems. Row polymorphism is one popular approach to implementing effect systems for effect handlers. Links [22] use Rémy-style row polymorphism with presence types [46], while Koka [32] and Frank [34] use scoped rows [31] which allow duplicated labels. Morris and McKinna [37] proposes a general framework for comparing different kinds of row types, and Yoshioka et al. [58] proposes a similar framework focusing on comparing effect rows. MET adopts Leijen-style scoped rows meanwhile supports absent operation signatures similar to presence types.

Subtyping-based Effect Systems. Eff [3, 45] is equipped with an effect system with both effect variables and sub-effecting based on the type inference and elaboration described in Karachalias et al. [28]. The effect system of Helium [6] is based on finite sets, offering a natural sub-effecting relation corresponding to set-inclusion. As such, their system aligns closely with Lucassen and

Gifford [36]-style effect systems. Tang et al. [51] proposes a calculus for effect handlers with effect polymorphism and sub-effecting via qualified types [26, 37].

Modal Types and Effects. Choudhury and Krishnaswami [11] proposes to use the necessity modality to recover purity from an effectful calculus, which is similar to our empty absolute modality. Zyuzin and Nanevski [61] extends contextual modal types [38] to algebraic effects and handlers. Their system does not have similar constructs to our relative modality and cannot benefit from ambient effect contexts due to strict syntactic restrictions. As a result, they cannot provide concise modular types for higher-order functions and handlers as MET does.

Effects in Call-By-Push-Value. In CBPV [33], effects are usually tracked on typing judgements for computations and captured into types when switching to values [18, 27, 52]. MET tracks effect contexts as modes in typing judgements for all terms to fully make use of ambient effects.

7.5 Future Work

Future work includes: implementing our system as an extension to OCaml; exploring extensions of modal effect types with Fitch-style unboxing, named handlers, generative effects, and capabilities; combining modal effect types with control-flow linearity; and developing a denotational semantics.

References

- [1] Andreas Abel and Jean-Philippe Bernardy. 2020. A unified view of modalities in type systems. *Proc. ACM Program. Lang.* 4, ICFP, Article 90 (aug 2020), 28 pages. <https://doi.org/10.1145/3408972>
- [2] Danel Ahman. 2023. When Programs Have to Watch Paint Dry. In *Foundations of Software Science and Computation Structures*, Orna Kupferman and Pawel Sobocinski (Eds.). Springer Nature Switzerland, Cham, 1–23.
- [3] Andrej Bauer and Matija Pretnar. 2013. An Effect System for Algebraic Effects and Handlers. In *Algebra and Coalgebra in Computer Science*, Reiko Heckel and Stefan Milius (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 1–16.
- [4] Andrej Bauer and Matija Pretnar. 2015. Programming with algebraic effects and handlers. *J. Log. Algebraic Methods Program.* 84, 1 (2015), 108–123.
- [5] Dariusz Biernacki, Maciej Piróg, Piotr Polesiuk, and Filip Sieczkowski. 2018. Handle with care: relational interpretation of algebraic effects and handlers. *Proc. ACM Program. Lang.* 2, POPL (2018), 8:1–8:30. <https://doi.org/10.1145/3158096>
- [6] Dariusz Biernacki, Maciej Piróg, Piotr Polesiuk, and Filip Sieczkowski. 2020. Binders by day, labels by night: effect instances via lexically scoped handlers. *Proc. ACM Program. Lang.* 4, POPL (2020), 48:1–48:29. <https://doi.org/10.1145/3371116>
- [7] Aleksander Boruch-Gruszecki, Martin Odersky, Edward Lee, Ondrej Lhoták, and Jonathan Immanuel Brachthäuser. 2023. Capturing Types. *ACM Trans. Program. Lang. Syst.* 45, 4 (2023), 21:1–21:52. <https://doi.org/10.1145/3618003>
- [8] Jonathan Immanuel Brachthäuser, Philipp Schuster, Edward Lee, and Aleksander Boruch-Gruszecki. 2022. Effects, capabilities, and boxes: from scope-based reasoning to type-based reasoning and back. *Proc. ACM Program. Lang.* 6, OOPSLA1 (2022), 1–30. <https://doi.org/10.1145/3527320>
- [9] Jonathan Immanuel Brachthäuser, Philipp Schuster, and Klaus Ostermann. 2020. Effects as capabilities: effect handlers and lightweight effect polymorphism. *Proc. ACM Program. Lang.* 4, OOPSLA (2020), 126:1–126:30. <https://doi.org/10.1145/3428194>
- [10] Pritam Choudhury, Harley Eades III, Richard A. Eisenberg, and Stephanie Weirich. 2021. A graded dependent type system with a usage-aware semantics. *Proc. ACM Program. Lang.* 5, POPL, Article 50 (jan 2021), 32 pages. <https://doi.org/10.1145/3434331>
- [11] Vikraman Choudhury and Neel Krishnaswami. 2020. Recovering purity with comonads and capabilities. *Proc. ACM Program. Lang.* 4, ICFP (2020), 111:1–111:28. <https://doi.org/10.1145/3408993>
- [12] Ranald Clouston. 2018. Fitch-Style Modal Lambda Calculi. In *Foundations of Software Science and Computation Structures - 21st International Conference, FOSSACS 2018, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2018, Thessaloniki, Greece, April 14–20, 2018, Proceedings (Lecture Notes in Computer Science, Vol. 10803)*, Christel Baier and Ugo Dal Lago (Eds.). Springer, 258–275. https://doi.org/10.1007/978-3-319-89366-2_14
- [13] Lukas Convent, Sam Lindley, Conor McBride, and Craig McLaughlin. 2020. Doo bee doo bee doo. *J. Funct. Program.* 30 (2020), e9. <https://doi.org/10.1017/S0956796820000039>
- [14] Chen Cui, Shengyi Ji, and Bruno C. d. S. Oliveira. 2023. Greedy Implicit Bounded Quantification. *Proc. ACM Program. Lang.* 7, OOPSLA2 (2023), 2083–2111. <https://doi.org/10.1145/3622871>

- 1177 [15] Paulo Emilio de Vilhena and François Pottier. 2023. A Type System for Effect Handlers and Dynamic Labels. In *Programming Languages and Systems - 32nd European Symposium on Programming, ESOP 2023, Held as Part of the*
1178 *European Joint Conferences on Theory and Practice of Software, ETAPS 2023, Paris, France, April 22-27, 2023, Proceedings*
1179 *(Lecture Notes in Computer Science, Vol. 13990)*, Thomas Wies (Ed.). Springer, 225–252. [https://doi.org/10.1007/978-3-](https://doi.org/10.1007/978-3-031-30044-8_9)
1180 [031-30044-8_9](https://doi.org/10.1007/978-3-031-30044-8_9)
- 1181 [16] Jana Dunfield and Neel Krishnaswami. 2022. Bidirectional Typing. *ACM Comput. Surv.* 54, 5 (2022), 98:1–98:38.
1182 <https://doi.org/10.1145/3450952>
- 1183 [17] Jana Dunfield and Neelakantan R. Krishnaswami. 2013. Complete and easy bidirectional typechecking for higher-rank
1184 polymorphism. In *ACM SIGPLAN International Conference on Functional Programming, ICFP'13, Boston, MA, USA -*
1185 *September 25 - 27, 2013*, Greg Morrisett and Tarmo Uustalu (Eds.). ACM, 429–442. [https://doi.org/10.1145/2500365.](https://doi.org/10.1145/2500365.2500582)
1186 [2500582](https://doi.org/10.1145/2500365.2500582)
- 1187 [18] Yannick Forster, Ohad Kammar, Sam Lindley, and Matija Pretnar. 2019. On the expressive power of user-defined
1188 effects: Effect handlers, monadic reflection, delimited control. *J. Funct. Program.* 29 (2019), e15. [https://doi.org/10.](https://doi.org/10.1017/S0956796819000121)
1189 [1017/S0956796819000121](https://doi.org/10.1017/S0956796819000121)
- 1190 [19] Jeffrey S. Foster, Manuel Fähndrich, and Alexander Aiken. 1999. A Theory of Type Qualifiers. In *Proceedings of the*
1191 *1999 ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI), Atlanta, Georgia, USA,*
1192 *May 1-4, 1999*, Barbara G. Ryder and Benjamin G. Zorn (Eds.). ACM, 192–203. <https://doi.org/10.1145/301618.301665>
- 1193 [20] Daniel Gratzer. 2023. *Syntax and semantics of modal type theory*. Ph.D. Dissertation. Aarhus University.
- 1194 [21] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. 2020. Multimodal Dependent Type Theory. In *LICS*
1195 *'20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8-11, 2020*, Holger
1196 Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller (Eds.). ACM, 492–506. [https://doi.org/10.1145/3373718.](https://doi.org/10.1145/3373718.3394736)
1197 [3394736](https://doi.org/10.1145/3373718.3394736)
- 1198 [22] Daniel Hillerström and Sam Lindley. 2016. Liberating Effects with Rows and Handlers (*TyDe 2016*). Association for
1199 Computing Machinery, New York, NY, USA, 15–27. <https://doi.org/10.1145/2976022.2976033>
- 1200 [23] Daniel Hillerström and Sam Lindley. 2018. Shallow Effect Handlers. In *Programming Languages and Systems - 16th*
1201 *Asian Symposium, APLAS 2018, Wellington, New Zealand, December 2-6, 2018, Proceedings (Lecture Notes in Computer*
1202 *Science, Vol. 11275)*, Sukyoung Ryu (Ed.). Springer, 415–435. https://doi.org/10.1007/978-3-030-02768-1_22
- 1203 [24] Daniel Hillerström, Sam Lindley, and Robert Atkey. 2020. Effect handlers via generalised continuations. *J. Funct.*
1204 *Program.* 30 (2020), e5. <https://doi.org/10.1017/S0956796820000040>
- 1205 [25] Daniel Hillerström. 2022. *Foundations for Programming and Implementing Effect Handlers*. Ph.D. Dissertation. The
1206 University of Edinburgh, UK. <https://doi.org/10.7488/era/2122>
- 1207 [26] Mark P. Jones. 1994. A Theory of Qualified Types. *Sci. Comput. Program.* 22, 3 (1994), 231–256. [https://doi.org/10.](https://doi.org/10.1016/0167-6423(94)00005-0)
1208 [1016/0167-6423\(94\)00005-0](https://doi.org/10.1016/0167-6423(94)00005-0)
- 1209 [27] Ohad Kammar, Sam Lindley, and Nicolas Oury. 2013. Handlers in action. In *ACM SIGPLAN International Conference on*
1210 *Functional Programming, ICFP'13, Boston, MA, USA - September 25 - 27, 2013*, Greg Morrisett and Tarmo Uustalu (Eds.).
1211 ACM, 145–158. <https://doi.org/10.1145/2500365.2500590>
- 1212 [28] Georgios Karachalias, Matija Pretnar, Amr Hany Saleh, Stien Vanderhallen, and Tom Schrijvers. 2020. Explicit effect
1213 subtyping. *J. Funct. Program.* 30 (2020), e15. <https://doi.org/10.1017/S0956796820000131>
- 1214 [29] Shin-ya Katsumata. 2018. A Double Category Theoretic Analysis of Graded Linear Exponential Comonads. In *Foundations*
1215 *of Software Science and Computation Structures*, Christel Baier and Ugo Dal Lago (Eds.). Springer International
1216 Publishing, Cham, 110–127.
- 1217 [30] G. A. Kavvos and Daniel Gratzer. 2023. Under Lock and Key: a Proof System for a Multimodal Logic. *Bull. Symb. Log.*
1218 29, 2 (2023), 264–293. <https://doi.org/10.1017/BSL.2023.14>
- 1219 [31] Daan Leijen. 2005. Extensible records with scoped labels. In *Revised Selected Papers from the Sixth Symposium on*
1220 *Trends in Functional Programming, TFP 2005, Tallinn, Estonia, 23-24 September 2005 (Trends in Functional Programming,*
1221 *Vol. 6)*, Marko C. J. D. van Eekelen (Ed.). Intellect, 179–194.
- 1222 [32] Daan Leijen. 2017. Type Directed Compilation of Row-Typed Algebraic Effects. In *Proceedings of the 44th ACM*
1223 *SIGPLAN Symposium on Principles of Programming Languages (Paris, France) (POPL '17)*. Association for Computing
1224 Machinery, New York, NY, USA, 486–499. <https://doi.org/10.1145/3009837.3009872>
- 1225 [33] Paul Blain Levy. 2004. *Call-By-Push-Value: A Functional/Imperative Synthesis*. Semantics Structures in Computation,
Vol. 2. Springer.
- [34] Sam Lindley, Conor McBride, and Craig McLaughlin. 2017. Do Be Do Be Do. In *Proceedings of the 44th ACM SIGPLAN*
Symposium on Principles of Programming Languages (Paris, France) (POPL 2017). Association for Computing Machinery,
New York, NY, USA, 500–514. <https://doi.org/10.1145/3009837.3009897>
- [35] Anton Lorenzen, Leo White, Stephen Dolan, Richard A. Eisenberg, and Sam Lindley. 2024. Oxidizing OCaml with
Modal Memory Management. *Proc. ACM Program. Lang.* 8, ICFP (2024). [https://antonlorenzen.de/oxidizing-ocaml-](https://antonlorenzen.de/oxidizing-ocaml-modal-memory-management.pdf)
[modal-memory-management.pdf](https://antonlorenzen.de/oxidizing-ocaml-modal-memory-management.pdf)

- 1226 [36] John M. Lucassen and David K. Gifford. 1988. Polymorphic Effect Systems. In *POPL*. ACM Press, 47–57. <https://doi.org/10.1145/73560.73564>
- 1227
- 1228 [37] J. Garrett Morris and James McKinna. 2019. Abstracting Extensible Data Types: Or, Rows by Any Other Name. *Proc. ACM Program. Lang.* 3, POPL, Article 12 (jan 2019), 28 pages. <https://doi.org/10.1145/3290325>
- 1229
- 1230 [38] Aleksandar Nanevski, Frank Pfenning, and Brigitte Pientka. 2008. Contextual modal type theory. *ACM Trans. Comput. Log.* 9, 3 (2008), 23:1–23:49. <https://doi.org/10.1145/1352582.1352591>
- 1231
- 1232 [39] Max S. New, Eric Giovannini, and Daniel R. Licata. 2023. Gradual Typing for Effect Handlers. *Proc. ACM Program. Lang.* 7, OOPSLA2 (2023), 1758–1786. <https://doi.org/10.1145/3622860>
- 1233
- 1234 [40] Dominic Orchard, Vilem-Benjamin Liepelt, and Harley Eades III. 2019. Quantitative program reasoning with graded modal types. *Proc. ACM Program. Lang.* 3, ICFP, Article 110 (jul 2019), 30 pages. <https://doi.org/10.1145/3341714>
- 1235
- 1236 [41] Tomas Petricek, Dominic Orchard, and Alan Mycroft. 2014. Coeffects: a calculus of context-dependent computation. In *Proceedings of the 19th ACM SIGPLAN International Conference on Functional Programming (Gothenburg, Sweden) (ICFP '14)*. Association for Computing Machinery, New York, NY, USA, 123–135. <https://doi.org/10.1145/2628136.2628160>
- 1237
- 1238 [42] Luna Phipps-Costin, Andreas Rossberg, Arjun Guha, Daan Leijen, Daniel Hillerström, KC Sivaramakrishnan, Matija Pretnar, and Sam Lindley. 2023. Continuing WebAssembly with Effect Handlers. *Proc. ACM Program. Lang.* 7, OOPSLA2 (2023), 460–485. <https://doi.org/10.1145/3622814>
- 1239
- 1240 [43] Benjamin C. Pierce and David N. Turner. 2000. Local type inference. *ACM Trans. Program. Lang. Syst.* 22, 1 (2000), 1–44. <https://doi.org/10.1145/345099.345100>
- 1241
- 1242 [44] Gordon D. Plotkin and Matija Pretnar. 2013. Handling Algebraic Effects. *Log. Methods Comput. Sci.* 9, 4 (2013).
- 1243
- 1244 [45] Matija Pretnar. 2014. Inferring Algebraic Effects. *Log. Methods Comput. Sci.* 10, 3 (2014). [https://doi.org/10.2168/LMCS-10\(3:21\)2014](https://doi.org/10.2168/LMCS-10(3:21)2014)
- 1245
- 1246 [46] Didier Rémy. 1994. Type Inference for Records in a Natural Extension of ML. In *Theoretical Aspects of Object-Oriented Programming: Types, Semantics, and Language Design*. Citeseer.
- 1247
- 1248 [47] Dennis Ritchie and Ken Thompson. 1974. The UNIX Time-Sharing System. *Commun. ACM* 17, 7 (1974), 365–375.
- 1249
- 1250 [48] Michael Shulman. 2018. Brouwer’s fixed-point theorem in real-cohesive homotopy type theory. *Math. Struct. Comput. Sci.* 28, 6 (2018), 856–941. <https://doi.org/10.1017/S0960129517000147>
- 1251
- 1252 [49] Michael Shulman. 2023. Semantics of multimodal adjoint type theory. In *Proceedings of the 39th Conference on the Mathematical Foundations of Programming Semantics, MFPS XXXIX, Indiana University, Bloomington, IN, USA, June 21-23, 2023 (EPTICS, Vol. 3)*, Marie Kerjean and Paul Blain Levy (Eds.). EpiSciences. <https://doi.org/10.46298/ENTICS.12300>
- 1253
- 1254 [50] K. C. Sivaramakrishnan, Stephen Dolan, Leo White, Tom Kelly, Sadiq Jaffer, and Anil Madhavapeddy. 2021. Retrofitting effect handlers onto OCaml. In *PLDI '21: 42nd ACM SIGPLAN International Conference on Programming Language Design and Implementation, Virtual Event, Canada, June 20-25, 2021*, Stephen N. Freund and Eran Yahav (Eds.). ACM, 206–221. <https://doi.org/10.1145/3453483.3454039>
- 1255
- 1256 [51] Wenhao Tang, Daniel Hillerström, Sam Lindley, and J. Garrett Morris. 2024. Soundly Handling Linearity. *Proc. ACM Program. Lang.* 8, POPL, Article 54 (jan 2024), 29 pages. <https://doi.org/10.1145/3632896>
- 1257
- 1258 [52] Cassia Torczon, Emmanuel Suárez Acevedo, Shubh Agrawal, Joey Velez-Ginorio, and Stephanie Weirich. 2023. Effects and Coeffects in Call-By-Push-Value (Extended Version). arXiv:2311.11795 [cs.PL] <https://arxiv.org/abs/2311.11795>
- 1259
- 1260 [53] Birthe van den Berg and Tom Schrijvers. 2023. A Framework for Higher-Order Effects & Handlers. *CoRR* abs/2302.01415 (2023). <https://doi.org/10.48550/arXiv.2302.01415> arXiv:2302.01415
- 1261
- 1262 [54] Nicolas Wu, Tom Schrijvers, and Ralf Hinze. 2014. Effect Handlers in Scope. *SIGPLAN Not.* 49, 12 (Sept. 2014), 1–12. <https://doi.org/10.1145/2775050.2633358>
- 1263
- 1264 [55] Ningning Xie, Youyou Cong, Kazuki Ikemori, and Daan Leijen. 2022. First-class names for effect handlers. *Proc. ACM Program. Lang.* 6, OOPSLA2 (2022), 30–59. <https://doi.org/10.1145/3563289>
- 1265
- 1266 [56] Xu Xue and Bruno C. d. S. Oliveira. 2024. Contextual Typing. *Proc. ACM Program. Lang.* 8, ICFP, Article 266 (Aug. 2024), 29 pages. <https://doi.org/10.1145/3674655>
- 1267
- 1268 [57] Zhixuan Yang and Nicolas Wu. 2023. Modular Models of Monoids with Operations. *Proc. ACM Program. Lang.* 7, ICFP (2023), 566–603. <https://doi.org/10.1145/3607850>
- 1269
- 1270 [58] Takuma Yoshioka, Taro Sekiyama, and Atsushi Igarashi. 2024. Abstracting Effect Systems for Algebraic Effect Handlers. *CoRR* abs/2404.16381 (2024). <https://doi.org/10.48550/ARXIV.2404.16381> arXiv:2404.16381
- 1271
- 1272 [59] Yizhou Zhang and Andrew C. Myers. 2019. Abstraction-safe effect handlers via tunneling. *Proc. ACM Program. Lang.* 3, POPL (2019), 5:1–5:29. <https://doi.org/10.1145/3290318>
- 1273
- 1274 [60] Jinxu Zhao and Bruno C. d. S. Oliveira. 2022. Elementary Type Inference. In *36th European Conference on Object-Oriented Programming, ECOOP 2022, June 6-10, 2022, Berlin, Germany (LIPIcs, Vol. 222)*, Karim Ali and Jan Vitek (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2:1–2:28. <https://doi.org/10.4230/LIPICS.ECOOP.2022.2>
- [61] Nikita Zyzun and Aleksandar Nanevski. 2021. Contextual modal types for algebraic effects and handlers. *Proc. ACM Program. Lang.* 5, ICFP (2021), 1–29. <https://doi.org/10.1145/3473580>

1275 A Full Specification of MET with Extensions

1276 We provide the full specification of MET including all extensions.

1278 A.1 More Extensions

1279 We first present three more extensions of modal effect types: richer forms of handlers, boxing pure
1280 computations, and commuting modalities with type abstraction. We discuss the key ideas of these
1281 extensions below and show their full specification in the following sub-sections.

1282
1283 *Absolute and Shallow Handlers.* Up to now we have considered only *deep* handlers of the form
1284 **handle** M **with** H where M depends on the ambient effect contexts. Deep handlers automatically
1285 wrap the handler around the body of the continuation r captured in a handler clause, and thus
1286 r depends on the ambient effect context. Though this usually suffices in practice, in some cases
1287 we may want the computation M or the continuation to be absolute, i.e., independent from the
1288 ambient effect context. We call such handlers *absolute* handlers. This situation is more prevalent
1289 with effect polymorphism.

1290 To support absolute handlers, we extend the handler syntax and typing rules as follows.

$$\begin{array}{c}
 \text{T-HANDLER}^{\text{A}} \\
 D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, \blacksquare_{[D+E]_F} \vdash M : A @ D + E \\
 \Gamma, \blacksquare_{[E]_F}, x : [D + E]A \vdash N : B @ F \\
 \frac{[\Gamma, \blacksquare_{[E]_F}, p_i : A_i, r_i : [E](B_i \rightarrow B) \vdash N_i : B @ E]_i \quad [E]_F \Rightarrow \mathbb{1}_F}{\Gamma \vdash \mathbf{handle}^{\text{A}} M \mathbf{with} \{\mathbf{return} \ x \mapsto N\} \uplus \{\ell_i \ p_i \ r_i \mapsto N_i\}_i : B @ F}
 \end{array}$$

1297 The T-HANDLER^A rule extends the context with an absolute lock $\blacksquare_{[D+E]_F}$ specifying the effect
1298 context for M , and boxes the continuation r with the absolute modality $[E]$, where E exactly
1299 gives the effect context after handling. We put the lock $\blacksquare_{[E]_F}$ in handler clauses as deep handlers
1300 capture themselves into continuations. We also extend the handler syntax with *shallow* handlers
1301 **handle**[†] M **with** H , in which the handler is not automatically wrapped around the body of
1302 continuations, and *absolute shallow* handlers **handle**^{A†} M **with** H [23, 27].

1304 *Boxing Computations under Empty Effect Contexts.* We have restricted boxes to values in order
1305 to guarantee effect safety. This restriction is not essential for \square . For example, suppose we have
1306 $f : \square_F (A \rightarrow B)$ and $x : \square_F A$, it is sound to treat $\mathbf{mod}_{\square} (f x)$ as a computation which returns a
1307 value of type $\square B$. As $f x$ is evaluated under the empty effect context, we can guarantee that it
1308 cannot get stuck on unhandled operations.

1309 We extend the introduction rule for the empty absolute modality to allow non-value terms with
1310 the following typing rule.

$$\begin{array}{c}
 \text{T-MODABS} \\
 \Gamma, \blacksquare_{\square_F} \vdash M : A @ \cdot \\
 \hline
 \Gamma \vdash \mathbf{mod}_{\square} M : \square A @ F
 \end{array}$$

1315 As an example, we can write the following *app* function.

$$\begin{array}{l}
 1317 \mathit{app} \quad : \quad \forall \alpha. \forall \beta. \square (\alpha \rightarrow \beta) \rightarrow \square \alpha \rightarrow \square \beta \\
 1318 \mathit{app} \quad = \quad \Lambda \alpha. \Lambda \beta. \lambda f. \lambda x. \mathbf{let} \ \mathbf{mod}_{\square} f = f \ \mathbf{in} \ \mathbf{let} \ \mathbf{mod}_{\square} x = x \ \mathbf{in} \ \mathbf{mod}_{\square} (f x) \\
 1319
 \end{array}$$

1320 The formula corresponding to the type of this function is commonly referred to as Axiom K in modal
1321 logic and is also satisfied by other similar modalities such as the safety modality of Choudhury and
1322 Krishnaswami [11].

1324 *Commuting Modalities and Type Abstraction.* Crisp elimination rules in Section 5.1 allow us
 1325 to commute modalities and data types. Similarly, it is also sound and useful to commute type
 1326 abstractions and modalities. However, the current modality elimination rule cannot do so, for a
 1327 similar reason to why it is not possible to transform $\forall\alpha.A + B$ to $(\forall\alpha.A) + (\forall\alpha.B)$ in System F.
 1328 We extend modality elimination to the form $\mathbf{let}_v \mathbf{mod}_\mu \overline{\Lambda\alpha^K} x = V$ in M which allows V to use
 1329 additional type variables in $\overline{\alpha^K}$ which are abstracted when bound to x . The extended typing and
 1330 reduction rules are as follows.

$$\begin{array}{c}
 \text{T-LETMOD}' \\
 \frac{v_F : E \rightarrow F \quad \Gamma, \mathbf{lock}_{v_F}, \overline{\alpha : K} \vdash V : \mu A @ E \quad \Gamma, x : v_F \circ \mu_E \overline{\forall\alpha^K}. A \vdash M : B @ F}{\Gamma \vdash \mathbf{let}_v \mathbf{mod}_\mu \overline{\Lambda\alpha^K}. x = V \mathbf{in} M : B @ F} \\
 \\
 \text{E-LETMOD}' \quad \mathbf{let}_v \mathbf{mod}_\mu \overline{\Lambda\alpha^K}. x = \mathbf{mod}_\mu U \mathbf{in} M \rightsquigarrow M[(\overline{\Lambda\alpha^K}. U)/x]
 \end{array}$$

1337 For instance, we can now write a function of type $\forall\alpha^K. \mu A \rightarrow \mu(\forall\alpha.A)$ where $\alpha \notin \text{ftv}(\mu)$ as follows.

$$\lambda x^{\forall\alpha^K. \mu A}. \mathbf{let} \mathbf{mod}_\mu \overline{\Lambda\alpha^K}. y = x \alpha \mathbf{in} \mathbf{mod}_\mu y$$

1340 A.2 Syntax

1341 Figure 6 gives the syntax of MET with all extensions including data types, polymorphism, and
 1342 enriched handlers. We highlight the syntax not present in core MET.

1345	Types	$A, B ::= 1 \mid A \rightarrow B \mid \mu A \mid \alpha \mid \forall\alpha^K. A \mid (A, B) \mid A + B$
1346	Masks	$L ::= \cdot \mid \ell, L$
1347	Extensions	$D ::= \cdot \mid \ell : P, D$
1348	Effect Contexts	$E, F ::= \cdot \mid \ell : P, E \mid \varepsilon \mid E \setminus L$
1349	Signatures	$P ::= A \rightarrow B \mid -$
1350	Modalities	$\mu ::= [E] \mid \langle L \mid D \rangle$
1351	Kinds	$K ::= \text{Abs} \mid \text{Any} \mid \text{Effect}$
1352	Contexts	$\Gamma ::= \cdot \mid \Gamma, x : \mu_F A \mid \Gamma, \mathbf{lock}_{\mu_F} \mid \Gamma, \alpha : K$
1353	Terms	$M, N ::= x \mid \lambda x^A. M \mid MN \mid \overline{\Lambda\alpha^K}. V \mid MA$ $\mid \mathbf{mod}_\mu V \mid \mathbf{let}_v \mathbf{mod}_\mu x = V \mathbf{in} M$ $\mid \mathbf{do} \ell M \mid \mathbf{mask}_L M \mid \mathbf{handle}^\delta M \mathbf{with} H$ $\mid (M, N) \mid \mathbf{case}_v V \mathbf{of} (x, y) \mapsto M$ $\mid \mathbf{inl} M \mid \mathbf{inr} M \mid \mathbf{case}_v V \mathbf{of} \{\mathbf{inl} x \mapsto M, \mathbf{inr} y \mapsto N\}$ $\mid \mathbf{let}_v \mathbf{mod}_\mu \overline{\Lambda\alpha^K}. x = V \mathbf{in} M$
1360	Values	$V, W ::= x \mid \lambda x^A. M \mid \mathbf{mod}_\mu V \mid \overline{\Lambda\alpha^K}. V \mid VA \mid (V, W) \mid \mathbf{inl} V \mid \mathbf{inr} V$
1361	Handlers	$H ::= \{\mathbf{return} x \mapsto M\} \mid \{\ell p r \mapsto M\} \uplus H$
1362	Decorations	$\delta ::= \cdot \mid \mathbb{A} \mid \dagger \mid \mathbb{A}\dagger$

1365 Fig. 6. Syntax of MET with all extensions.

1369 A.3 Kinding, Well-Formedness, Type Equivalence and Sub-effecting

1370 The full kinding and well-formedness rules for MET are shown in Figure 7. The type equivalence
 1371 and sub-effecting rules are shown in Figure 8. We highlight the special rule that allows us to add or
 1372 remove absent labels from the right of effect contexts.

1373 A.4 Auxiliary Operations

1374 Since we extend the syntax of effect contexts E , we also need to define a new case for the operation
 1375 $E - L$ as follows. The definitions of other operations $D + E$ and $L \bowtie D$ remain unchanged from
 1376 those in Section 3.2. We include the full definition here for easy reference.

$$1377 \boxed{D + E} \quad \boxed{E - L} \quad \boxed{L \bowtie D}$$

$$1378 \quad D + E = D, E$$

$$1379 \quad \cdot - L = \cdot$$

$$1380 \quad (\ell : P, E) - L = \begin{cases} E - L' & \text{if } L \equiv \ell, L' \\ \ell : P, (E - L) & \text{otherwise} \end{cases}$$

$$1381 \quad \varepsilon \setminus L - L' = \varepsilon \setminus (L, L')$$

$$\cdot \bowtie D = (\cdot, D)$$

$$1382 \quad (\ell, L) \bowtie D = \begin{cases} (L', D'') & \text{if } D' \equiv \ell : P, D'' \\ ((\ell, L'), D') & \text{otherwise} \end{cases}$$

$$1383 \quad \text{where } (L', D') = L \bowtie D$$

1384 Since we extend the syntax of contexts, we need to extend $\text{locks}(\Gamma)$ with one extra trivial case.

$$1385 \quad \text{locks}(\cdot) = \mathbb{1}$$

$$1386 \quad \text{locks}(\Gamma, \mu_{\mu_F}) = \text{locks}(\Gamma) \circ \mu_F$$

$$1387 \quad \text{locks}(\Gamma, x :_{\mu_F} A) = \text{locks}(\Gamma)$$

$$1388 \quad \text{locks}(\Gamma, \alpha : K) = \text{locks}(\Gamma)$$

$$1389 \quad \boxed{\Gamma \vdash A : K}$$

$$1390 \quad \frac{\Gamma \ni \alpha : K}{\Gamma \vdash \alpha : K}$$

$$1391 \quad \frac{\Gamma \vdash A : \text{Abs}}{\Gamma \vdash A : \text{Any}}$$

$$1392 \quad \frac{\Gamma \vdash [E] \quad \Gamma \vdash A : \text{Any}}{\Gamma \vdash [E]A : \text{Abs}}$$

$$1393 \quad \frac{\Gamma \vdash \langle L|D \rangle \quad \Gamma \vdash A : K}{\Gamma \vdash \langle L|D \rangle A : K}$$

$$1394 \quad \frac{\Gamma \vdash A : \text{Any}}{\Gamma \vdash B : \text{Any}}$$

$$1395 \quad \frac{\Gamma, \alpha : K \vdash A : K'}{\Gamma \vdash \forall \alpha^K. A : K'}$$

$$1396 \quad \frac{}{\Gamma \vdash 1 : \text{Abs}}$$

$$1397 \quad \frac{\Gamma \vdash A : K}{\Gamma \vdash (A, B) : K}$$

$$1398 \quad \frac{\Gamma \vdash A : K}{\Gamma \vdash A + B : K}$$

$$1399 \quad \boxed{\Gamma \vdash \mu} \quad \boxed{\Gamma \vdash E : K} \quad \boxed{\Gamma \vdash L} \quad \boxed{\Gamma \vdash D} \quad \boxed{\Gamma \vdash P}$$

$$1400 \quad \frac{\Gamma \vdash L \quad \Gamma \vdash D}{\Gamma \vdash \langle L|D \rangle}$$

$$1401 \quad \frac{\Gamma \vdash E : \text{Effect}}{\Gamma \vdash [E]}$$

$$1402 \quad \frac{}{\Gamma \vdash \cdot : \text{Effect}}$$

$$1403 \quad \frac{\Gamma \vdash P \quad \Gamma \vdash E : \text{Effect}}{\Gamma \vdash \ell : P, E : \text{Effect}}$$

$$1404 \quad \frac{\Gamma \vdash E : \text{Effect} \quad \Gamma \vdash L}{\Gamma \vdash E \setminus L : \text{Effect}}$$

$$1405 \quad \frac{}{\Gamma \vdash L}$$

$$1406 \quad \frac{}{\Gamma \vdash \cdot}$$

$$1407 \quad \frac{\Gamma \vdash P \quad \Gamma \vdash D}{\Gamma \vdash \ell : P, D}$$

$$1408 \quad \frac{}{\Gamma \vdash -}$$

$$1409 \quad \frac{\Gamma \vdash A : \text{Abs} \quad \Gamma \vdash B : \text{Abs}}{\Gamma \vdash A \rightarrow B}$$

$$1410 \quad \boxed{\Gamma \vdash (\mu, A) \Rightarrow v @ F}$$

$$1411 \quad \frac{\Gamma \vdash A : \text{Abs}}{\Gamma \vdash (\mu, A) \Rightarrow v @ F}$$

$$1412 \quad \frac{\mu_F \Rightarrow v_F}{\Gamma \vdash (\mu, A) \Rightarrow v @ F}$$

$$1413 \quad \boxed{\Gamma @ E}$$

$$1414 \quad \frac{}{\cdot @ E}$$

$$1415 \quad \frac{\Gamma @ F \quad \mu_F : E \rightarrow F \quad \Gamma \vdash A : K}{\Gamma, x :_{\mu_F} A @ F}$$

$$1416 \quad \frac{\Gamma @ E}{\Gamma, \alpha : K @ E}$$

$$1417 \quad \frac{\Gamma @ F \quad \mu_F : E \rightarrow F}{\Gamma, \mu_{\mu_F} @ E}$$

1419 Fig. 7. Kinding, well-formedness, and auxiliary rules for MET.

1422	$L \equiv L'$	$D \equiv D'$			
1423					
1424	$\frac{L_1 \equiv L_2 \quad L_2 \equiv L_3}{L_1 \equiv L_3}$				
1425	$\cdot \equiv \cdot$	$\frac{L \equiv L'}{\ell, L \equiv \ell, L'}$		$\frac{\ell \neq \ell' \quad L \equiv L'}{\ell, \ell', L \equiv \ell', \ell, L}$	
1426					
1427	$\frac{D_1 \equiv D_2 \quad D_2 \equiv D_3}{D_1 \equiv D_3}$		$\frac{P \equiv P' \quad D \equiv D'}{\ell : P, D \equiv \ell : P', D'}$		$\frac{\ell \neq \ell'}{\ell : P, \ell' : P', D \equiv \ell' : P', \ell : P, D}$
1428	$\cdot \equiv \cdot$				
1429					
1430	$E \equiv F$				
1431					
1432	$\frac{E_1 \equiv E_2 \quad E_2 \equiv E_3}{E_1 \equiv E_3}$		$\frac{P \equiv P' \quad E \equiv E'}{\ell : P, E \equiv \ell : P', E'}$		$\frac{\ell \neq \ell'}{\ell : P, \ell' : P', E \equiv \ell' : P', \ell : P, E}$
1433	$\cdot \equiv \cdot$				
1434					
1435					
1436	$E, \ell : - \equiv E$	$\varepsilon \setminus L \equiv \varepsilon \setminus L$	$\frac{E \equiv E' \quad L \equiv L'}{E \setminus L \equiv E' \setminus L'}$	$\overline{E \setminus \cdot} \equiv E$	$\overline{\cdot \setminus L} \equiv \cdot$
1437					
1438					
1439	$\ell \notin L$				
1440	$\overline{(\ell : P, E) \setminus (\ell, L) \equiv E \setminus L}$	$\overline{(\ell : P, E) \setminus L \equiv \ell : P, E \setminus L}$	$\overline{(\varepsilon \setminus L) \setminus L' \equiv \varepsilon \setminus (L, L')}$		
1441	$P \equiv P'$	$\mu \equiv \nu$			
1442					
1443	$\frac{A \equiv A' \quad B \equiv B'}{A \rightarrow B \equiv A' \rightarrow B'}$		$\overline{- \equiv -}$	$\frac{E \equiv F}{[E] \equiv [F]}$	$\frac{L \equiv L' \quad D \equiv D'}{\langle L D \rangle \equiv \langle L' D' \rangle}$
1444					
1445					
1446	$A \equiv B$				
1447					
1448	$\overline{\alpha \equiv \alpha}$	$\frac{\mu \equiv \nu \quad A \equiv B}{\mu A \equiv \nu B}$	$\frac{A \equiv A' \quad B \equiv B'}{A \rightarrow B \equiv A' \rightarrow B'}$	$\frac{A \equiv B}{\forall \alpha^K. A \equiv \forall \alpha^K. B}$	
1449					
1450					
1451	$\frac{A \equiv A' \quad B \equiv B'}{(A, B) \equiv (A', B')}$		$\frac{A \equiv A' \quad B \equiv B'}{A + B \equiv A' + B'}$		
1452					
1453					
1454	$E \leq F$				
1455					
1456	$\overline{\cdot \leq E}$		$\frac{E \equiv F}{E \leq F}$	$\frac{E_1 \equiv \ell : P_1, E'_1 \quad E_2 \equiv \ell : P_2, E'_2}{P_1 \leq P_2 \quad E'_1 \leq E'_2}{E_1 \leq E_2}$	
1457					
1458					
1459	$P \leq P'$	$D \leq D'$			
1460					
1461					
1462	$\overline{P \leq P}$		$\overline{- \leq P}$	$\overline{\cdot \leq \cdot}$	$\frac{D_1 \equiv \ell : P_1, D'_1 \quad D_2 \equiv \ell : P_2, D'_2}{P_1 \leq P_2 \quad D'_1 \leq D'_2}{D_1 \leq D_2}$
1463					
1464					
1465					
1466					
1467					
1468					
1469					
1470					

Fig. 8. Type equivalence and sub-effecting for MET.

1471 A.5 Typing Rules

1472 Figure 9 gives the typing rules for MET. We only show the extended rules with respect to the typing
 1473 rules of core MET in Figure 2.
 1474

$$\begin{array}{c}
 1475 \boxed{\Gamma \vdash M : A @ E} \\
 1476 \\
 1477 \text{T-TABS} \quad \frac{\Gamma, \alpha : K \vdash V : A @ E}{\Gamma \vdash \Lambda \alpha^K . V : \forall \alpha^K . A @ E} \\
 1478 \\
 1479 \text{T-TAPP} \quad \frac{\Gamma \vdash M : \forall \alpha^K . B @ E \quad \Gamma \vdash A : K}{\Gamma \vdash M A : B[A/\alpha] @ E} \\
 1480 \\
 1481 \text{T-PAIR} \quad \frac{\Gamma \vdash M : A @ E \quad \Gamma \vdash N : B @ E}{\Gamma \vdash (M, N) : (A, B) @ E} \\
 1482 \\
 1483 \text{T-INL} \quad \frac{\Gamma \vdash M : A @ E}{\Gamma \vdash \mathbf{inl} M : A + B @ E} \quad \text{T-INR} \quad \frac{\Gamma \vdash M : B @ E}{\Gamma \vdash \mathbf{inr} M : A + B @ E} \\
 1484 \\
 1485 \text{T-CRISPPAIR} \quad \frac{v_F : E \rightarrow F \quad \Gamma, \mathbf{lock}_{v_F} \vdash V : (A, B) @ E \quad \Gamma, x :_{v_F} A, y :_{v_F} B \vdash M : A' @ F}{\Gamma \vdash \mathbf{case}_v V \mathbf{of} (x, y) \mapsto M : A' @ F} \\
 1486 \\
 1487 \text{T-CRISPSUM} \quad \frac{v_F : E \rightarrow F \quad \Gamma, \mathbf{lock}_{v_F} \vdash V : A + B @ E \quad \Gamma, x :_{v_F} A \vdash M_1 : A' @ F \quad \Gamma, y :_{v_F} B \vdash M_2 : A' @ F}{\Gamma \vdash \mathbf{case}_v V \mathbf{of} \{\mathbf{inl} x \mapsto M_1, \mathbf{inr} y \mapsto M_2\} : A' @ F} \\
 1488 \\
 1489 \text{T-HANDLER}^A \quad \frac{D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, \mathbf{lock}_{[D+E]_F} \vdash M : A @ D + E \quad \Gamma, \mathbf{lock}_{[E]_F}, x : [D + E]A \vdash N : B @ F \quad [\Gamma, \mathbf{lock}_{[E]_F}, p_i : A_i, r_i : [E](B_i \rightarrow B) \vdash N_i : B @ E]_i \quad [E]_F \Rightarrow \mathbb{1}_F}{\Gamma \vdash \mathbf{handle}^A M \mathbf{with} \{\mathbf{return} x \mapsto N\} \uplus \{\ell_i p_i r_i \mapsto N_i\}_i : B @ F} \\
 1490 \\
 1491 \text{T-SHALLOWHANDLER} \quad \frac{D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, \mathbf{lock}_{\langle D \rangle} \vdash M : A @ D + F \quad \Gamma, x : \langle D \rangle A \vdash N : B @ F \quad [\Gamma, p_i : A_i, r_i : \langle D \rangle(B_i \rightarrow A) \vdash N_i : B @ F]_i}{\Gamma \vdash \mathbf{handle}^\dagger M \mathbf{with} \{\mathbf{return} x \mapsto N\} \uplus \{\ell_i p_i r_i \mapsto N_i\}_i : B @ F} \\
 1492 \\
 1493 \text{T-SHALLOWHANDLER}^A \quad \frac{D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, \mathbf{lock}_{[D+E]_F} \vdash M : A @ D + E \quad \Gamma, x : [D + E]A \vdash N : B @ F \quad [\Gamma, p_i : A_i, r_i : [D + E](B_i \rightarrow A) \vdash N_i : B @ F]_i \quad [E]_F \Rightarrow \mathbb{1}_F}{\Gamma \vdash \mathbf{handle}^{A\dagger} M \mathbf{with} \{\mathbf{return} x \mapsto N\} \uplus \{\ell_i p_i r_i \mapsto N_i\}_i : B @ F} \\
 1494 \\
 1495 \\
 1496 \\
 1497 \\
 1498 \\
 1499 \\
 1500 \\
 1501 \\
 1502 \\
 1503 \\
 1504 \\
 1505 \\
 1506 \text{T-LETMOD}' \quad \frac{v_F : E \rightarrow F \quad \Gamma, \mathbf{lock}_{v_F}, \alpha : \bar{K} \vdash V : \mu A @ E}{\Gamma, x :_{v_F \circ \mu E} \forall \alpha^K . A \vdash M : B @ F} \\
 1507 \\
 1508 \text{T-MODABS} \quad \frac{\Gamma, \mathbf{lock}_{[]_F} \vdash M : A @ \cdot}{\Gamma \vdash \mathbf{mod}_{[]} M : []A @ F} \quad \frac{\Gamma, \mathbf{lock}_{v_F}, \alpha : \bar{K} \vdash V : \mu A @ E \quad \Gamma, x :_{v_F \circ \mu E} \forall \alpha^K . A \vdash M : B @ F}{\Gamma \vdash \mathbf{let}_v \mathbf{mod}_\mu \Lambda \alpha^K . x = V \mathbf{in} M : B @ F} \\
 1509 \\
 1510 \\
 1511 \\
 1512 \\
 1513 \\
 1514 \\
 1515 \\
 1516 \\
 1517 \\
 1518 \\
 1519
 \end{array}$$

Fig. 9. Typing rules for MET (only showing extensions to core MET in Figure 2).

1514 A.6 Operational Semantics

1515 As type application are treated as values and can reduce, we first define value normal forms U that
 1516 cannot reduce further as follows.
 1517

1518 Value normal forms $U ::= x \mid \lambda x^A . M \mid \Lambda \alpha^K . V \mid \mathbf{mod}_\mu U \mid (U_1, U_2) \mid \mathbf{inl} U \mid \mathbf{inr} U$
 1519

1520	E-APP	$(\lambda x^A.M) U \rightsquigarrow M[U/x]$
1521	E-TAPP	$(\Lambda \alpha.V) A \rightsquigarrow V[A/\alpha]$
1522	E-LETMOD	$\mathbf{let}_\nu \mathbf{mod}_\mu x = \mathbf{mod}_\mu U \mathbf{in} M \rightsquigarrow M[U/x]$
1523	E-LETMOD'	$\mathbf{let}_\nu \mathbf{mod}_\mu \Lambda \alpha^K.x = \mathbf{mod}_\mu U \mathbf{in} M \rightsquigarrow M[(\Lambda \alpha^K.U)/x]$
1524	E-MASK	$\mathbf{mask}_L U \rightsquigarrow \mathbf{mod}_{\langle L \rangle} U$
1525	E-PAIR	$\mathbf{case}_\mu (U_1, U_2) \mathbf{of} (x, y) \mapsto N \rightsquigarrow N[U_1/x, U_2/y]$
1526	E-INL	$\mathbf{case}_\mu \mathbf{inl} U \mathbf{of} \{\mathbf{inl} x \mapsto N_1, \dots\} \rightsquigarrow N_1[U/x]$
1527	E-INR	$\mathbf{case}_\mu \mathbf{inr} U \mathbf{of} \{\mathbf{inr} y \mapsto N_2, \dots\} \rightsquigarrow N_2[U/y]$
1528	E-RET	$\mathbf{handle} U \mathbf{with} H \rightsquigarrow N[(\mathbf{mod}_{\langle D \rangle} U)/x],$
1530	E-OP	$\mathbf{handle} \mathcal{E}[\mathbf{do} \ell U] \mathbf{with} H \rightsquigarrow N[U/p, (\lambda y. \mathbf{handle} \mathcal{E}[y] \mathbf{with} H)/r],$
1531		$\text{where } 0\text{-free}(\ell, \mathcal{E}) \text{ and } (\ell p r \mapsto N) \in H$
1532	E-RET ^A	$\mathbf{handle} U \mathbf{with} H \rightsquigarrow N[(\mathbf{mod}_{[D+E]} U)/x]$
1533		$\text{where } (\mathbf{return} x \mapsto N) \in H$
1534	E-OP ^A	$\mathbf{handle}^A \mathcal{E}[\mathbf{do} \ell U] \mathbf{with} H \rightsquigarrow$
1535		$N[U/p, (\mathbf{mod}_{[E]} (\lambda y. \mathbf{handle}^A \mathcal{E}[y] \mathbf{with} H)/r)]$
1536		$\text{where } 0\text{-free}(\ell, \mathcal{E}) \text{ and } (\ell p r \mapsto N) \in H$
1537	E-RET [†]	$\mathbf{handle}^\dagger U \mathbf{with} H \rightsquigarrow N[(\mathbf{mod}_{\langle D \rangle} U)/x]$
1538		$\text{where } (\mathbf{return} x \mapsto N) \in H$
1539	E-OP [†]	$\mathbf{handle}^\dagger \mathcal{E}[\mathbf{do} \ell U] \mathbf{with} H \rightsquigarrow N[U/p, (\lambda y. \mathcal{E}[y])/r]$
1540		$\text{where } 0\text{-free}(\ell, \mathcal{E}) \text{ and } (\ell p r \mapsto N) \in H$
1541	E-RET ^{A†}	$\mathbf{handle}^{A\dagger} U \mathbf{with} H \rightsquigarrow N[(\mathbf{mod}_{[D+E]} U)/x]$
1542		$\text{where } (\mathbf{return} x \mapsto N) \in H$
1543	E-OP ^{A†}	$\mathbf{handle}^{A\dagger} \mathcal{E}[\mathbf{do} \ell U] \mathbf{with} H \rightsquigarrow N[U/p, (\mathbf{mod}_{[D+E]} (\lambda y. \mathcal{E}[y]))/r]$
1544		$\text{where } 0\text{-free}(\ell, \mathcal{E}) \text{ and } (\ell p r \mapsto N) \in H$
1545	E-LIFT	$\mathcal{E}[M] \rightsquigarrow \mathcal{E}[N],$
1546		$\text{if } M \rightsquigarrow N$

Fig. 10. Operational semantics for MET.

We also extend the definition of evaluation contexts. The full definition is given as follows. Notice that we use value normal forms instead of values.

Evaluation contexts $\mathcal{E} ::= [] \mid \mathcal{E} A \mid \mathcal{E} N \mid U \mathcal{E} \mid \mathbf{do} \ell \mathcal{E} \mid \mathbf{mask}_L \mathcal{E} \mid \mathbf{handle}^\delta \mathcal{E} \mathbf{with} H$
 $\mid \mathbf{mod}_\mu \mathcal{E} \mid \mathbf{let}_\nu \mathbf{mod}_\mu x = \mathcal{E} \mathbf{in} M \mid \mathbf{let}_\nu \mathbf{mod}_\mu \Lambda \alpha^K.x = \mathcal{E} \mathbf{in} M$
 $\mid (\mathcal{E}, N) \mid (U, \mathcal{E}) \mid \mathbf{case}_\nu \mathcal{E} \mathbf{of} (x, y) \mapsto M$
 $\mid \mathbf{inl} \mathcal{E} \mid \mathbf{inr} \mathcal{E} \mid \mathbf{case}_\nu \mathcal{E} \mathbf{of} \{\mathbf{inl} x \mapsto M, \mathbf{inr} y \mapsto N\}$

Figure 10 shows the operational semantics of MET.

B Meta Theory and Proofs for MET

We provide meta theory and proofs for MET in Sections 3 and 5 including all extensions. We also prove the encoding in Section 4.

B.1 The Double Category of Effects

A double category extends a 2-category with an additional kind of morphisms. Alongside the regular morphisms, now called *horizontal* morphisms, there are also *vertical* morphisms that connect the objects of the 2-category. This makes it possible to generalise the 2-cells to transform arbitrary



Fig. 11. 2-cells in a 2-category compared to 2-cells in a double category.

morphisms, whose source and target are connected by vertical morphisms. Figure 11 shows the differences between 2-cells in a 2-category and those in a double category using syntax of MET.

In MET, objects/modes are given by effect contexts, the horizontal morphisms by modalities, the vertical morphisms by the sub-effecting relation, and 2-cells by the modality transformations.

Now we show that it indeed has the structure of a double category.

Since the sub-effecting relation is a preorder, effect contexts (objects) E and sub-effecting (vertical morphisms) $E \leq F$ obviously form a category given by the poset.

We repeat the definition of modalities and modality composition from Section 3.3 here for easy reference. We define them directly in terms of morphisms between modes.

$$\begin{aligned} [E]_F & : & E & \rightarrow F \\ \langle L|D \rangle_F & : & D + (F - L) & \rightarrow F \end{aligned}$$

$$\begin{aligned} [E']_F \circ [E]_{E'} & = [E]_F \\ \langle L|D \rangle_F \circ [E]_{D+(F-L)} & = [E]_F \\ [E]_F \circ \langle L|D \rangle_E & = [D + (E - L)]_F \\ \langle L_1|D_1 \rangle_F \circ \langle L_2|D_2 \rangle_{D_1+(F-L_1)} & = \langle L_1 + L_2 | D_1 + D_2 \rangle_F \quad \text{where } (L, D) = L_2 \bowtie D_1 \end{aligned}$$

The effect contexts (objects) and modalities (horizontal morphisms) also form a category since modality composition possesses associativity and identity. We have the following lemma.

LEMMA B.1 (MODES AND MODALITIES FORM A CATEGORY). *Modes and modalities form a category with the identity morphism $\mathbb{1}_E = \langle | \rangle_E : E \rightarrow E$ and the morphism composition $\mu_F \circ \nu_{F'}$ such that*

- (1) *Identity: $\mathbb{1}_F \circ \mu_F = \mu_F = \mu_F \circ \mathbb{1}_E$ for $\mu_F : E \rightarrow F$.*
- (2) *Associativity: $(\mu_{E_1} \circ \nu_{E_2}) \circ \xi_{E_3} = \mu_{E_1} \circ (\nu_{E_2} \circ \xi_{E_3})$ for $\mu_{E_1} : E_2 \rightarrow E_1$, $\nu_{E_2} : E_3 \rightarrow E_2$, and $\xi_{E_3} : E \rightarrow E_3$.*

PROOF. By inlining the definitions of modalities and checking each case. \square

In Section 3, we only define the modality transformations of shape $\mu_F \Rightarrow \nu_F$ where the targets of μ and ν are required to be the same effect context F . This is enough for presenting the calculus, but we can further extend it to allow $\mu_F \Rightarrow \nu_{F'}$ where $F \leq F'$. This is used in the meta theory for MET such as the lock weakening lemma (Lemma B.11.3).

The extended modality transformation relation is defined by the transitive closure of the following rules. Compared to the definition in Section 3.3, the only new rule is MT-MONO.

MT-ABS $\frac{\mu_F : E' \rightarrow F \quad E \leq E'}{[E]_F \Rightarrow \mu_F}$	MT-UPCAST $\frac{D \leq D'}{\langle L D \rangle_F \Rightarrow \langle L D' \rangle_F}$	MT-EXPAND $\frac{(F - L) \equiv \ell : P, E}{\langle \ell, L D, \ell : P \rangle_F \Leftrightarrow \langle L D \rangle_F}$	MT-MONO $\frac{F \leq F'}{\mu_F \Rightarrow \mu_{F'}}$
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The following lemmas shows that the transformation $\mu_F \Rightarrow \nu_{F'}$ satisfies the requirement of being 2-cells in the double category of effects with well-defined vertical and horizontal composition.

LEMMA B.2 (MODALITY TRANSFORMATIONS ARE 2-CELLS). *If $\mu_F \Rightarrow \nu_{F'}$, $\mu_F : E \rightarrow F$, and $\nu_{F'} : E' \rightarrow F'$, then $E \leq E'$ and $F \leq F'$. Moreover, the transformation relation is closed under vertical and horizontal composition as shown by the following admissible rules.*

$$\frac{\mu_{F_1} \Rightarrow \nu_{F_2} \quad \nu_{F_2} \Rightarrow \xi_{F_3}}{\mu_{F_1} \Rightarrow \xi_{F_3}} \quad \frac{\mu_F \Rightarrow \mu'_{F'} \quad \nu_E \Rightarrow \nu'_{E'} \quad \mu_F : E \rightarrow F \quad \mu'_{F'} : E' \rightarrow F'}{\mu_F \circ \nu_E \Rightarrow \mu'_{F'} \circ \nu'_{E'}}$$

PROOF. To make proving easier, we give the resulting rules by taking the transitive closure.

$$\frac{\mu_{F'} : E' \rightarrow F' \quad E \leq E' \quad F \leq F'}{[E]_F \Rightarrow \mu_{F'}}$$

$$\frac{L = \text{dom}(D) \quad D_1 \leq D'_1 \quad (F' - L_1) \equiv D, E \quad F \leq F' \quad \text{present}(D)}{\langle L_1 | D_1 \rangle_F \Rightarrow \langle L, L_1 | D'_1, D \rangle_{F'}}$$

$$\frac{L = \text{dom}(D) \quad D_1 \leq D'_1 \quad (F' - L_1) \equiv D, E \quad F \leq F'}{\langle L, L_1 | D_1, D \rangle_F \Rightarrow \langle L_1 | D'_1 \rangle_{F'}}$$

The predicate $\text{present}(D)$ checks if all labels in D are present. It is easy to see that sources and targets of morphisms increase. Vertical composition follows directly from the fact that we take the transitive closure. Horizontal compositions follows from case analysis on shapes of modalities being composed. \square

More on Relationships between MET and Multimodal Type Theory. In addition to extending to a double category, MET also differs from MTT in the usage of morphism families. In types and terms we use μ , indexed families of morphisms between modes, instead of concrete morphisms μ_F . We do not lose any information. Given a typing judgement $\Gamma \vdash M : A @ E$, the indexes for all modalities in M and A are determined by E . Similarly, given a variable binding $x :_{\mu_F} A$, the indexes of all modalities in A are determined by μ_F .

Using indexed families of modalities in types and terms is very useful to allow term variables to be used flexibly in different effect contexts larger than where they are defined. This greatly simplifies the support of subeffecting; we do not need to update all indexes of modalities in a term or type when upcasting this term or type to a larger effect context. As a result, every type is always well-defined at any modes, which means that we do not need to define the well-formedness judgement $A @ E$ as in MTT. Moreover, one important benefit of having types well-defined at any modes is that when adding value polymorphism, type quantifiers do not need to carry the additional information about the modes at which the type variables can be used, greatly simplifying the type system. Otherwise, polymorphic types would need to have forms $\forall \alpha^K @ E.A$, where E indicates the mode of the type variable α .

In contexts, we still keep concrete morphisms μ_F , which makes the proof trees of terms much more structured than using morphism families.

B.2 Lemmas for Modes and Modalities

Beyond the structure and properties of double categories shown in Appendix B.1, we have some extra properties on modes and modalities in MET.

The most important one is that horizontal morphisms (sub-effecting) act functorially on vertical ones (modalities). In other words, the action of μ on effect contexts gives a total monotone function.

LEMMA B.3 (MONOTONE MODALITIES). *If $\mu_F : E \rightarrow F$ and $F \leq F'$, then $\mu_{F'} : E' \rightarrow F'$ with $E \leq E'$.*

PROOF. By definition. \square

We prove the lemma on the equivalence between syntactic and semantic definition of modality transformation in Section 3.3. This lemma can be generalised to the general form of 2-cells in a double category $\mu_F \Rightarrow \nu_{F'}$, where $F \leq F'$.

LEMMA 3.1 (SEMANTICS OF MODALITY TRANSFORMATION). *We have $\mu_F \Rightarrow \nu_F$ if and only if $\mu(F') \leq \nu(F')$ for all F' with $F \leq F'$.*

PROOF. From left to right, it is obvious that the semantics is preserved after taking the transitive closure. We only need to show the transformation given by each rule satisfies the semantics.

Case MT-ABS. Follow from Lemma B.3.

Case MT-UPCAST. Since $D \leq D'$, we have $D + (F - L) \leq D' + (F - L)$ for any F .

Case MT-EXPAND. Since $(F - L) \equiv \ell : A \rightarrow B, E$, for any $F \leq F'$ we have $(F' - L) \equiv \ell : A \rightarrow B, E'$ for some E' . Both sides act on F' give $D, \ell : A \rightarrow B, E'$. Notice that it is important for ℓ to not be absent here; otherwise, in F' we could upcast the absent signature of ℓ to any concrete signatures, which then breaks the condition $\mu(F') \leq \nu(F')$.

Case MT-SHRINK. Similar to the above case.

From left to right, we need to show that for all pairs μ_F and ν_F satisfying the semantic definition, we have $\mu_F \Rightarrow \nu_F$ in the transitive closure of the three syntactic rules. This obviously holds for those transformation starting from absolute modalities. For those transformation starting from relative modalities, observe that they can only be transformed other relative modalities by the semantic definition. By taking the transitive closure of the last two rules, we have

$$\frac{\text{MT-MULTIEXPAND} \quad L = \text{dom}(D) \quad D_1 \leq D'_1 \quad (F - L_1) \equiv D, E \quad \text{present}(D)}{\langle L_1 | D_1 \rangle_F \Rightarrow \langle L, L_1 | D'_1, D \rangle_F}$$

$$\frac{\text{MT-MULTISHRINK} \quad L = \text{dom}(D) \quad D_1 \leq D'_1 \quad (F - L_1) \equiv D, E}{\langle L, L_1 | D_1, D \rangle_F \Rightarrow \langle L_1 | D'_1 \rangle_F}$$

The predicate $\text{present}(D)$ checks if all labels in D are present. Suppose $\langle L_1 | D_1 \rangle_F$ and $\langle L_2 | D_2 \rangle_F$ satisfies that $D_1 + (F' - L_1) \leq D_2 + (F' - L_2)$ (1) for all $F \leq F'$. Case analysis on the relationship between D_1 and D_2 .

Case D_2 is longer than D_1 . By (1) we have $D_2 \equiv D'_1, D$ for $D_1 \leq D'_1$. Let $L = \text{dom}(D)$. Using proof by contradiction, we can show that $L_2 \equiv L, L_1$, $\text{present}(D)$, and $(F - L_1) \equiv D, E$ for some E ; otherwise, we can always properly set F' to violate (1) meanwhile satisfying $F \leq F'$. Thus, this case is covered by MT-MULTIEXPAND.

Case D_1 is longer than D_2 . We have $D_1 \equiv D'_2, D$ for $D'_2 \leq D_2$. Similar to the above case, using proof by contradiction we can show that it is covered by MT-MULTISHRINK.

□

Our proofs for type soundness and effect safety do not use ad-hoc case analysis on shapes of modalities or rely on any specific properties about the definition of composition and transformation (except for the parts about effect handlers since they specify the required modalities in the typing rules). As a result, it should be possible to generalise our calculus and proofs to other mode theories satisfying certain extra properties. We state some properties of the mode theory as the following lemmas for easier references in proofs. Most of them directly follow from the definition.

LEMMA B.4 (VERTICAL COMPOSITION). *If $\mu_{F_1} \Rightarrow \nu_{F_2}$ and $\nu_{F_2} \Rightarrow \xi_{F_3}$, then $\mu_{F_1} \Rightarrow \xi_{F_3}$.*

PROOF. Follow from Lemma B.2

□

1716 LEMMA B.5 (HORIZONTAL COMPOSITION). *If $\mu_F : E \rightarrow F$, $\mu'_{F'} : E' \rightarrow F'$, $\mu_F \Rightarrow \mu'_{F'}$, and $\nu_E \Rightarrow \nu'_{E'}$,*
 1717 *then $\mu_F \circ \nu_E \Rightarrow \mu'_{F'} \circ \nu'_{E'}$.*

1718 PROOF. Follow from Lemma B.2 □

1720 LEMMA B.6 (MONOTONE MODALITY TRANSFORMATION). *If $\mu_F \Rightarrow \nu_F$ and $F \leq F'$, then $\mu_{F'} \Rightarrow \nu_{F'}$.*

1721 PROOF. Follow from Lemma 3.1 □

1723 LEMMA B.7 (ASYMMETRIC REFLEXIVITY OF MODALITY TRANSFORMATION). *If $F \leq F'$ and $\mu_F : E \rightarrow$*
 1724 *F , then $\mu_F \Rightarrow \mu_{F'}$.*

1726 PROOF. By definition. □

1728 B.3 Lemmas for the Calculus

1729 We prove structural and substitution lemmas for MET as well as some other auxiliary lemmas for
 1730 proving type soundness.

1731 LEMMA B.8 (CANONICAL FORMS).

- 1733 1. *If $\vdash U : \mu A @ E$, then U is of shape $\mathbf{mod}_\mu U'$.*
- 1734 2. *If $\vdash U : A \rightarrow B @ E$, then U is of shape $\lambda x^A.M$.*
- 1735 3. *If $\vdash U : \forall \alpha.A @ E$, then U is of shape $\Lambda \alpha.V$.*
- 1736 4. *If $\vdash U : (A, B) @ E$, then U is of shape (U_1, U_2) .*
- 1737 5. *If $\vdash U : A + B @ E$, then U is either of shape $\mathbf{inl} U'$ or of shape $\mathbf{inr} U'$.*

1738 PROOF. Directly follows from the typing rules. □

1740 In order to define the lock weakening lemma, we first define a context update operation $(\Gamma)_{F'}$
 1741 which gives a new context derived from updating the indexes of all locks and variable bindings in
 1742 Γ such that $(\Gamma)_{F'} @ F'$.

$$\begin{aligned}
 1744 & (\cdot)_F = \cdot \\
 1745 & (\mathbf{lock}[E]_{F'}, \Gamma')_F = \mathbf{lock}[E]_F, \Gamma' \\
 1746 & (\mathbf{lock}\langle L|D \rangle_{F'}, \Gamma')_F = \mathbf{lock}\langle L|D \rangle_F, (\Gamma')_{D+(F-L)} \\
 1747 & (x : \mu_{F'} A, \Gamma')_F = x : \mu_F A, (\Gamma')_F \\
 1748 & (\alpha : K, \Gamma')_F = \alpha : K, (\Gamma')_F
 \end{aligned}$$

1749 The have the following lemma showing that the index update operation preserves the locks(-)
 1750 operation except for updating the index.

1752 LEMMA B.9 (INDEX UPDATE PRESERVES COMPOSITION). *If $\mu_F = \mathbf{locks}(\Gamma) : E \rightarrow F$, $F \leq F'$, and*
 1753 *$\mathbf{locks}((\Gamma)_{F'}) : E' \rightarrow F'$, then $\mathbf{locks}((\Gamma)_{F'}) = \mu_{F'}$.*

1754 PROOF. By straightforward induction on the context and using the property that $(\mu \circ \nu)_F = \mu_F \circ \nu_E$
 1755 for $\mu_F : E \rightarrow F$. □

1757 COROLLARY B.10 (INDEX UPDATE PRESERVES TRANSFORMATION). *If $\mathbf{locks}(\Gamma) : E \rightarrow F$, $F \leq F'$, and*
 1758 *$\mathbf{locks}((\Gamma)_{F'}) : E' \rightarrow F'$, then $\mathbf{locks}(\Gamma) \Rightarrow \mathbf{locks}((\Gamma)_{F'})$.*

1760 PROOF. Immediately follow from Lemma B.9 and Lemma B.7. □

1761 We have the following structural lemmas.

1763 LEMMA B.11 (STRUCTURAL RULES). *The following structural rules are admissible.*

1764

1765 1. Variable weakening.

$$1766 \frac{\Gamma, \Gamma' \vdash M : B @ E \quad \Gamma, x : \mu_F A, \Gamma' @ E}{1767 \Gamma, x : \mu_F A, \Gamma' \vdash M : B @ E} \quad 1768$$

1769 2. Variable swapping.

$$1770 \frac{\Gamma, x : \mu_F A, y : \nu_F B, \Gamma' \vdash M : A' @ E}{1771 \Gamma, y : \nu_F B, x : \mu_F A, \Gamma' \vdash M : A' @ E} \quad 1772$$

1773 3. Lock weakening.

$$1774 \frac{\Gamma, \blacksquare_{\mu_F}, \Gamma' \vdash M : A @ E \quad \mu_F \Rightarrow \nu_F \quad \nu_F : F' \rightarrow F \quad \text{locks}(\langle \Gamma' \rangle_{F'}) : E' \rightarrow F'}{1775 \Gamma, \blacksquare_{\nu_F}, \langle \Gamma' \rangle_{F'} \vdash M : A @ E'} \quad 1776$$

1777 4. Type variable weakening.

$$1778 \frac{\Gamma, \Gamma' \vdash M : B @ E}{1779 \Gamma, \alpha : K, \Gamma' \vdash M : B @ E} \quad 1780$$

1781 5. Type variable swapping.

$$1782 \frac{\Gamma_1, \Gamma_2, \alpha : K, \Gamma_3 \vdash M : A @ E}{1783 \Gamma_1, \alpha : K, \Gamma_3 \vdash M : A @ E} \quad \alpha \notin \text{ftv}(\Gamma_2) \quad \Gamma_1, \alpha : K, \Gamma_3 \vdash M : A @ E$$

$$1784 \frac{\Gamma_1, \Gamma_2, \alpha : K, \Gamma_3 \vdash M : A @ E}{\Gamma_1, \Gamma_2, \alpha : K, \Gamma_3 \vdash M : A @ E} \quad 1785$$

1786 PROOF. 1, 2, 4, and 5 follow from straightforward induction on the typing derivation. For 3, we
 1787 also proceed by induction on the typing derivation. The most interesting case is T-VAR. Other cases
 1788 mostly follow from IHs.

1789 Case

$$1790 \frac{\text{T-VAR} \quad \nu'_{F_1} = \text{locks}(\Gamma_2) : E \rightarrow F_1 \quad \mu'_{F_1} \Rightarrow \nu'_{F_1} \text{ (1) or } \Gamma \vdash A : \text{Abs}}{1791 \Gamma_1, x : \mu'_{F_1}, \Gamma_2 \vdash x : A @ E} \quad 1792$$

1793 Trivial when A is pure. Otherwise, case analysis on where the lock weakening happens.

1794 Case Γ . Supposing $\Gamma_1 = \Gamma, \blacksquare_{\mu_F}, \Gamma_0$ and after lock weakening we have $\Gamma, \blacksquare_{\nu_F}, \Gamma'_0, x : \mu'_{F_1}, \Gamma'_2$
 1795 where $\Gamma'_2 = \langle \Gamma_2 \rangle_{F'_1} : E' \rightarrow F'_1$ and $\Gamma'_0 = \langle \Gamma_0 \rangle_{F'} : F'_1 \rightarrow F'$. By Lemma B.9 on $\Gamma_0, F \leq F'$,
 1796 and Lemma B.3, we have $F_1 \leq F'_1$. Then by (1) and Lemma B.6, we have $\mu'_{F'_1} \Rightarrow \nu'_{F'_1}$.
 1797 Then by Lemma B.9 we have $\nu'_{F'_1} = \text{locks}(\Gamma'_2)$. Finally by T-VAR we have

$$1800 \Gamma, \blacksquare_{\nu_F}, \Gamma'_0, x : \mu'_{F'_1}, \Gamma'_2 \vdash x : A @ E'$$

1801 Case Γ_2 . Suppose $\Gamma_2 = \Gamma_0, \blacksquare_{\mu_F}, \Gamma'$. is weakened to $\Gamma'_2 = \Gamma_0, \blacksquare_{\nu_F}, \langle \Gamma' \rangle_{F'}$. By Corollary B.10 we
 1802 have $\text{locks}(\Gamma') \Rightarrow \text{locks}(\langle \Gamma' \rangle_{F'})$. Then by Lemma B.5 we have we have $\text{locks}(\Gamma_2) \Rightarrow$
 1803 $\text{locks}(\Gamma'_2)$. By Lemma B.4 and (1), we have $\mu'_{F_1} \Rightarrow \text{locks}(\Gamma'_2)$. Finally by T-VAR we have

$$1804 \Gamma, x : \mu'_{F_1}, \Gamma'_2 \vdash x : A @ E' \quad 1805$$

1806 Case

$$1807 \frac{\text{T-MOD} \quad \mu'_E : F_1 \rightarrow E \quad \Gamma, \blacksquare_{\mu_F}, \Gamma', \blacksquare_{\mu'_E} \vdash V : A @ F_1 (1)}{1808 \Gamma, \blacksquare_{\mu_F}, \Gamma' \vdash \text{mod}_{\mu'} V : \mu' A @ E} \quad 1809$$

1810 We have

$$1811 \langle \Gamma', \blacksquare_{\mu'_E} \rangle_{F'} = \langle \Gamma' \rangle_{F'}, \langle \blacksquare_{\mu'_E} \rangle_{E'} = \langle \Gamma' \rangle_{F'}, \blacksquare_{\mu'_E}.$$

1812

1813

Supposing $\mu'_{E'} : F'_1 \rightarrow E'$, by $\text{locks}(\langle \Gamma' \rangle_{F'}, \mathbf{\mu}_{\mu'_{E'}}) : F'_1 \rightarrow F'$ and IH on (1), we have

$$\Gamma, \mathbf{\mu}_{\mu_F}, \langle \Gamma' \rangle_{F'}, \mathbf{\mu}_{\mu'_{E'}} \vdash V : A @ F'_1.$$

Then by T-MOD we have

$$\Gamma, \mathbf{\mu}_{\mu_F}, \langle \Gamma' \rangle_{F'} \vdash \mathbf{mod}_{\mu'} V : \mu' A @ E'.$$

Case

T-LETMOD

$$\frac{\begin{array}{c} v'_E : F_1 \rightarrow E \\ \Gamma, \mathbf{\mu}_{\mu_F}, \Gamma', \mathbf{\mu}_{v'_E} \vdash V : \mu' A @ F_1 \text{ (1)} \quad \Gamma, \mathbf{\mu}_{\mu_F}, \Gamma', x : v'_E \circ \mu'_{F_1} A \vdash M : B @ E \text{ (2)} \end{array}}{\Gamma, \mathbf{\mu}_{\mu_F}, \Gamma' \vdash \mathbf{let}_{v'} \mathbf{mod}_{\mu'} x = V \mathbf{in} M : B @ E}$$

By IH on (1), we have

$$\Gamma, \mathbf{\mu}_{v_F}, \langle \Gamma' \rangle_{F'}, \mathbf{\mu}_{v'_E} \vdash V : \mu' A @ F'_1$$

where $v'_E : F'_1 \rightarrow E'$. By IH on (2), we have

$$\Gamma, \mathbf{\mu}_{v_F}, \langle \Gamma' \rangle_{F'}, x : v'_E \circ \mu'_{F'_1} A \vdash M : B @ E'.$$

Then by T-LETMOD, we have

$$\Gamma, \mathbf{\mu}_{\mu_F}, \langle \Gamma' \rangle_{F'} \vdash \mathbf{let}_{v'} \mathbf{mod}_{\mu'} x = V \mathbf{in} M : B @ E'$$

Case

T-LETMOD'

$$\frac{\begin{array}{c} v'_E : F_1 \rightarrow E \\ \Gamma, \mathbf{\mu}_{\mu_F}, \Gamma', \mathbf{\mu}_{v'_E}, \overline{\alpha : K} \vdash V : \mu' A @ F_1 \text{ (1)} \quad \Gamma, \mathbf{\mu}_{\mu_F}, \Gamma', x : v'_E \circ \mu'_{F_1} \forall \overline{\alpha^K}. A \vdash M : B @ E \text{ (2)} \end{array}}{\Gamma, \mathbf{\mu}_{\mu_F}, \Gamma' \vdash \mathbf{let}_{v'} \mathbf{mod}_{\mu'} \overline{\Lambda \alpha^K}. x = V \mathbf{in} M : B @ E}$$

Similar to the case for T-LETMOD. BY IH on (1), we have

$$\Gamma, \mathbf{\mu}_{v_F}, \langle \Gamma' \rangle_{F'}, \mathbf{\mu}_{v'_E}, \overline{\alpha : K} \vdash V : \mu' A @ F'_1$$

where $v'_E : F'_1 \rightarrow E'$. By IH on (2), we have

$$\Gamma, \mathbf{\mu}_{v_F}, \langle \Gamma' \rangle_{F'}, x : v'_E \circ \mu'_{F'_1} \forall \overline{\alpha^K}. A \vdash M : B @ E'.$$

Then by T-LETMOD', we have

$$\Gamma, \mathbf{\mu}_{\mu_F}, \langle \Gamma' \rangle_{F'} \vdash \mathbf{let}_{v'} \mathbf{mod}_{\mu'} \overline{\Lambda \alpha^K}. x = V \mathbf{in} M : B @ E'$$

Case T-TABS, T-ABS, T-TAPP, T-APP, T-DO, T-MASK, T-HANDLER, T-MODABS, other handlers and data types. Follow from IH. Similar to other cases we have shown. \square

As a corollary of Lemma B.11.3, the following sub-effecting rule is admissible.

COROLLARY B.12 (SUB-EFFECTING). *The following rule is admissible.*

$$\frac{\Gamma \vdash M : A @ E \quad \text{locks}(\Gamma) : E \rightarrow F \quad F \leq F' \quad \text{locks}(\langle \Gamma \rangle_{F'}) : E' \rightarrow F'}{\langle \Gamma \rangle_{F'} \vdash M : A @ E'}$$

PROOF. Follow from Lemma B.11.3 by adding the lock $\mathbf{\mu}_{[F]}$ to the left of Γ in $\Gamma \vdash M : A @ E$, and weaken it to $\mathbf{\mu}_{[F']}$. Note that typing judgements still hold after adding a lock to or removing a lock from the left of the context, as long as the new contexts are still well-defined. \square

The following lemma reflects the intuition that pure values can be used in any effect context.

LEMMA B.13 (PURE PROMOTION). *The following promotion rule is admissible.*

$$\frac{\Gamma_1, \Gamma \vdash V : A @ E \quad \Gamma_1 \vdash A : \text{Abs}}{\text{locks}(\Gamma) : E \rightarrow F \quad \text{locks}(\Gamma') : E' \rightarrow F \quad \text{fv}(V) \cap \text{dom}(\Gamma') = \emptyset} \Gamma_1, \Gamma' \vdash V : A @ E'$$

PROOF. By induction on the typing derivation of V .

Case T-VAR. Trivial.

Case

$$\frac{\text{T-MOD} \quad \mu_E : F_1 \rightarrow E \quad \Gamma_1, \Gamma, \blacksquare_{\mu_E} \vdash V : A @ F_1 (1)}{\Gamma_1, \Gamma \vdash \mathbf{mod}_{\mu} V : \mu A @ E}$$

Case analysis on the shape of μ .

Case μ is relative. By kinding, A is also pure. By IH on (1), we have

$$\Gamma_1, \Gamma', \blacksquare_{\mu_{E'}} \vdash V : A @ F'_1$$

where $\mu_{E'} : F'_1 \rightarrow E'$. Then by T-MOD we have

$$\Gamma_1, \Gamma' \vdash \mathbf{mod}_{\mu} V : \mu A @ E'$$

Case μ is absolute. We have $\mu = [F_1]$ and $\text{locks}(\Gamma', \blacksquare_{\mu_{E'}}) = [F_1]_F = \text{locks}(\Gamma, \blacksquare_{\mu_E})$. Thus, replacing the context $(\Gamma, \blacksquare_{\mu_E})$ with $(\Gamma', \blacksquare_{\mu_{E'}})$ in (1) does not influence all usages of T-VAR in the derivation tree of (1). We have

$$\Gamma_1, \Gamma', \blacksquare_{\mu_{E'}} \vdash V : A @ F_1$$

Then by T-MOD we have

$$\Gamma_1, \Gamma' \vdash \mathbf{mod}_{\mu} V : \mu A @ E'$$

Case T-TABS. Follow from IH and Lemma B.11.5.

Case T-ABS. Impossible since function types are impure.

Case Data types. Follow from IHs.

□

LEMMA B.14 (SUBSTITUTION). *The following substitution rules are admissible.*

1. *Preservation of kinds under type substitution.*

$$\frac{\Gamma \vdash A : K \quad \Gamma, \alpha : K, \Gamma' \vdash B : K'}{\Gamma, \Gamma' \vdash B[A/\alpha] : K'}$$

2. *Preservation of types under type substitution.*

$$\frac{\Gamma \vdash A : K \quad \Gamma, \alpha : K, \Gamma' \vdash M : B @ E}{\Gamma, \Gamma' \vdash M[A/\alpha] : B[A/\alpha] @ E}$$

3. *Preservation of types under value substitution.*

$$\frac{\Gamma, \blacksquare_{\mu_F} \vdash V : A @ F' \quad \Gamma, x :_{\mu_F} A, \Gamma' \vdash M : B @ E}{\Gamma, \Gamma' \vdash M[V/x] : B @ E}$$

1912 PROOF.

- 1913 1. By straightforward induction on the kinding derivation.
 1914 2. By straightforward induction on the typing derivation of M .
 1915 3. By induction on the typing derivation of M . Trivial when variable x is not used. In the following
 1916 induction we always assume x is used.

1917 Case

$$\frac{\text{T-VAR}}{v_F = \text{locks}(\Gamma') : E \rightarrow F \quad \mu_F \Rightarrow v_F \text{ (1) or } \Gamma \vdash A : \text{Abs}} \\ \Gamma, x :_{\mu_F} A, \Gamma' \vdash x : A @ E$$

1918 Case analysis on the purity of A

1919 Case Impure. By $\Gamma, \blacktriangle_{\mu_F} \vdash V : A @ F'$, (1), and Lemma B.11.3, we have

$$\Gamma, \blacktriangle_{v_F} \vdash V : A @ E.$$

1920 Then, by context equivalence, Lemma B.11.1, and Lemma B.11.4, we have

$$\Gamma, \Gamma' \vdash V : A @ E.$$

1921 Case Pure. By $\Gamma, \blacktriangle_{\mu_F} \vdash V : A @ F'$ and Lemma B.13, we have

$$\Gamma, \Gamma' \vdash V : A @ E.$$

1922 Case

$$\frac{\text{T-MOD}}{\mu'_E : F_1 \rightarrow E \quad \Gamma, x :_{\mu_F} A, \Gamma', \blacktriangle_{\mu'_E} \vdash W : B @ F_1 \text{ (1)}} \\ \Gamma, x :_{\mu_F} A, \Gamma' \vdash \mathbf{mod}_{\mu'} W : \mu' B @ E$$

1923 By IH on (1) we have

$$\Gamma, \Gamma', \blacktriangle_{\mu'_E} \vdash W[V/x] : B @ F_1.$$

1924 Then by T-MOD we have

$$\Gamma, \Gamma' \vdash (\mathbf{mod}_{\mu'} W)[V/x] : \mu' B @ E$$

1925 Case

1926 T-LETMOD

$$\frac{v_E : F_1 \rightarrow E \quad \Gamma, x :_{\mu_F} A, \Gamma', \blacktriangle_{v_E} \vdash W : \mu' A' @ F_1 \text{ (1)} \quad \Gamma, x :_{\mu_F} A, \Gamma', y :_{v_E \circ \mu'_{F_1}} A' \vdash M : B @ E \text{ (2)}} \\ \Gamma, x :_{\mu_F} A, \Gamma' \vdash \mathbf{let}_v \mathbf{mod}_{\mu'} y = W \mathbf{in} M : B @ E$$

1927 By IH on (1), we have

$$\Gamma, \Gamma', \blacktriangle_{v_E} \vdash W[V/x] : \mu' A' @ F_1.$$

1928 By IH on (2), we have

$$\Gamma, \Gamma', y :_{v_E \circ \mu'_{F_1}} A' \vdash M[V/x] : B @ E.$$

1929 Then by T-LETMOD, we have

$$\Gamma, \Gamma' \vdash (\mathbf{let}_v \mathbf{mod}_{\mu'} y = W \mathbf{in} M)[V/x] : B @ E$$

1930 Case

$$\frac{\text{T-LETMOD}' \quad v_E : F_1 \rightarrow E \quad \Gamma, x :_{\mu_F} A, \Gamma', \blacktriangle_{v_E}, \overline{\alpha : K} \vdash V : \mu' A' @ F_1 \text{ (1)} \\ \Gamma, x :_{\mu_F} A, \Gamma', y :_{v_E \circ \mu'_{F_1}} \forall \overline{\alpha^K}. A' \vdash M : B @ E \text{ (2)}} \\ \Gamma, x :_{\mu_F} A, \Gamma' \vdash \mathbf{let}_v \mathbf{mod}_{\mu'} \overline{\Lambda \alpha^K}. y = V \mathbf{in} M : B @ E$$

1931

1961 Similar to the case for T-LETMOD. Our goal follows from IH on (1), IH on (2), and T-LETMOD'.

1962 Case

$$\begin{array}{c}
 \text{T-MASK} \\
 \frac{\Gamma, x : \mu_F A, \Gamma', \blacktriangleleft_{\langle L \rangle E} \vdash M : B @ E - L \text{ (1)}}{\Gamma, x : \mu_F A, \Gamma' \vdash \mathbf{mask}_L M : \langle L \rangle B @ E}
 \end{array}$$

1966 By IH on (1) we have

$$\Gamma, \Gamma', \blacktriangleleft_{\langle L \rangle E} \vdash M[V/x] : B @ E - L.$$

1970 Then by T-MASK we have

$$\Gamma, \Gamma' \vdash (\mathbf{mask}_L M)[V/x] : \langle L \rangle B @ E$$

1973 Case

$$\begin{array}{c}
 \text{T-HANDLER} \\
 \frac{
 \begin{array}{c}
 D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, x : \mu_F A, \Gamma', \blacktriangleleft_{\langle D \rangle E} \vdash M : A_0 @ D + E \text{ (1)} \\
 \Gamma, x : \mu_F A, \Gamma', y : \langle D \rangle A_0 \vdash N : B @ E \text{ (2)} \\
 [\Gamma, x : \mu_F A, \Gamma', p_i : A_i, r_i : B_i \rightarrow B \vdash N_i : B @ E \text{ (3)}]_i
 \end{array}
 }{
 \Gamma, x : \mu_F A, \Gamma' \vdash \mathbf{handle} M \mathbf{with} \{\mathbf{return} y \mapsto N\} \uplus \{\ell_i p_i r_i \mapsto N_i\}_i : B @ E
 }
 \end{array}$$

1978 Follow from IH on (1),(2),(3), and reapplying T-HANDLER.

1980 Case T-TABS, T-TAPP, T-ABS, T-APP, T-Do. Follow from IH.

1981 Case T-MODABS, other handlers and data types. Follow from IH.

1982 □

1984 B.4 Progress

1986 THEOREM 3.3 (PROGRESS). *If $\vdash M : A @ E$, then either there exists N such that $M \rightsquigarrow N$ or M is in a normal form with respect to E .*

1988 PROOF. By induction on the typing derivation $\vdash M : A @ E$. The most non-trivial cases are T-MASK and T-HANDLER. Other cases follow from IHs and reduction rules, using Lemma B.8.

1991 Case M is in a value normal form U . Trivial. Base case.

1992 Case T-Do. Trivial. Base case.

1993 Case T-MOD. $\mathbf{mod}_\mu V$. By IH on V .

1994 Case T-LETMOD. $\mathbf{let}_v \mathbf{mod}_\mu x = V \mathbf{in} N$. By IH on V , if V is reducible then M is reducible; otherwise, V is in a value normal form, then by Lemma B.8 we have that M is reducible by E-LETMOD.

1997 Case T-LETMOD'. Similar to the case for T-LETMOD.

1998 Case T-TAPP. MA . Similarly by IH on M , Lemma B.8, and E-TAPP.

1999 Case T-APP. MN . Similarly by IH on M and N , Lemma B.8, and E-APP.

2000 Case T-MASK. $\mathbf{mask}^E M$. By IH on M .

2001 Case M is reducible. Trivial.

2002 Case M is in a value normal form. By E-MASK.

2003 Case $M = \mathcal{E}[\mathbf{do} \ell U]$ with n -free(ℓ, \mathcal{E}). The whole term is in a normal form.

2004 Case Handlers. The general form is $\mathbf{handle}^\delta M \mathbf{with} H$. By IH on M .

2005 Case M is reducible. Trivial.

2006 Case M is in a value normal form. By E-RET.

2007 Case $M = \mathcal{E}[\mathbf{do} \ell U]$ with n -free(ℓ, \mathcal{E}). If $n = 0$ and $\ell \in H$, then reducible by E-Op.

2008 Otherwise, the whole term is in a normal form.

2009

2010 Case T-MODABS. $\mathbf{mod}[] M$. If $M \rightsquigarrow N$, follow by IH on M . Otherwise, M must be in a value normal
 2011 form because the T-MODABS requires M to have the empty effect. In this case, $\mathbf{mod}[] M$ is
 2012 also in a value normal form.

2013 Case Other handlers and data types. Similar to other cases.

2014

2015

2016 B.5 Subject Reduction

2017 THEOREM 3.4 (SUBJECT REDUCTION). *If $\Gamma \vdash M : A @ E$ and $M \rightsquigarrow N$, then $\Gamma \vdash N : A @ E$.*

2018

2019 PROOF. By induction on the typing derivation $\Gamma \vdash M : A @ E$.

2020 Case T-VAR. Impossible as there is no further reduction.

2021 Case

2022

2023

2024

2025

$$\frac{\text{T-MOD} \quad \mu_F : E \rightarrow F \quad \Gamma, \mathfrak{A}_{\mu_F} \vdash V : A @ E (1)}{\Gamma \vdash \mathbf{mod}_{\mu} V : \mu A @ F}$$

2026 The only way to reduce is by E-LIFT and $V \rightsquigarrow W$. IH on (1) gives

2027

2028

$$\Gamma, \mathfrak{A}_{\mu_F} \vdash W : A @ E.$$

2029 Then by T-MOD we have

2030

2031

$$\Gamma \vdash \mathbf{mod}_{\mu} W : \mu A @ F.$$

2032 Case

2033

2034

2035

2036

$$\frac{\text{T-LETMOD} \quad v_F : E \rightarrow F \quad \Gamma, \mathfrak{A}_{v_F} \vdash V : \mu A @ E (1) \quad \Gamma, x : v_F \circ \mu_E A \vdash M : B @ F (2)}{\Gamma \vdash \mathbf{let}_v \mathbf{mod}_{\mu} x = V \mathbf{in} M : B @ F}$$

2037

2038

2039

By case analysis on the reduction.

Case E-LIFT with $V \rightsquigarrow W$. By IH on (1) and reapplying T-LETMOD.

Case E-LETMOD. We have $V = \mathbf{mod}_{\mu} U$ and

2040

$$\mathbf{let}_v \mathbf{mod}_{\mu} x = \mathbf{mod}_{\mu} U \mathbf{in} M \rightsquigarrow M[U/x].$$

2041

2042

Inversion on (1) gives

2043

$$\Gamma, \mathfrak{A}_{v_F}, \mathfrak{A}_{\mu_E} \vdash U : A @ E'$$

2044

2045

where $\mu_E : E' \rightarrow E$. By context equivalence, we have

2046

2047

$$\Gamma, \mathfrak{A}_{v_F \circ \mu_E} \vdash U : A @ E'$$

2048

where $v_F \circ \mu_E : E' \rightarrow F$. By Lemma B.14.3 and (2), we have

2049

2050

$$\Gamma \vdash M[U/x] : B @ F.$$

2051 Case

2052

2053

2054

2055

$$\frac{\text{T-LETMOD}' \quad v_F : E \rightarrow F \quad \Gamma, \mathfrak{A}_{v_F}, \overline{\alpha} : \overline{K} \vdash V : \mu A @ E (1) \quad \Gamma, x : v_F \circ \mu_E \sqrt{\alpha}^{\overline{K}}. A \vdash M : B @ F (2)}{\Gamma \vdash \mathbf{let}_v \mathbf{mod}_{\mu} \overline{\Lambda \alpha}^{\overline{K}}. x = V \mathbf{in} M : B @ F}$$

2056

2057

2058

Similar to the case for T-LETMOD'. By case analysis on the reduction.

Case E-LIFT with $V \rightsquigarrow W$. By IH on (1) and reapplying T-LETMOD'.

Case E-LETMOD'. We have $V = \mathbf{mod}_\mu U$ and

$$\mathbf{let}_v \mathbf{mod}_\mu \Lambda \overline{\alpha^K}.x = \mathbf{mod}_\mu U \text{ in } M \rightsquigarrow M[(\forall \overline{\alpha^K}.U)/x].$$

Inversion on (1) gives

$$\Gamma, \mathbf{\mu}_{v_F}, \overline{\alpha : K}, \mathbf{\mu}_{\mu_E} \vdash U : \mu A @ E'.$$

where $\mu_E : E' \rightarrow E$. By Lemma B.11.5 we have

$$\Gamma, \mathbf{\mu}_{v_F}, \mathbf{\mu}_{\mu_E}, \overline{\alpha : K} \vdash U : A @ E'.$$

By context equivalence, we have

$$\Gamma, \mathbf{\mu}_{v_F \circ \mu_E}, \overline{\alpha : K} \vdash U : A @ E'.$$

where $v_F \circ \mu_E : E' \rightarrow F$. By T-TABS we have

$$\Gamma, \mathbf{\mu}_{v_F \circ \mu_E} \vdash \Lambda \overline{\alpha^K}.U : \forall \overline{\alpha^K}.A @ E'.$$

By Lemma B.14.3 and (2), we have

$$\Gamma \vdash M[U/x] : B @ F.$$

Case T-TABS, T-ABS. Impossible as there is no further reduction.

Case

$$\frac{\text{T-TAPP} \quad \Gamma \vdash M : \forall \overline{\alpha^K}.B @ E (1) \quad \Gamma \vdash A : K (2)}{\Gamma \vdash MA : B[A/\alpha] @ E}$$

By case analysis on the reduction.

Case E-LIFT with $M \rightsquigarrow N$. By IH on (1) and reapplying T-TAPP.

Case E-TAPP. We have $M = \Lambda \overline{\alpha^K}.V$ and

$$(\Lambda \overline{\alpha^K}.V) A \rightsquigarrow V[A/\alpha].$$

Inversion on (1) gives

$$\Gamma, \alpha : K \vdash V : B @ E.$$

Then by Lemma B.14.2 on (2), we have

$$\Gamma \vdash V[A/\alpha] : B[A/\alpha] @ E.$$

Case

$$\frac{\text{T-APP} \quad \Gamma \vdash M : A \rightarrow B @ E (1) \quad \Gamma \vdash N : A @ E (2)}{\Gamma \vdash MN : B @ E}$$

By case analysis on the reduction.

Case E-LIFT with $M \rightsquigarrow M'$. By IH on (1) and reapplying T-APP.

Case E-LIFT with $N \rightsquigarrow N'$. By IH on (2) and reapplying T-APP.

Case E-APP. We have $M = \lambda x^A.M'$, $N = U$, and

$$MN \rightsquigarrow M'[U/x].$$

Inversion on (1) gives

$$\Gamma, x : A \vdash M' : B @ E.$$

Then by Lemma B.14.3 we have

$$\Gamma \vdash M'[U/x] : B @ E.$$

2108 Case T-Do. The only way to reduce is by E-LIFT. Follow from IH and reapplying T-Do.
 2109 Case

$$\frac{\text{T-MASK}}{\Gamma, \mathbf{\blacktriangleleft}_{\langle L \rangle_F} \vdash M : A @ F - L (1)} \\ \Gamma \vdash \mathbf{mask}_L M : \langle L \rangle A @ F$$

2113 By case analysis on the reduction.

2114 Case E-LIFT with $M \rightsquigarrow N$. By IH on (1) and reapplying T-MASK.

2115 Case E-MASK. We have $M = U$ and

$$\mathbf{mask}_L U \rightsquigarrow \mathbf{mod}_{\langle L \rangle} U.$$

2118 By $\langle L \rangle_F : F - L \rightarrow F$ and T-MOD, we have

$$\Gamma \vdash \mathbf{mod}_{\langle L \rangle} U : \langle L \rangle A @ F.$$

2121 Case

2122 T-HANDLER

$$\frac{\begin{array}{l} H = \{\mathbf{return} \ x \mapsto N\} \uplus \{\ell_i \ p_i \ r_i \mapsto N_i\}_i \\ D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, \mathbf{\blacktriangleleft}_{\langle D \rangle_F} \vdash M : A @ D + F (1) \\ \Gamma, x : \langle D \rangle A \vdash N : B @ F (2) \quad [\Gamma, p_i : A_i, r_i : B_i \rightarrow B \vdash N_i : B @ F (3)]_i \end{array}}{\Gamma \vdash \mathbf{handle} \ M \ \mathbf{with} \ H : B @ F}$$

2128 By case analysis on the reduction.

2129 Case E-LIFT with $M \rightsquigarrow M'$. By IHs and reapplying T-HANDLER.

2130 Case E-RET. We have $M = U$ and

$$\mathbf{handle} \ U \ \mathbf{with} \ H \rightsquigarrow N[(\mathbf{mod}_{\langle D \rangle} U)/x].$$

2132 By (1), $\langle D \rangle_F : F \rightarrow D + F$, and T-MOD, we have

$$\Gamma \vdash \mathbf{mod}_{\langle D \rangle} U : A @ F.$$

2135 Then by (2) and Lemma B.14.3 we have

$$\Gamma \vdash N[(\mathbf{mod}_{\langle D \rangle} U)/x] : B @ F.$$

2138 Case E-OP. We have $M = \mathcal{E}[\mathbf{do} \ \ell_j \ U]$, 0-free(ℓ_j, \mathcal{E}), $\ell_j \ p_j \ r_j \mapsto N_j$, and

$$\mathbf{handle} \ M \ \mathbf{with} \ H \rightsquigarrow N_j[U/p, (\lambda y. \mathbf{handle} \ \mathcal{E}[y] \ \mathbf{with} \ H)/r].$$

2141 Since D is well-kinded, A_j and B_j are pure. By inversion on $\mathbf{do} \ \ell_j \ U$ we have

$$\Gamma, \mathbf{\blacktriangleleft}_{\langle D \rangle_F} \vdash U : A_j @ D + F.$$

2144 By A_j is pure and Lemma B.13, we have

$$\Gamma, \mathbf{\blacktriangleleft}_{\langle D \rangle_F}, \mathbf{\blacktriangleleft}_{\langle L \rangle_{D+F}} \vdash U : A_j @ F$$

2147 where $L = \text{dom}(D)$. By context equivalence, we have

$$\Gamma \vdash U : A_j @ F (4)$$

2150 Observe that B_j being pure allows $y : B_j$ to be accessed in any context. By (1) and a
 2151 straightforward induction on \mathcal{E} we have

$$\Gamma, y : B_j, \mathbf{\blacktriangleleft}_{\langle D \rangle_F} \vdash \mathcal{E}[y] : A @ D + F.$$

2154 Then by T-HANDLER and T-ABS we have

$$\Gamma \vdash \lambda y. \mathbf{handle} \ \mathcal{E}[y] \ \mathbf{with} \ H : B_j \rightarrow B @ F (5).$$

2156

2157 Finally, by (3), (4), (5), and Lemma B.14.3 we have

$$2158 \quad \Gamma \vdash N_j[U/p, (\lambda y. \mathbf{handle} \mathcal{E}[y] \mathbf{with} H)/r] : B @ F.$$

2159 Case

$$2160 \quad \text{T-HANDLER}^A$$

$$2161 \quad D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, \mathfrak{A}_{[D+E]_F} \vdash M : A @ D + E \text{ (1)}$$

$$2162 \quad \Gamma, \mathfrak{A}_{[E]_F}, x : [D + E]A \vdash N : B @ F \text{ (2)}$$

$$2163 \quad \frac{[\Gamma, \mathfrak{A}_{[E]_F}, p_i : A_i, r_i : [E](B_i \rightarrow B) \vdash N_i : B @ E \text{ (3)}]_i \quad [E]_F \Rightarrow \mathbb{1}_F}{\Gamma \vdash \mathbf{handle}^A M \mathbf{with} \{\mathbf{return} x \mapsto N\} \uplus \{\ell_i p_i r_i \mapsto N_i\}_i : B @ F}$$

2164 By case analysis on the reduction.

2165 Case E-LIFT with $M \rightsquigarrow M'$. By IHs and reapplying T-HANDLER^A.

2166 Case E-RET^A. We have $M = U$ and

$$2167 \quad \mathbf{handle} M \mathbf{with} H \rightsquigarrow N[(\mathbf{mod}_{[D+E]} U)/x].$$

2168 By (1) and $[D + E]_F = [E]_F \circ [D + E]_E$ we have

$$2169 \quad \Gamma, \mathfrak{A}_{[E]_F}, \mathfrak{A}_{[D+E]_E} \vdash U : A @ E.$$

2170 By $[D + E]_E : D + E \rightarrow E$ and T-MOD, we have

$$2171 \quad \Gamma, \mathfrak{A}_{[E]_F} \vdash \mathbf{mod}_{[D+E]} U : [D + E]A @ E.$$

2172 Then by (2) and Lemma B.14.3 we have

$$2173 \quad \Gamma, \mathfrak{A}_{[E]_F} \vdash N[(\mathbf{mod}_{[D+E]} U)/x] : B @ E.$$

2174 By $[E]_F \Rightarrow \mathbb{1}_F$ and Lemma B.11.3 we have

$$2175 \quad \Gamma \vdash N[(\mathbf{mod}_{[D+E]} U)/x] : B @ F.$$

2176 Case E-OP^A. We have $M = \mathcal{E}[\mathbf{do} \ell_j U]$, 0-free(ℓ_j, \mathcal{E}), $\ell_j p_j r_j \mapsto N_j$, and

$$2177 \quad \mathbf{handle}^A M \mathbf{with} H \rightsquigarrow N_j[U/p, (\mathbf{mod}_{[E]} (\lambda y. \mathbf{handle}^A \mathcal{E}[y] \mathbf{with} H))/r].$$

2178 Since D is well-kinded, A_j and B_j are pure. By inversion on $\mathbf{do} \ell_j U$, we have

$$2179 \quad \Gamma, \mathfrak{A}_{[D+E]_F} \vdash U : A_j @ D + E.$$

2180 By A_j is pure and Lemma B.13, we have

$$2181 \quad \Gamma \vdash U : A_j @ F \text{ (4)}.$$

2182 Observe that B_j being pure allows y to be accessed in any context. By (1) and a straightforward induction on \mathcal{E} we have

$$2183 \quad \Gamma, y : B_j, \mathfrak{A}_{[D+E]_F} \vdash \mathcal{E}[y] : A @ D + E.$$

2184 By $[E]_F \circ [E]_E \circ [D + E]_E = [D + E]_F$ and context equivalence, we have

$$2185 \quad \Gamma, y : B_j, \mathfrak{A}_{[E]_F}, \mathfrak{A}_{[E]_E}, \mathfrak{A}_{[D+E]_E} \vdash \mathcal{E}[y] : A @ D + E.$$

2186 Since B_j is pure, we can swap $y : B_j$ with locks and derive

$$2187 \quad \Gamma, \mathfrak{A}_{[E]_F}, \mathfrak{A}_{[E]_E}, y : B_j, \mathfrak{A}_{[D+E]_E} \vdash \mathcal{E}[y] : A @ D + E.$$

2188 By T-HANDLER^A, we have

$$2189 \quad \Gamma, \mathfrak{A}_{[E]_F}, \mathfrak{A}_{[E]_E}, y : B_j \vdash \mathbf{handle}^A \mathcal{E}[y] \mathbf{with} H : B @ E.$$

2190

Notice that we can put H after absolute locks because all clauses in H have an absolute lock $\mathbf{lock}_{[E]_E}$ in their contexts. Then by T-ABS and T-MOD we have

$$\Gamma, \mathbf{lock}_{[E]_F} \vdash \mathbf{mod}_{[E]} (\lambda y. \mathbf{handle}^A \mathcal{E}[y] \mathbf{with} H) : [E](B_j \rightarrow B) @ E (5).$$

By (3), (4), (5), and Lemma B.14.3 we have

$$\Gamma, \mathbf{lock}_{[E]_F} \vdash N_j[U/p, (\mathbf{mod}_{[F]} (\lambda y. \mathbf{handle} \mathcal{E}[y] \mathbf{with} H))/r] : B @ E.$$

Finally, by $[E]_F \Rightarrow \mathbb{1}_F$ and Lemma B.11.3 we have

$$\Gamma \vdash N_j[U/p, (\mathbf{mod}_{[F]} (\lambda y. \mathbf{handle} \mathcal{E}[y] \mathbf{with} H))/r] : B @ F.$$

Case Shallow handlers. Similar to the cases of deep handlers.

Case Data types. Nothing more special than the cases we have already shown. Introduction rules follows from IHs and reapplying the same typing rules. Elimination rules require to additionally consider their corresponding reduction rules.

□

B.6 Proof of Encoding

We prove the encoding from F_{eff}^1 into MET in Section 4.

Definition 4.1 (Well-scoped). A typing judgement $\Gamma_1, x :_{\varepsilon} A, \Gamma_2 \vdash M : B!E$ is *well-scoped for x* if either $x \notin \text{fv}(M)$ or $\diamond_F^A \notin \Gamma_2$ or $A = \forall.A'$. A typing judgement $\Gamma \vdash M : A!E$ is *well-scoped* if it is well-scoped for all $x \in \Gamma$.

LEMMA B.15 (WELL-SCOPEDNESS OF DERIVATION TREES). *If the judgement at the bottom of a derivation tree is well-scoped, then every judgement in the derivation tree is well-scoped.*

PROOF. Assume the contrary. Let $\Gamma_1, x :_{\varepsilon} A, \Gamma_2 \vdash M : B!E$ be the top-most judgement in the derivation tree with $x \in \text{fv}(M)$ and $\diamond_F^A \in \Gamma_2$ and $A \neq \forall.A'$. By case analysis on whether $\diamond_F^A \in \Gamma_2$ was introduced in the derivation tree.

Case not introduced in the derivation tree: Then the judgement at the bottom of the derivation tree must contain both the marker and x and is not well-scoped for x . Contradiction.

Case introduced in the derivation tree: since we chose the top-most judgement, the judgement must have introduced the marker by an application of the R-EABS rule. Let ε' be the effect variable introduced at this judgement. Then $\varepsilon \neq \varepsilon'$ by the side-condition of the R-EABS rule. We have that ε is the ambient effect at the R-VAR rule where x is used as a free variable, since we chose the top-most judgement. By the side-condition of the R-VAR rule, then $\varepsilon = \varepsilon'$ or $A = \forall.A'$. Contradiction.

□

In the special case we consider there are no absent signatures. This implies that submoding on effects can only add labels to the end. Furthermore, all labels are drawn from a global environment and thus have the same signatures. This allows us to freely permute them in the effect row. In this case, we can strengthen the statement to the following:

COROLLARY B.16 (TRANSFORMATION FROM INDEX). *If $\langle L_1 | D_1 \rangle(F) \leq \langle L_2 | D_2 \rangle(F)$ and $L_1 \leq F$ and $L_2 \leq F$ and $L_1 \bowtie D_1 = L_2 \bowtie D_2$, then $\langle L_1 | D_1 \rangle_F \Rightarrow \langle L_2 | D_2 \rangle_F$.*

2255 **PROOF.** We show that for all F' with $F \leq F'$, we have $\langle L_1 | D_1 \rangle(F') \leq \langle L_2 | D_2 \rangle(F')$. Since all
 2256 signatures are present in F , we have that $F' = F + \bar{l}$ for some collection of labels with signatures \bar{l} .
 2257 Then we use that $L_1 \leq F$:

$$\begin{aligned} 2258 \quad \langle L_1 | D_1 \rangle(F') &= \langle L_1 | D_1 \rangle(F + \bar{l}) \\ 2259 &= D_1 + ((F + \bar{l}) - L_1) \\ 2260 &= D_1 + ((F - L_1) + \bar{l}) \\ 2261 &= \langle L_1 | D_1 \rangle(F) + \bar{l} \end{aligned}$$

2262 and the same for $\langle L_2 | D_2 \rangle(F')$. Since $\langle L_1 | D_1 \rangle(F) \leq \langle L_2 | D_2 \rangle(F)$ and we can freely permute labels,
 2263 we have that $(\langle L_1 | D_1 \rangle(F) + \bar{l}) \leq (\langle L_2 | D_2 \rangle(F) + \bar{l})$. \square

2264 The condition that $L_1 \bowtie D_1 = L_2 \bowtie D_2$ can be checked easily, where for the composition of
 2265 modalities we use the fact that for $\langle L | D \rangle = \langle L_1 | D_1 \rangle \circ \langle L_2 | D_2 \rangle$, we have $L \bowtie D = (L_1, L_2) \bowtie (D_1, D_2)$.
 2266

2267 **LEMMA B.17 (FIRST MODALITY TRANSFORMATION).** For all E_1, E_2, E_3 :

$$2268 \quad (\langle E_1 - E_2 | E_2 - E_1 \rangle \circ \langle E_2 - E_3 | E_3 - E_2 \rangle)_{E_1} \Leftrightarrow \langle E_1 - E_3 | E_3 - E_1 \rangle_{E_1}$$

2269 **PROOF.** We can use Corollary B.16 since $(E_1 - E_3) \leq E_1$ and $(E_1 - E_2) + L \leq E_1$ where $(L, D) =$
 2270 $(E_2 - E_3) \bowtie (E_2 - E_1)$. We have:

$$\begin{aligned} 2271 \quad \langle E_1 - E_3 | E_3 - E_1 \rangle(E_1) &= (E_3 - E_1) + (E_1 - (E_1 - E_3)) \\ 2272 &= (E_3 - E_1) + (E_1 \cap E_3) \\ 2273 &= E_3 \end{aligned}$$

2274 and using this calculation:

$$\begin{aligned} 2275 \quad \langle E_1 - E_2 | E_2 - E_1 \rangle \circ \langle E_2 - E_3 | E_3 - E_2 \rangle(E_1) &= \langle E_2 - E_3 | E_3 - E_2 \rangle(\langle E_1 - E_2 | E_2 - E_1 \rangle(E_1)) \\ 2276 &= \langle E_2 - E_3 | E_3 - E_2 \rangle(E_2) \\ 2277 &= E_3 \end{aligned}$$

2278 \square

2279 **LEMMA B.18 (SECOND MODALITY TRANSFORMATION).** For all L, E, F :

$$2280 \quad \langle L + (E - F) | F - E \rangle_{L+E} \Rightarrow \langle (L + E) - F | F - (L + E) \rangle_{L+E}$$

2281 **PROOF.** We can use Corollary B.16 since $(L + E) - F \leq L + E$ and $L + (E - F) \leq L + E$. We have:

$$\begin{aligned} 2282 \quad \langle (L + E) - F | F - (L + E) \rangle(L + E) &= (F - (L + E)) + ((L + E) - (L + E - F)) \\ 2283 &= (F - (L + E)) + ((L + E) \cap F) \\ 2284 &= F \end{aligned}$$

2285 and:

$$\begin{aligned} 2286 \quad \langle L + (E - F) | F - E \rangle(L + E) &= (F - E) + ((L + E) - (L + (E - F))) \\ 2287 &= (F - E) + (E - (E - F)) \\ 2288 &= (F - E) + (E \cap F) \\ 2289 &= F \end{aligned}$$

2290 \square

LEMMA B.19 (THIRD MODALITY TRANSFORMATION). For all $\bar{\ell}_i, E, F$:

$$(\langle \bar{\ell}_i \rangle \circ \langle \bar{\ell}_i, E - F | F - \bar{\ell}_i, E \rangle)_E \Rightarrow \langle E - F | F - E \rangle_E$$

PROOF. We can use Corollary B.16 since $(\langle \bar{\ell}_i \rangle \circ \langle \bar{\ell}_i, E - F | F - \bar{\ell}_i, E \rangle) = \langle \bar{\ell}_i, E - F | F - \bar{\ell}_i, E \rangle(\bar{\ell}_i, E)$ and $\bar{\ell}_i, E - F \leq \bar{\ell}_i, E$ and $E - F \leq E$. We have $\langle E - F | F - E \rangle(E) = F$ and:

$$\begin{aligned} \langle \bar{\ell}_i \rangle \circ \langle \bar{\ell}_i, E - F | F - \bar{\ell}_i, E \rangle(E) &= \langle \bar{\ell}_i, E - F | F - \bar{\ell}_i, E \rangle(\langle \bar{\ell}_i \rangle(E)) \\ &= \langle \bar{\ell}_i, E - F | F - \bar{\ell}_i, E \rangle(\bar{\ell}_i, E) \\ &= F \end{aligned}$$

□

LEMMA B.20 (TRANSLATING INSTANTIATED TYPES). For all F_{eff}^1 types A : $\llbracket A \rrbracket_E = \llbracket A[E'/] \rrbracket_{E, E'}$.

PROOF. By induction on the type A .

Case $A = \text{Int}$. Trivial.

Case $A = \forall. A'$. Trivial.

Case $A = A' \rightarrow^F B'$. Then:

$$\begin{aligned} \llbracket A \rrbracket_E &= \langle E - F | F - E \rangle(\llbracket A' \rrbracket_F \rightarrow \llbracket B' \rrbracket_F) \\ \llbracket A[E'/] \rrbracket_{E, E'} &= \langle E, E' - F, E' | F, E' - E, E' \rangle(\llbracket A'[E'/] \rrbracket_{F, E'} \rightarrow \llbracket B'[E'/] \rrbracket_{F, E'}) \end{aligned}$$

By the induction hypothesis we have:

$$\begin{aligned} \llbracket A' \rrbracket_F &= \llbracket A'[E'/] \rrbracket_{F, E'} \\ \llbracket B' \rrbracket_F &= \llbracket B'[E'/] \rrbracket_{F, E'} \end{aligned}$$

Since we can freely permute labels:

$$\begin{aligned} \langle E, E' - F, E' | F, E' - E, E' \rangle &= \langle E', E - E', F | E', F - E', E \rangle \\ &= \langle E - F | F - E \rangle \end{aligned}$$

□

LEMMA 4.2 (TYPE PRESERVATION OF ENCODING). If $\Gamma \vdash M : A! \{E|\varepsilon\}$ is well-scoped, then $M : A! E \rightarrow M'$ and $\llbracket \Gamma \rrbracket_E \vdash M' : \llbracket A \rrbracket_E @ E$.

PROOF. By induction on the typing derivation $\Gamma \vdash M : A! E$. We prove this for each rule of the translation. As a visual aid, we repeat each rule where we replace the translation premises by the MET judgement implied by the induction hypothesis and the translation in the conclusion by the MET judgement we need to prove.

Case

R-VAR

$$\frac{}{\llbracket \Gamma_1, x : A, \Gamma_2 \rrbracket_E \vdash \text{rebox}(x; A; E) : \llbracket A \rrbracket_E @ E}$$

We use the $\text{rebox}(x; A; E)$ function defined as follows:

$$\text{rebox}(x; A; E) = \begin{cases} \mathbf{mod}_{\{\}} x, & \text{if } A = \text{Int} \\ \mathbf{mod}_{\langle E-F | F-E \rangle} x, & \text{if } A = A' \rightarrow^F B' \\ \mathbf{mod}_{\{\}} x, & \text{if } A = \forall. A' \end{cases}$$

This function is exactly equivalent to $\mathbf{mod}_\mu x$ where $\mu = \text{topmod}(\llbracket A \rrbracket_E)$. We use the T-MOD rule to introduce the box. By cases on the type A :

2353 Case $A = \text{Int}$. We can use the T-VAR rule since $\cdot \vdash \text{Int} : \text{Abs}$.

2354 Case $A = \forall.A'$. Then $\llbracket A \rrbracket_F = \llbracket \llbracket A' \rrbracket \rrbracket$. for all F . By rule MT-ABS, the pure modality transforms
2355 into any other modality and so we can use the T-VAR rule.

2356 Case $A = A' \rightarrow^F B'$. Since the F_{eff}^1 judgement is well-scoped, we have that $\text{locks}(\Gamma_2)$
2357 is the composition of transition modalities. Furthermore, $\text{locks}(\Gamma') \circ \langle E - F | F - E \rangle :$
2358 $F \rightarrow F'$ for the context F' where x as introduced and x is annotated by the modality
2359 $\langle F' - F | F - F' \rangle_{F'} : F \rightarrow F'$. By Lemma B.17, we can use the T-VAR rule.

2360 Case

$$2361 \quad \text{R-APP}$$

$$2362 \quad \frac{\llbracket \Gamma \rrbracket_E \vdash M' : \llbracket A \rightarrow^E B \rrbracket_E @ E \quad \llbracket \Gamma \rrbracket_E \vdash N' : \llbracket A \rrbracket_E @ E \quad x \text{ fresh}}{\llbracket \Gamma \rrbracket_E \vdash \mathbf{let mod}_{\langle \cdot \rangle} x = M' \mathbf{ in } x N' : \llbracket B \rrbracket_E @ E}$$

2366 We have $\llbracket A \rightarrow^E B \rrbracket_E = \langle \cdot \rangle (\llbracket A \rrbracket_E \rightarrow \llbracket B \rrbracket_E)$. The claim follows by the T-LETMOD and T-APP
2367 rules.

2368 Case

$$2369 \quad \text{R-ABS}$$

$$2370 \quad \frac{\llbracket \Gamma, \blacklozenge_E x : A \rrbracket_F \vdash M' : \llbracket B \rrbracket_F @ F \quad v := \langle E - F | F - E \rangle \quad \mu := \text{topmod}(\llbracket A \rrbracket_F)}{\llbracket \Gamma \rrbracket_E \vdash \mathbf{mod}_v (\lambda x^{\llbracket A \rrbracket_F} . \mathbf{let mod}_\mu x = x \mathbf{ in } M') : \llbracket A \rightarrow^F B \rrbracket_E @ E}$$

2374 We have $\llbracket \Gamma, \blacklozenge_E x : A \rrbracket_F = \llbracket \Gamma \rrbracket_E, \blacklozenge_{\langle E - F | F - E \rangle} x :_{\mu_F} A'$ where $\mu A' = \llbracket A \rrbracket_F$. Further $\llbracket A \rightarrow^F$
2375 $B \rrbracket_E = \langle E - F | F - E \rangle (\llbracket A \rrbracket_F \rightarrow \llbracket B \rrbracket_F)$. The claim follows from the T-LETMOD, T-ABS and
2376 T-MOD rules.

2377 Case

$$2378 \quad \text{R-EABS}$$

$$2379 \quad \frac{\llbracket \Gamma, \blacklozenge_E \cdot \rrbracket \vdash V' : \llbracket A \rrbracket \cdot @ \cdot}{\llbracket \Gamma \rrbracket_E \vdash \mathbf{mod}_{\square} V' : \llbracket \forall.A \rrbracket_E @ E}$$

2382 We have $\llbracket \Gamma, \blacklozenge_E \cdot \rrbracket = \llbracket \Gamma \rrbracket_E, \blacklozenge_{\square}$. Further, $\llbracket \forall.A \rrbracket_E = \llbracket \llbracket A \rrbracket \rrbracket$. The claim follows from the T-MOD
2383 rule.

2384 Case

$$2385 \quad \text{R-EAPP}$$

$$2386 \quad \frac{\llbracket \Gamma \rrbracket_E \vdash M' : \llbracket \forall.A \rrbracket_E @ E \quad x \text{ fresh}}{\llbracket \Gamma \rrbracket_E \vdash \mathbf{let mod}_{\square} x = M' \mathbf{ in } x : \llbracket A[E/\cdot] \rrbracket_E @ E}$$

2389 We have $\llbracket \forall.A \rrbracket_E = \llbracket \llbracket A \rrbracket \rrbracket$. By Lemma B.20, $\llbracket A \rrbracket \cdot = \llbracket A[E/\cdot] \rrbracket_E$. The claim follows by the
2390 T-LETMOD rule.

2391 Case

$$2392 \quad \text{R-Do}$$

$$2393 \quad \ell : A \rightarrow B \in \Sigma$$

$$2394 \quad \frac{\llbracket \Gamma \rrbracket_{\ell, E} \vdash M' : \llbracket A \rrbracket_{\ell, E} @ \ell, E}{\llbracket \Gamma \rrbracket_{\ell, E} \vdash \mathbf{do } \ell M' : \llbracket B \rrbracket_{\ell, E} @ \ell, E}$$

2398 Because we only allow pure values in the effect signatures of F_{eff}^1 , we have that $\llbracket A \rrbracket_{\ell, E} = \llbracket A \rrbracket$.
2399 and $\llbracket B \rrbracket_{\ell, E} = \llbracket B \rrbracket \cdot$, where $\ell : \llbracket A \rrbracket \cdot \rightarrow \llbracket B \rrbracket \cdot$ in MET. The claim follows directly by the T-DO
2400 rule.

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2402 Case

2403 R-MASK

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\begin{array}{c}
\llbracket \Gamma, \blacklozenge_{L+E} \rrbracket_E \vdash M' : \llbracket A \rrbracket_E @ E \\
\mu_1 := \text{topmod}(\llbracket A \rrbracket_E) \quad \mu_2 := \text{topmod}(\llbracket A \rrbracket_{L+E})
\end{array}
}{
\llbracket \Gamma \rrbracket_{L+E} \vdash \mathbf{let\ mod}_{\langle L \rangle; \mu_1} x = \mathbf{mask}_L M' \mathbf{in\ mod}_{\mu_2} x : \llbracket A \rrbracket_{L+E} @ L + E
}$$

2409 We have $\llbracket \Gamma, \blacklozenge_{L+E} \rrbracket_E = \llbracket \Gamma \rrbracket_{L+E}, \blacklozenge_{\langle (L+E) - E | E - (L+E) \rangle}$. By permuting labels, we have
2410 $\langle (L+E) - E | E - (L+E) \rangle = \langle L \rangle$. The goal follows by the T-LETMOD, T-MASK and T-MOD
2411 rules if we can show that x can be used under the box. This is clear for integers, since they
2412 are pure and otherwise we need to show that $\langle \langle L \rangle \circ \mu_1 \rangle_{L+E} \Rightarrow \langle \mu_2 \rangle_{L+E}$. For $A = \forall. A'$ this is
2413 clear since $\mu_1 = \mu_2 = []$ and $\langle L \rangle \circ [] = []$. For functions, this follows from Lemma B.18.

2414 Case

2415 R-HANDLER

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\end{array}
\frac{
\begin{array}{c}
\llbracket \Gamma, \blacklozenge_{\bar{\ell}_i, E} \rrbracket_{\bar{\ell}_i, E} \vdash M' : \llbracket A \rrbracket_{\bar{\ell}_i, E} @ \bar{\ell}_i, E \\
\llbracket \Gamma, x : A \rrbracket_E \vdash N' : \llbracket B \rrbracket_E @ E \quad \llbracket \llbracket \Gamma, p_i : A_i, r_i : B_i \rightarrow^E B \rrbracket_E \vdash N'_i : \llbracket B \rrbracket_E @ E \rrbracket_i \\
\mu := \text{topmod}(\llbracket A \rrbracket_{\bar{\ell}_i, E}) \quad \mu' := \text{topmod}(\llbracket A \rrbracket_E) \\
N'' := \mathbf{let\ mod}_{\langle \bar{\ell}_i \rangle; \mu} x = x \mathbf{in\ let}_{\mu'} \mathbf{mod}_{\langle \rangle} x = \mathbf{mod}_{\langle \rangle} x \mathbf{in\ } N' \\
[\mu_i := \text{topmod}(\llbracket A_i \rrbracket_{\cdot}) \quad N''_i := \mathbf{let\ mod}_{\mu_i} p_i = p_i \mathbf{in\ } N'_i]_i \\
H = \{\mathbf{return\ } x \mapsto N\} \uplus \{\ell_i\ p_i\ r_i \mapsto N_i\}_i \\
H' = \{\mathbf{return\ } x \mapsto N''\} \uplus \{\ell_i\ p_i\ r_i \mapsto N'_i\}_i
\end{array}
}{
\llbracket \Gamma \rrbracket_E \vdash \mathbf{handle\ } M' \mathbf{with\ } H' : \llbracket B \rrbracket_E @ E
}$$

2427 We have $\llbracket \Gamma, \blacklozenge_{\bar{\ell}_i, E} \rrbracket_{\bar{\ell}_i, E} = \llbracket \Gamma \rrbracket_E, \blacklozenge_{\langle E - \bar{\ell}_i, E | \bar{\ell}_i, E - E \rangle}$. By permuting labels, we have $\langle E - \bar{\ell}_i, E | \bar{\ell}_i, E - E \rangle =$
2428 $\langle \bar{\ell}_i \rangle$. In the operation clauses, we have that $\llbracket B_i \rightarrow^E B \rrbracket_E = \langle \rangle (\llbracket B_i \rrbracket_E \rightarrow \llbracket B \rrbracket_E)$. Because the
2429 argument and return of effects are pure, we have that $\llbracket B_i \rrbracket_E = \llbracket B_i \rrbracket_{\cdot}$ and $\llbracket A_i \rrbracket_E = \llbracket A_i \rrbracket_{\cdot}$. We
2430 need to unbox the argument p_i though. In the return clause, MET gives us $x : \langle \bar{\ell}_i \rangle \llbracket A \rrbracket_{\bar{\ell}_i, E}$,
2431 but we need $x : \llbracket A \rrbracket_E$. We achieve this by unboxing x fully and then re-boxing it with
2432 the modality μ' . This is possible for integers because they are pure, for \forall s because of the
2433 MT-ABS rule and for functions due to the modality transformation in Lemma B.19.

2434 □

2437 C Specification and Implementation of METL

2438 In this section, we provide the syntax and typing rules for METL and the elaboration from it to MET.
2439 We also briefly discuss our prototype implementation.

2441 C.1 Syntax and Typing Rules

2442 The syntax of METL is shown in Figure 12. We include extensions of data types and polymorphism
2443 in Section 5. For the latter we require explicit type application $M A$.

2444 The bidirectional typing rules for METL are shown in Figure 13.

2445 We repeat the definition of across used in B-VAR here for easy reference.

$$\begin{array}{c}
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\end{array}
\text{across}(\Gamma, A, \nu, F) = \begin{cases} A, & \text{if } \Gamma \vdash A : \text{Abs} \\ \zeta G, & \text{otherwise, where } A = \bar{\mu}G \text{ and } \nu_F \bar{\mu}_F = \zeta_E \end{cases}$$

2451	Types	$A, B ::= A \rightarrow B \mid \mu A \mid \alpha \mid (A, B) \mid A + B$
2452	Guarded Types	$G ::= A \rightarrow B \mid \alpha \mid (A, B) \mid A + B$
2453	Masks	$L ::= \cdot \mid \ell, L$
2454	Extensions	$D ::= \cdot \mid \ell : P, D$
2455	Effect Contexts	$E, F ::= \cdot \mid \ell : P, E$
2456	Modalities	$\mu ::= [E] \mid \langle L \mid D \rangle$
2457	Kinds	$K ::= \text{Abs} \mid \text{Any} \mid \text{Effect}$
2458	Contexts	$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \alpha : K \mid \Gamma, \mu_F$
2459	Terms	$M, N ::= x \mid \lambda x. M \mid M N \mid M : A \mid M A \mid \mathbf{do} \ell M$ $\mid \mathbf{mask}_L M \mid \mathbf{handle} M \mathbf{with} H$ $\mid (M, N) \mid \mathbf{case} M \mathbf{of} (x, y) \mapsto N$ $\mid \mathbf{inl} M \mid \mathbf{inr} M \mid \mathbf{case} M \mathbf{of} \{ \mathbf{inl} x \mapsto M_1, \mathbf{inr} y \mapsto M_2 \}$
2463	Values	$V, W ::= x \mid \lambda x. M \mid V : A \mid V A \mid (V, W) \mid \mathbf{inl} V \mid \mathbf{inr} V$
2464	Handlers	$H ::= \{ \mathbf{return} x \mapsto M \} \mid \{ (\ell : A \rightarrow B) p r \mapsto M \} \uplus H$

Fig. 12. Syntax of METL.

For $\mu_F : E \rightarrow F$ and $\nu_F : F' \rightarrow F$, the right residual $\nu_F \setminus \mu_F$ is a partial operation defined as follows.

$$\begin{aligned}
 \nu_F \setminus [E]_F &= [E]_{F'} \\
 \langle L' \mid D' \rangle_F \setminus \langle L \mid D \rangle_F &= \begin{cases} \langle \langle L' \rangle + (L - L') \mid D + F|_{L'-L} \rangle_{D'+(F-L)}, & \text{if present}(F|_{L'-L}) \\ \text{fail}, & \text{otherwise} \end{cases} \\
 [E]_F \setminus \langle L \mid D \rangle_F &= \text{fail}
 \end{aligned}$$

We define $[D]$ as converting an extension to a mask by taking the multiset of all its labels. We define $E|_L$ as the extension derived by extracting the entries in E with labels in L from the left. The predicate $\text{present}(D)$ checks if all labels in D are present.

$$\begin{aligned}
 [\cdot] &= \cdot & (\ell : P, E)|_{\ell, L} &= \ell : P, E|_L \\
 [\ell : P, D] &= \ell, [D] & (\ell' : P, E)|_{\ell, L} &= E|_{\ell, L} \\
 & & \varepsilon|_{\ell, L} &= \text{fail}
 \end{aligned}$$

B-MOD introduces a lock and B-FORALL introduces a type variable into the context, respectively.

B-ANNOTATION is standard for bidirectional typing. B-SWITCH not only switches the direction from checking to inference, but also transforms the top-level modalities when there is a mismatch. B-ABS is standard. Both B-APP and B-TAPP unbox the eliminated term M when it has top-level modalities. B-DO is standard. For masks and handlers, we have typing rules in both checking and inference modes. For B-HANDLERINFERR, we use a partial join operation $A \vee_{\Gamma, F} B$ to join the potentially different types of different branches. The join operation fails when A and B are different types modulo top-level modalities; otherwise, it tries to transform the top-level modalities of A and B to the same one. As a special case, if A and B give some absolute guarded type G after removing top-level modalities, the join operation succeeds and directly returns G , which is a general enough solution because an absolute type has no restriction on its accessibility.

We define join on types as follows.

$$\bar{\mu}G \vee_{\Gamma, F} \bar{\nu}G = \begin{cases} G, & \text{if } \Gamma \vdash G : \text{Abs} \\ (\bar{\mu} \vee_F \bar{\nu})G, & \text{otherwise} \end{cases}$$

In order to define $\mu \vee_F \nu$, we first define some auxiliary join operations.

We define the join of operation signatures $P \vee P'$ as follows, in order to obtain the minimal signature P'' such that $P \leq P''$ and $P' \leq P''$.

$$\begin{aligned} - \vee P &= P \\ P \vee - &= P \\ P \vee P' &= \begin{cases} P, & \text{if } P \equiv P' \\ \text{fail}, & \text{otherwise} \end{cases} \end{aligned}$$

We define the join of effect contexts $E \vee E'$ as follows, in order to obtain the minimal effect context F such that $E \leq F$ and $E' \leq F$.

$$\begin{aligned} \cdot \vee \cdot &= \cdot \\ \ell : P, E \vee \ell : P', E' &= \ell : (P \vee P'), (E \vee E') \end{aligned}$$

We define the extraction $E|_D$ of an extension from an effect context as follows, where all labels in D must be present, creating an extension by extracting the part of E that matches D from E .

$$\begin{aligned} E|. &= \cdot \\ (\ell : P, E)|_{\ell, P', D} &= \begin{cases} \ell : P, E|_D, & \text{if } P \equiv P' \equiv A \rightarrow B \\ \text{fail}, & \text{otherwise} \end{cases} \end{aligned}$$

We define the join of an extension and an effect context $D \vee E$ as follows, creating an extension based on D by joining the signatures with those of corresponding labels in E .

$$\begin{aligned} \cdot \vee E &= \cdot \\ \ell : P, D \vee \ell : P', E' &= \ell : (P \vee P'), (D \vee E') \end{aligned}$$

We define the join of extensions $D \vee D'$ as follows.

$$\begin{aligned} \cdot \vee D &= D \\ \ell : P, D \vee \ell : P', D' &= \ell : (P \vee P'), (D \vee D') \\ \ell : P, D \vee D' &= \ell : P, (D \vee D'), \quad \text{where } \ell \notin \text{dom}(D') \end{aligned}$$

This is similar to the union of multisets except for taking the union of signatures and being order-sensitive.

We write $L \cup L'$ and $L \cap L'$ for the standard union and intersection of multisets.

We define the join of modalities $\mu \vee_F \nu$ at mode F as follows, where if the join operation succeeds, we have $\mu_F \Rightarrow (\mu \vee_F \nu)_F$ and $\nu_F \Rightarrow (\mu \vee_F \nu)_F$, and for any other ζ such that $\mu_F \Rightarrow \zeta_F$ and $\nu_F \Rightarrow \zeta_F$, we have $(\mu \vee_F \nu)_F \Rightarrow \zeta_F$.

$$\begin{aligned} [E] \vee_F [E'] &= [E \vee E'] \\ [E] \vee_F \langle L|D \rangle &= \begin{cases} \langle L|D \vee E \rangle, & \text{if } E \leq D \vee E + (F - L) \\ \text{fail}, & \text{otherwise} \end{cases} \\ \langle L|D \rangle \vee_F \langle L'|D' \rangle &= \begin{cases} \langle L \cup L' | (D \vee D')|_{L_3} + (F - L \cap L')|_{(D \vee D' - L_3)} \rangle, & \\ \text{if } L_1 \equiv L_2 \text{ and } [D_1] \equiv [D_2] & \\ \text{where } (L_1, D_1) = L \bowtie D \text{ and } (L_2, D_2) = L' \bowtie D' & \\ \text{and } L_3 = \text{dom}(D) \cap \text{dom}(D') & \\ \text{fail}, & \text{otherwise} \end{cases} \end{aligned}$$

The third case may look involved. It basically just joins the masks and extensions of the two relative modalities together. The side conditions are used to guarantee that both relative modalities can be transformed to the join result. We define $D - L$ similarly to $E - L$ as follows.

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$$\begin{aligned} \cdot - L &= \cdot \\ (\ell : P, D) - L &= \begin{cases} D - L' & \text{if } L \equiv \ell, L' \\ \ell : P, (D - L) & \text{otherwise} \end{cases} \end{aligned}$$

The introduction rules of data types are standard. For their elimination, we have both checking and inference version. Both B-CRISPPAIR and B-CRISPSUM extract the top-level modalities of the data values and distribute them to the types of the variables for their components. We also use the join operation to unify the different branches of B-CRISPSUMINFERR.

C.2 Elaboration and Implementation

It is easy to elaborate METL terms into MET terms in a type-directed manner. We formally define the elaboration alongside the typing rules in Figures 14 and 15. We have $\Gamma \vdash M \Rightarrow A @ E \dashrightarrow N$ for the inference mode and $\Gamma \vdash M \Leftarrow A @ E \dashrightarrow N$ for the checking mode, both of which elaborates M in METL to N in MET.

We use the following auxiliary functions in the encoding.

$$\begin{aligned} \mathbf{let\ mod}_{\bar{\mu}} x = V \mathbf{in\ } N &= \begin{cases} (\lambda x. N) V & \text{if } \bar{\mu} = \cdot \\ \mathbf{let\ mod}_{\nu} x = V \mathbf{in\ let}_{\nu} \mathbf{mod}_{\bar{\mu}'} x = x \mathbf{in\ } N & \text{if } \bar{\mu} = \nu, \bar{\mu}' \end{cases} \\ \mathbf{unmod}(M; \bar{\mu}) &= (\lambda y. \mathbf{let\ mod}_{\bar{\mu}} y = y \mathbf{in\ } y) M \\ \mathbf{unvar}(x; A; M) &= \mathbf{let\ mod}_{\bar{\mu}} x = x \mathbf{in\ } M \quad \text{where } A = \bar{\mu}G \\ \mathbf{join}_{\Gamma, E}(M : \bar{\mu}G, N : \bar{\nu}G) &= (\mathbf{let\ mod}_{\bar{\mu}} x = M \mathbf{in\ mod}_{\zeta} x, \mathbf{let\ mod}_{\bar{\nu}} x = N \mathbf{in\ mod}_{\zeta} x) \\ &\quad \text{where } \bar{\mu}G \vee_{\Gamma, E} \bar{\nu}G = \zeta G \\ \mathbf{join}_{\Gamma, E}(M : A, N_1 : B_1, \dots) &= \mathbf{join}_{\Gamma, E}(\mathbf{join}_{\Gamma, E}(M : A, N_1 : B_1), \dots) \end{aligned}$$

The core idea of the elaboration is similar to the greedy unboxing strategy of the encoding we have in Section 4.2. For B-ABS (and also handler rules), we immediately fully unbox the bound variables. For B-MOD, we insert explicitly boxing. For B-VAR, we re-box them with the appropriate modality ζ . For B-SWITCH, we insert an unboxing followed by boxing. For B-APP and B-TAPP, we unbox $\bar{\mu}$. For B-HANDLERCHECK and B-HANDLERINFERR, we unbox the bound variables in handler clauses. We do not need to unbox the continuation functions since they have no top-level modality. For B-CRISPPAIRCHECK/INFERR and B-CRISPSUMCHECK/INFERR, we unbox V before case splitting. Also, when the \wedge_E operation is used (such as in B-CRISPSUMINFERR and B-HANDLERINFERR), we use join to unbox the terms correspondingly.

As an example, the handler `asList = fun f → handle f () with ...` in Section 2.3 is elaborated to the following term in MET.

$$\lambda f \langle \mathbf{yield: Int} \rightarrow 1 \rangle^{(1 \rightarrow 1)}. \mathbf{let\ mod}_{\langle \mathbf{yield: Int} \rightarrow 1 \rangle} f = f \mathbf{in\ handle\ } f \text{ () with } \dots$$

We implement a prototype of METL with all features mentioned above as well as algebraic data types and pattern matching. We do not encounter any challenges in generalising the pairs and sums to algebraic data types. In our implementation, we do not strictly follow the conventional bidirectional typing approach, which distinguishes between the checking and inference mode as in the above rules. Instead, we use the form $\Gamma \vdash M \Leftarrow S \Rightarrow A @ E$ where each rule has an input shape and output type, similar to contextual typing [56]. When S is empty, we are in inference mode; when S is a complete type, we are in checking mode; otherwise, we can still use S to pass in partial type information which could allow us to type check more programs.

2598	$\Gamma \vdash M \Rightarrow A @ E$	$\Gamma \vdash M \Leftarrow A @ E$		
2599				
2600	B-VAR			
2601	$v_F = \text{locks}(\Gamma') : E \rightarrow F$	B-MOD	$\mu_F : E \rightarrow F$	B-FORALL
2602	$B = \text{across}(\Gamma, A, v, F)$	$\Gamma, \mu_F \vdash V \Leftarrow A @ E$	$\mu_F : E \rightarrow F$	$\Gamma, \alpha : K \vdash V \Leftarrow A @ E$
2603	$\Gamma, x : A, \Gamma' \vdash x \Rightarrow B @ E$	$\Gamma \vdash V \Leftarrow \mu A @ F$		$\Gamma \vdash V \Leftarrow \forall \alpha : K.A @ E$
2604				
2605	B-ANNOTATION		B-SWITCH	
2606	$\Gamma \vdash V \Leftarrow A @ E$		$\Gamma \vdash M \Rightarrow \bar{\mu}A @ E \quad \Gamma \vdash (\bar{\mu}, A) \Rightarrow \bar{v} @ E$	
2607	$\Gamma \vdash V : A \Rightarrow A @ E$		$\Gamma \vdash M \Leftarrow \bar{v}A @ E$	
2608				
2609		B-APP		B-TAPP
2610	B-ABS	$\Gamma \vdash M \Rightarrow \bar{\mu}(A \rightarrow B) @ E$		$\Gamma \vdash M \Rightarrow \bar{\mu}(\forall \alpha : K.B) @ E$
2611	$\Gamma, x : A \vdash M \Leftarrow B @ E$	$\bar{\mu}_E \Rightarrow \mathbb{1}_E \quad \Gamma \vdash N \Leftarrow A @ E$		$\bar{\mu}_E \Rightarrow \mathbb{1}_E \quad \Gamma \vdash A : K$
2612	$\Gamma \vdash \lambda x.M \Leftarrow A \rightarrow B @ E$	$\Gamma \vdash M N \Rightarrow B @ E$		$\Gamma \vdash M A \Rightarrow B[A/\alpha] @ E$
2613				
2614	B-DO			
2615	$E = \ell : A \rightarrow B, F$	B-MASKCHECK		B-MASKINFER
2616	$\Gamma \vdash M \Leftarrow A @ E$	$\Gamma, \mu_{\langle L \rangle_F} \vdash M \Leftarrow A @ F - L$		$\Gamma, \mu_{\langle L \rangle_F} \vdash M \Rightarrow A @ F - L$
2617	$\Gamma \vdash \text{do } \ell M \Rightarrow B @ E$	$\Gamma \vdash \text{mask}_L M \Leftarrow \langle L \rangle A @ F$		$\Gamma \vdash \text{mask}_L M \Rightarrow \langle L \rangle A @ F$
2618				
2619	B-HANDLERINFER			
2620	$D = \{\ell_i : A_i \rightarrow B_i\}_i$	$\Gamma, \mu_{\langle D \rangle_E} \vdash M \Rightarrow A @ D + E$	$\Gamma, x : \langle D \rangle A \vdash N \Rightarrow B' @ E$	
2621		$[\Gamma, p_i : A_i, r_i : B_i \rightarrow B' \vdash N_i \Rightarrow B_i @ E]_i$	$B = B'(\bigvee_{\Gamma, E} B_i)_i$	
2622	$\Gamma \vdash \text{handle } M \text{ with } \{\text{return } x \mapsto N\} \cup \{(\ell_i : A_i \rightarrow B_i) p_i r_i \mapsto N_i\}_i \Rightarrow B @ E$			
2623				
2624	B-HANDLERCHECK			
2625	$D = \{\ell_i : A_i \rightarrow B_i\}_i$	$\Gamma, \mu_{\langle D \rangle_E} \vdash M \Rightarrow A @ D + E$	$\Gamma, x : \langle D \rangle A \vdash N \Leftarrow B @ E$	
2626		$[\Gamma, p_i : A_i, r_i : B_i \rightarrow B \vdash N_i \Leftarrow B @ E]_i$		
2627	$\Gamma \vdash \text{handle } M \text{ with } \{\text{return } x \mapsto N\} \cup \{(\ell_i : A_i \rightarrow B_i) p_i r_i \mapsto N_i\}_i \Leftarrow B @ E$			
2628				
2629	B-PAIR		B-INL	B-INR
2630	$\Gamma \vdash M \Leftarrow A @ E$	$\Gamma \vdash N \Leftarrow B @ E$	$\Gamma \vdash M \Leftarrow A @ E$	$\Gamma \vdash M \Leftarrow B @ E$
2631	$\Gamma \vdash (M, N) \Leftarrow (A, B) @ E$		$\Gamma \vdash \text{inl } M \Leftarrow A + B @ E$	$\Gamma \vdash \text{inr } M \Leftarrow A + B @ E$
2632				
2633	B-CRISPPAIRINFER		B-CRISPPAIRCHECK	
2634	$\Gamma \vdash V \Rightarrow \bar{\mu}(A, B) @ E$		$\Gamma \vdash V \Rightarrow \bar{\mu}(A, B) @ E$	
2635	$\Gamma, x : \bar{\mu}A, y : \bar{\mu}B \vdash M \Rightarrow A' @ E$		$\Gamma, x : \bar{\mu}A, y : \bar{\mu}B \vdash M \Leftarrow A' @ E$	
2636	$\Gamma \vdash \text{case } V \text{ of } (x, y) \mapsto M \Rightarrow A' @ E$		$\Gamma \vdash \text{case } V \text{ of } (x, y) \mapsto M \Leftarrow A' @ E$	
2637				
2638	B-CRISPΣINFER			
2639	$\Gamma \vdash V \Rightarrow \bar{\mu}(A + B) @ E$			
2640	$\Gamma, x : \bar{\mu}A \vdash M_1 \Rightarrow A_1 @ E$	$\Gamma, y : \bar{\mu}B \vdash M_2 \Rightarrow A_2 @ E$		
2641	$\Gamma \vdash \text{case } V \text{ of } \{\text{inl } x \mapsto M_1, \text{inr } y \mapsto M_2\} \Rightarrow A_1 \vee_{\Gamma, E} A_2 @ E$			
2642				
2643	B-CRISPΣCHECK			
2644	$\Gamma \vdash V \Rightarrow \bar{\mu}(A + B) @ E$			
2645	$\Gamma, x : \bar{\mu}A \vdash M_1 \Leftarrow A' @ E$	$\Gamma, y : \bar{\mu}B \vdash M_2 \Leftarrow A' @ E$		
2646	$\Gamma \vdash \text{case } V \text{ of } \{\text{inl } x \mapsto M_1, \text{inr } y \mapsto M_2\} \Leftarrow A' @ E$			

Fig. 13. Bidirectional typing rules for METL.

<div style="border: 1px solid black; padding: 5px; margin-bottom: 10px;"> $\Gamma \vdash M \Rightarrow A @ E \dashrightarrow N \quad \Gamma \vdash M \Leftarrow A @ E \dashrightarrow N$ </div> <div style="margin-bottom: 10px;"> <p>B-VAR</p> $\frac{v_F = \text{locks}(\Gamma') : E \rightarrow F \quad B = \text{across}(\Gamma, A, v, F) \quad A = \bar{\mu}G \quad B = \zeta G}{\Gamma, x : A, \Gamma' \vdash x \Rightarrow B @ E \dashrightarrow \text{mod}_\zeta x}$ </div> <div> <p>B-FORALL</p> $\frac{\Gamma, \alpha : K \vdash V \Leftarrow A @ E \dashrightarrow V'}{\Gamma \vdash V \Leftarrow \forall \alpha : K. A @ E \dashrightarrow \Lambda \alpha^K. V'}$ </div>	<div style="margin-bottom: 10px;"> <p>B-Mod</p> $\frac{\Gamma, \blacksquare_{\mu_F} \vdash V \Leftarrow A @ E \dashrightarrow V' \quad \mu_F : E \rightarrow F}{\Gamma \vdash V \Leftarrow \mu A @ F \dashrightarrow \text{mod}_\mu V'}$ </div> <div> <p>B-ANNOTATION</p> $\frac{\Gamma \vdash V \Leftarrow A @ E \dashrightarrow V'}{\Gamma \vdash V : A \Rightarrow A @ E \dashrightarrow V'}$ </div>
<div style="margin-bottom: 10px;"> <p>B-SWITCH</p> $\frac{\Gamma \vdash M \Rightarrow \bar{\mu}A @ E \dashrightarrow V' \quad (A, \bar{\mu}) \Rightarrow \bar{v} @ E \quad W = \text{let mod}_{\bar{\mu}} x = V' \text{ in mod}_{\bar{v}} x}{\Gamma \vdash M \Leftarrow \bar{v}A @ E \dashrightarrow W}$ </div> <div> <p>B-APP</p> $\frac{\Gamma \vdash M \Rightarrow \bar{\mu}(A \rightarrow B) @ E \dashrightarrow M' \quad \bar{\mu}_E \Rightarrow \mathbb{1}_E \quad M'' = \text{unmod}(M'; \bar{\mu}) \quad \Gamma \vdash N \Leftarrow A @ E \dashrightarrow N'}{\Gamma \vdash M N \Rightarrow B @ E \dashrightarrow M'' N'}$ </div>	<div> <p>B-ABS</p> $\frac{\Gamma, x : A \vdash M \Leftarrow B @ E \dashrightarrow M' \quad M'' = \lambda x. \text{unvar}(x; A; M')}{\Gamma \vdash \lambda x. M \Leftarrow A \rightarrow B @ E \dashrightarrow M''}$ </div>
<div> <p>B-TAPP</p> $\frac{\Gamma \vdash M \Rightarrow \bar{\mu}(\forall \alpha : K. B) @ E \dashrightarrow M' \quad \bar{\mu}_E \Rightarrow \mathbb{1}_E \quad M'' = \text{unmod}(M'; \bar{\mu}) \quad \Gamma \vdash A : K}{\Gamma \vdash M A \Rightarrow B[A/\alpha] @ E \dashrightarrow M'' A}$ </div>	<div> <p>B-Do</p> $\frac{E = \ell : A \rightarrow B, F \quad \Gamma \vdash M \Leftarrow A @ E \dashrightarrow M'}{\Gamma \vdash \text{do } \ell M \Rightarrow B @ E \dashrightarrow \text{do } \ell M'}$ </div>
<div> <p>B-MASKCHECK</p> $\frac{\Gamma, \blacksquare_{\langle L \rangle_F} \vdash M \Leftarrow A @ F - L \dashrightarrow M'}{\Gamma \vdash \text{mask}_L M \Leftarrow \langle L \rangle A @ F \dashrightarrow \text{mask}_L M'}$ </div>	<div> <p>B-MASKINFER</p> $\frac{\Gamma, \blacksquare_{\langle L \rangle_F} \vdash M \Rightarrow A @ F - L \dashrightarrow M'}{\Gamma \vdash \text{mask}_L M \Rightarrow \langle L \rangle A @ F \dashrightarrow \text{mask}_L M'}$ </div>
<div> <p>B-HANDLERCHECK</p> $\frac{D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, \blacksquare_{\langle D \rangle_E} \vdash M \Rightarrow A @ D + E \dashrightarrow M' \quad \Gamma, x : \langle D \rangle A \vdash N \Leftarrow B @ E \dashrightarrow N' \quad [\Gamma, p_i : A_i, r_i : B_i \rightarrow B \vdash N_i \Leftarrow B @ E]_i \dashrightarrow N'_i}{\Gamma \vdash \text{handle } M \text{ with } \{\text{return } x \mapsto N\} \uplus \{(\ell_i : A_i \rightarrow B_i) p_i r_i \mapsto N_i\}_i \Leftarrow B @ E \dashrightarrow \text{handle } M' \text{ with } \{\text{return } x \mapsto \text{unvar}(x; \langle D \rangle A; N')\} \uplus \{(\ell_i : A_i \rightarrow B_i) p_i r_i \mapsto \text{unvar}(p_i; A_i; N'_i)\}_i}$ </div>	
<div> <p>B-HANDLERINFER</p> $\frac{D = \{\ell_i : A_i \rightarrow B_i\}_i \quad \Gamma, \blacksquare_{\langle D \rangle_E} \vdash M \Rightarrow A @ D + E \dashrightarrow M' \quad \Gamma, x : \langle D \rangle A \vdash N \Rightarrow B' @ E \dashrightarrow N' \quad [\Gamma, p_i : A_i, r_i : B_i \rightarrow B' \vdash N_i \Rightarrow B_i @ E \dashrightarrow N'_i]_i \quad B = B' (\wedge_E B_i)_i \quad N'', (N'_i)_i = \text{join}_{\Gamma, E}(N' : B', (N'_i : B'_i))}{\Gamma \vdash \text{handle } M \text{ with } \{\text{return } x \mapsto N\} \uplus \{(\ell_i : A_i \rightarrow B_i) p_i r_i \mapsto N_i\}_i \Rightarrow B @ E \dashrightarrow \text{handle } M' \text{ with } \{\text{return } x \mapsto \text{unvar}(x; \langle D \rangle A; N'')\} \uplus \{(\ell_i : A_i \rightarrow B_i) p_i r_i \mapsto \text{unvar}(p_i; A_i; N'_i)\}_i}$ </div>	

Fig. 14. Elaboration from METL to MET (part I).

$\Gamma \vdash M \Rightarrow A @ E \dashrightarrow N$	$\Gamma \vdash M \Leftarrow A @ E \dashrightarrow N$
<p>2696</p> <p>2697</p> <p>2698</p> <p>2699</p> <p>2700</p> <p>2701</p> <p>2702</p> <p>2703</p> <p>2704</p> <p>2705</p> <p>2706</p> <p>2707</p> <p>2708</p> <p>2709</p> <p>2710</p> <p>2711</p> <p>2712</p> <p>2713</p> <p>2714</p> <p>2715</p> <p>2716</p> <p>2717</p> <p>2718</p> <p>2719</p> <p>2720</p> <p>2721</p> <p>2722</p> <p>2723</p> <p>2724</p> <p>2725</p> <p>2726</p> <p>2727</p> <p>2728</p> <p>2729</p> <p>2730</p> <p>2731</p> <p>2732</p> <p>2733</p> <p>2734</p> <p>2735</p> <p>2736</p> <p>2737</p> <p>2738</p> <p>2739</p> <p>2740</p> <p>2741</p> <p>2742</p> <p>2743</p> <p>2744</p>	<p>B-PAIR</p> $\frac{\Gamma \vdash M \Leftarrow A @ E \dashrightarrow M' \quad \Gamma \vdash N \Leftarrow B @ E \dashrightarrow N'}{\Gamma \vdash (M, N) \Leftarrow (A, B) @ E \dashrightarrow (M', N')}$ <p>B-INL</p> $\frac{\Gamma \vdash M \Leftarrow A @ E \dashrightarrow M'}{\Gamma \vdash \mathbf{inl} M \Leftarrow A + B @ E \dashrightarrow \mathbf{inl} M'}$ <p>B-CRISPPAIRINFER</p> $\frac{\Gamma \vdash V \Rightarrow \bar{\mu}(A, B) @ E \dashrightarrow V' \quad \Gamma, x : \bar{\mu}A, y : \bar{\mu}B \vdash M \Rightarrow A' @ E \dashrightarrow M'}{\Gamma \vdash \mathbf{case} V \mathbf{ of } (x, y) \mapsto M \Rightarrow A' @ E \dashrightarrow \mathbf{let mod}_{\bar{\mu}} x = V' \mathbf{ in case}_{\bar{\mu}} x \mathbf{ of } (x, y) \mapsto M'}$ <p>B-INR</p> $\frac{\Gamma \vdash M \Leftarrow B @ E \dashrightarrow M'}{\Gamma \vdash \mathbf{inr} M \Leftarrow A + B @ E \dashrightarrow \mathbf{inr} M'}$ <p>B-CRISPPAIRCHECK</p> $\frac{\Gamma \vdash V \Rightarrow \bar{\mu}(A, B) @ E \dashrightarrow V' \quad \Gamma, x : \bar{\mu}A, y : \bar{\mu}B \vdash M \Leftarrow A' @ E \dashrightarrow M'}{\Gamma \vdash \mathbf{case} V \mathbf{ of } (x, y) \mapsto M \Leftarrow A' @ E \dashrightarrow \mathbf{let mod}_{\bar{\mu}} x = V' \mathbf{ in case}_{\bar{\mu}} x \mathbf{ of } (x, y) \mapsto M'}$ <p>B-CRISPSUMINFER</p> $\frac{\Gamma \vdash V \Rightarrow \bar{\mu}(A + B) @ E \dashrightarrow V' \quad \Gamma, x : \bar{\mu}A \vdash M_1 \Rightarrow A_1 @ E \dashrightarrow M'_1 \quad \Gamma, y : \bar{\mu}B \vdash M_2 \Rightarrow A_2 @ E \dashrightarrow M'_2 \quad M''_1, M''_2 = \mathbf{join}_{\Gamma, E}(M_1 : A_1, M_2 : A_2)}{\Gamma \vdash \mathbf{case} V \mathbf{ of } \{\mathbf{inl} x \mapsto M_1, \mathbf{inr} y \mapsto M_2\} \Rightarrow A_1 \wedge_E A_2 @ E \dashrightarrow \mathbf{let mod}_{\bar{\mu}} x = V' \mathbf{ in case}_{\bar{\mu}} x \mathbf{ of } \{\mathbf{inl} x \mapsto M'_1, \mathbf{inr} y \mapsto M'_2\}}$ <p>B-CRISPSUMCHECK</p> $\frac{\Gamma \vdash V \Rightarrow \bar{\mu}(A + B) @ E \dashrightarrow V' \quad \Gamma, x : \bar{\mu}A \vdash M_1 \Leftarrow A' @ E \dashrightarrow M'_1 \quad \Gamma, y : \bar{\mu}B \vdash M_2 \Leftarrow A' @ E \dashrightarrow M'_2}{\Gamma \vdash \mathbf{case} V \mathbf{ of } \{\mathbf{inl} x \mapsto M_1, \mathbf{inr} y \mapsto M_2\} \Leftarrow A' @ E \dashrightarrow \mathbf{let mod}_{\bar{\mu}} x = V' \mathbf{ in case}_{\bar{\mu}} x \mathbf{ of } \{\mathbf{inl} x \mapsto M'_1, \mathbf{inr} y \mapsto M'_2\}}$

Fig. 15. Elaboration from METL to MET (part II).

Our implementation supports polymorphism with explicit type instantiation. As we have discussed in Section 6, it is natural to extend it with inference for polymorphism, following the literature on bidirectional typing [14, 17, 60]. We plan to explore this extension in the future.