Helmholtz’ and Longuet-Higgins’ Theories of Consonance and Harmony

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Draft, September 17, 2002

1 Introduction

The question of what constitutes musical experience and understanding is a very ancient one, like many important questions about the mind. The answers that have been offered over the years since the question was first posed have depended on the notion of mechanism that has been available as a metaphor for the mind.

For Aristotle, and for the Pythagoreans, the explanation of the musical faculty lay in the mathematics of integer ratios and the physics of simply vibrating strings. Helmholtz was able to draw upon nineteenth century physics, for a more properly mechanistic and complete explanation of the phenomenon of consonance. For him, a mechanism was a physical device such as a real resonator or oscillator. The principal tool that we have available, beyond those that Aristotle and Helmholtz knew of, is the computer.

Of course, it is often the algorithm that the computer executes that is of interest, rather than the computer itself, since for many interesting cases we can state the algorithm independently of any particular machine. However, the idea of an algorithm is not in itself novel. Algorithms (such as Euclid’s algorithm) were known to Helmholtz. It is the computer which transforms the notion of an algorithm from a procedure that needs a person to execute it to the status of a mechanism or explanation.

2 Consonance

Helmholtz (1862) explained the dimension of Consonance in terms of the coincidence and proximity of the overtones and difference tones that arise when simultaneously sounded notes excite real non-linear physical resonators, including the human ear. To the extent that an interval’s most powerful secondary tones exactly coincide, it is consonant or sweet-sounding. To the extent that any of its secondaries are separated in frequency by a small enough difference to “beat” at a rate which Helmoltz puts at around 33 *c/s*, it is dissonant, or harsh. Thus for the diatonic semitone, with a frequency ratio of 16/15, only very high, low-energy overtones coincide, so it is weakly consonant, while the two fundamentals themselves produce beats, in the usual musical ranges, so it is also strongly dissonant. For the perfect fifth, on the other hand, with a frequency ratio of 3/2, all its most powerful secondaries coincide, and only very weak ones are close enough to beat. The fifth is therefore strongly consonant and only weakly dissonant. This theory, which has survived (with an important modification due to Plomp and Levelt 1965) to the present day, successfully explains not only the subjective experience of consonance and dissonance in chords, and the effects of chord inversion, but also the possibility of Equal Temperament. The latter is the trick whereby by slightly mistuning all the semitones of the octave to the same ratio of $\sqrt[12]{2}$, one can make an instrument sound tolerably in tune in all twelve major and minor keys. Equal Temperament distorts the seconds and thirds (and their inverses the sevenths and sixths) more than the fourths and the fifths, and affects the octaves hardly at all. Helmholtz' theory predicts than distortion to the seconds and thirds will be less noticeable that distortion to the latter, so it explains why this works.

However, Helmholtz recognised very clearly that this success in explaining equal temperament raised a further question which his theory of consonance could not answer, namely what it is that makes the character of an augmented triad (C E G♯) or a diminished seventh chord (C E♭ G♭ B♭♭) so different from that of a major or minor triad. Consonance does not explain this effect, since all four chords when played on an equally-tempered instrument are entirely made up of minor and major thirds. He correctly observes that one of the equally-tempered major thirds in the augmented triad is always heard as the harmonically remote diminished fourth, and observes that "this chord is well adapted for showing that the original meaning of the intervals asserts itself even with the imperfect tuning of the piano, and determines the judgement of the ear." (Cf. Helmholtz 1862, as translated by Ellis 1885, p.213 and cf. p.338). But Helmholtz had no real explanation for how this could come about.

It is in no way to Helmholtz' discredit that this was so. He did in fact sketch an answer to the problem, and it is striking that his way of tackling it is essentially algorithmic, despite the fact that it implies a class of mechanism that he simply did not have a way of reifying. However, Helmholtz tried to approach the perceptual effect as one of dissonance, while in reality it concerns an entirely orthogonal relation between notes, namely the one that musicians usually refer to as the "harmonic" relation. This relation, which underlies phenomena like chord progression, key, and modulation, is quite independent of consonance, although both have their origin in the Pythagorean integer ratios.

3 Harmony

The first completely formal identification of the nature of the harmonic relation is in Longuet-Higgins (1962a, 1962b), although there are some earlier incomplete proposals, including work by Euler, Weber, Schoenberg, Hindemith, and in particular Ellis (1874,

E	B	F#	C#	G#	D#	A#	E#	B#
C	G	D	A	E	B	F#	C#	G#
Ab	Eb	Bb	F	C	G	D	A	E
Fb	Cb	Gb	Db	Ab	Eb	Bb	F	C
Dbb	Abb	Ebb	Bbb	Fb	Cb	Gb	Db	Ab

Figure 1: (Part of) The Space of Note-names (adapted from Longuet-Higgins 1962a)

1875) and Riemann (1914, see Hyer, 1995 and Cohn, 1997, 1998). Longuet-Higgins showed that the set of musical intervals relative to some fundamental frequency was the set of ratios definable as the product of powers of the prime factors 2, 3, and 5, and no others – that is as a ratio of the form $2^x.3^y.5^z$, where x, y, and z are positive or negative integers. (The fact that ratios involving factors of seven and higher primes do not contribute to this definition of harmony does not exclude them from the theory of consonance. In real resonators, overtones involving such factors do arise, and contribute to consonance. Helmholtz realised that the absence of such ratios from the chord system of tonal harmony represented a problem for his theory of chord function, and attempted an explanation in terms of consonance – see Ellis (translation) 1885, p.213).[1]

Longuet-Higgins' observation means that the intervals form a three-dimensional discrete space, with those factors as its generators, in which the musical intervals can be viewed as vectors. Since the ratio 2 corresponds to the musical octave, and since for most harmonic purposes, notes an octave apart are functionally equivalent, and have the same note-names, it is convenient to project the three dimensional space along this axis into the 3 x 5 plane, assigning each position its traditional note-name. It is convenient to plot the plane relative to a central C, when it appears as in Figure 1, adapted from Longuet-Higgins (1962a).

The traditional note names are ambiguous with respect to the intervals, and the pattern of names repeats itself in a south-easterly direction, although each position necessarily represents a unique frequency ratio when played in just intonation. (That is

[1]The history of these developments and some related developments in work of Balzano 1982, Shepard 1982 and Lerdahl 1988 is reviewed in greater detail by Steedman 1994.

III^{-}	VII^{-}	#IV^{-}	#I	#V	#II	#VI	#III^{+}	#VII^{+}
I^{-}	V^{-}	II^{-}	VI	III	VII	#IV	#I^{+}	#V^{+}
bVI^{-}	bIII^{-}	bVII^{-}	IV	I	V	II	VI^{+}	III^{+}
bIV^{-}	bI^{-}	bV^{-}	bII	bVI	bIII	bVII	IV^{+}	I^{+}
bbII^{-}	bbVI^{-}	bbIII^{-}	bbVII	bIV	bI	bV	bII^{+}	bVI^{+}

Figure 2: (Part of) The Space of Disambiguated Harmonic Intervals

to say that the note names "wrap" the plane of musically significant frequency ratios onto a cylinder of a kind discussed by Chew 2000, which is here projected back onto the plane. The 12 degrees of the even more ambiguous equally-tempered octave in turn wrap the cylinder into a torus, a fact that has received considerable attention in the "Neo-Riemannian" harmony literature—see Hyer 1995, and Cohn 1997, 1998.) Nevertheless, every vector in the infinite plane from some origin necessarily corresponds to a distinct frequency ratio, and potentially to a distinct musical function. There is a traditional nomenclature which distinguishes among the different functions corresponding for example to the two Ds relative to the central C in figure 1, as between the "major tone" and the "minor tone". However, this nomenclature is confusing and not entirely systematic. Instead we will display the intervals relative to an origin or tonic I using a standard roman numeral notation, as in Figure 2. In this figure the intervals are disambiguated. The prefix $\sharp$ and $\flat$ roughly correspond respectively to the traditional notions of "augmented" intervals, and to "minor" and/or "diminished" intervals, while the superscripts plus and minus roughly correspond to the "imperfect" intervals. (However the intervals here identified as II^{-}, $\flat VII^{-}$, and $\flat V^{-}$ would usually be referred to as the minor tone, dominant seventh, and minor fifth, rather than as imperfect intervals, and the interval shown as $\sharp IV$ should be referred to as the tritone, rather than the augmented fourth). The positions with no prefixes and suffixes are "major" and/or "perfect" intervals.

Crucially for our purpose, if we choose a particular position X in the plane of note-names of Figure 1 as origin, and then superimpose the plane of intervals in roman numeral notation of Figure 2, with the I over the X, then we can calculate note names

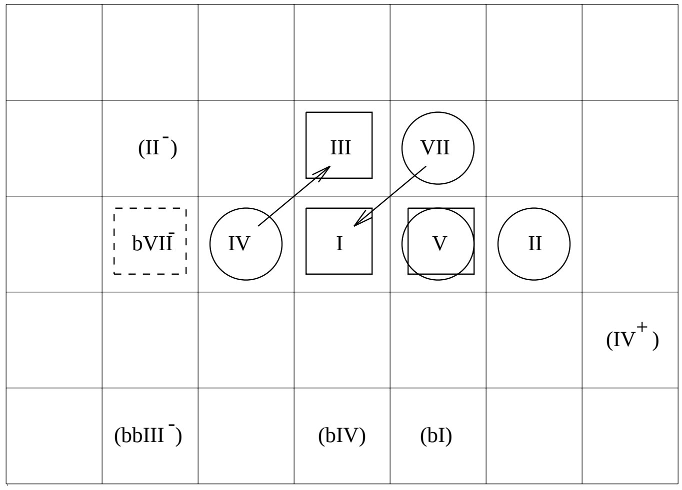

Figure 3: The Interpretation of the Dominant Seventh Chord (circles) and its resolution (squares)

corresponding to intervals like II_X, VII.[2]

Longuet-Higgins' harmonic representation therefore bears a strong resemblance to a "mental model" in the sense of Johnson-Laird 1983. That is to say that it builds directly into the representation some of the properties of the system that it represents. It will be obvious to musicians that the intervals that they refer to as harmonically remote, such as the imperfect and augmented intervals, are spatially distant from the origin in the representation. Similarly, the definition of musically coherent chord sequences such as the twelve-bar blues has something to with orderly progression to a destination by small steps in this space.

I^-7 is musically distinct from the original I, and if perfectly intoned (as opposed to being played on an equally tempered keyboard), would differ from the original in a ratio of 80:81. Nevertheless, we are able to treat it as the tonic.

This theory also explains why the dominant seventh chord creates such a strong expectation of a following chord to its left, whereas the same chord without does not. The major chord on a root V, shown in Figure 3 as made up of a circled V, VII, and II, is extremely unambiguous as to its interpretation, like all such triads. Thus, even if the major triad is played on an equally tempered instrument, obscuring the distinction between the frequency ratios of the pure intervals, having picked *that* V, the representation makes it obvious why the harmonically closest interpretations of the VII and the II are not any of the imperfect or diminished alternatives shown in brackets. However, it is the addition

[2] A simple analogue calculator for this purpose can readily be built by photocopying the roman numeral interval plane of Figure 2 onto transparent film, and then sliding it over the note-name plane, Figure 1.

of the dominant seventh of V, the circled IV, that makes the V chord have a hole in its middle, into which a triad on I (squared I, III, V) fits neatly, sharing one note with the first chord, and with the two remaining notes standing in semitone "leading note" relations with two other notes in the first chord.[3] A chord of I is indeed the expectation produced by a dominant seventh chord $V7$. Moreover the addition of a dominant seventh $\flat VII^-$ to the I major triad (dotted square) makes the I in turn lead onto the IV to its left. The effect of adding dominant seventh chords to *minor* triads is suggested as an exercise at this point. (Why is an alternation of major and minor dominant seventh chords so effective?)

4 Conclusion

I would like to return for a moment to the question of why Helmholtz did not manage to answer his own beautifully simple question concerning the nature of our experience of equal temperament.

Helmholtz actually had access to more of the crucial concepts that were needed for an answer than I have so far revealed. A very close relative of Longuet-Higgins' harmony theory was available during Helmhotz's lifetime. In fact it was presented to this Society, in a paper by Ellis (1874), entitled 'On Musical Duodenes', concerning the nature of modulation. We know that Helmholtz at least had access to this work, for the following curious reason. The translator of Helmholtz' 1862 book was none other than Ellis (1875), who greatly expanded the original by the addition of numerous appendices, mostly concerning a variety of novel keyboard instruments and tables of the precise frequencies of the pipes in the organs of the more significant churches of Europe – a fact of which we know that Helmholtz was aware, since he took exception to these rather extensive additions.

One of these appendices consisted of a fairly complete version of his paper on modulation to the Royal Society of the previous year, including the diagram reproduced in Figure 4 (taken from the second edition of Ellis' translation 1885, p.463, where he gives references to related even earlier work by Weber.).

We shall of course probably never know whether Helmholtz got as far as actually reading Appendix XX of Ellis' translation. But it is striking that neither he, nor Ellis, nor Riemann (1914, p.20, who offered a related triangular array), nor any of their contemporaries, seem to have seen that this diagram, which is in essence a reflection and a rotation of that proposed by Longuet-Higgins, needed only the notion of computation to breathe it into life as an answer to the question that Helmholtz had so clearly recognised.

[3] The addition of the new note also makes the V chord rather ambiguous. The added IV could be the south-easterly IV^+, making this a minor seventh $V(7')$ chord rather than a dominant seventh $V7$.

Appendix: Standard Chord Notation

The sequences (a) to (g) represent the 12-bar chord sequences. Vertical columns represent the 12 successive bars, further grouped into four-bar sections. Where only one chord symbol occurs in a bar it is to be understood to last for all four beats of the bar. Where there are two symbols, they each occupy two beats. The root of each chord is identified by a Roman numeral. This indicates a degree in the major scale of the keynote of the piece, *I* being the tonic and *VII* the seventh. The prefixes ♭ and ♯ identify the root of the chord in question as being one diatonic semitone above or below the degree in question. For example, ♭*III* indicates a chord whose root is the minor third of *I*. All chords are understood to be based on the major chord unless explicit indication is given that they are based on the minor by a small *m* immediately following the Roman numeral, as in ♭*IIIm*. Further numerical suffixes indicate that additional ”passing” notes are to be included with the notes of the basic minor or major chord. The ones in brackets are less harmonically significant, and are optional. Their identity is indicated in a rather obscure (but standard) way. The suffix 7 means that the ”dominant” seventh note, a tone below the tonic, is to be included, as in ♭*III7* and *IIIm7*. The nonstandard suffix (7') also denotes a keyboard tone below the tonic. However, in these chords the additional note functions as the *minor* seventh, rather than the dominant seventh – cf. footnote 3.) The suffix (M7), in contrast, indicates the inclusion of the leading note or major seventh, a semitone below the root, as in $IV(M7)$. The suffix $+5$ indicates the addition of the note an augmented fifth above the tonic (G♯ for the chord of C). It often occurs in combination with the dominant seventh, as in $V7+5$.

The suffix 6 indicates that the major sixth is added. The suffix ϕ7 indicates that the minor third, the diminished fifth (G♭ for the chord of Cϕ7), and the dominant seventh are included. The suffix $\circ$7 indicates that the minor third, the diminished fifth, and the diminished seventh (B♭♭ for the chord of C$\circ$7) are all included – this is the so-called diminished seventh chord.

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F♭	*A*♭	*C*	*E*	*G*♯	*B*♯	*Fx*
B♭♭	*D*♭	*F*	*A*	*C*♯	*E*♯	*Gx*
E♭♭	*G*♭	*B*♭	*D*	*F*♯	*A*♯	*Cx*
A♭♭	*C*♭	*E*♭	*G*	*B*	*D*♯	*Fx*
D♭♭	*F*♭	*A*♭	*C*	*E*	*G*♯	*B*♯
G♭♭	*B*♭♭	*D*♭	*F*	*A*	*C*♯	*E*♯
C♭♭	*E*♭♭	*G*♭	*B*♭	*D*	*F*♯	*A*♯
F♭♭	*A*♭♭	*C*♭	*E*♭	*G*	*B*	*D*♯
B♭♭♭	*D*♭♭	*F*♭	*A*♭	*C*	*E*	*G*♯

Figure 4: The Duodenarium (adapted from Ellis 1874, 1885)