Call-by-name is dual to call-by-value

Philip Wadler
University of Edinburgh
wadler@inf.ed.ac.uk

Part 1

A Deal with the Devil

Either

(a) I will give you \$1,000,000,000

or

(b) I will grant you one wish if you pay me \$1,000,000,000.

Either

(a) I will give you \$1,000,000,000

or

(b) I will grant you one wish if you pay me \$1,000,000,000.

Either

(a) I will give you \$1,000,000,000

or

(b) I will grant you one wish if you pay me \$1,000,000,000.

Part 2

A Question

$$\frac{\text{Lambda Calculus}}{\text{Natural Deduction}} = \frac{?}{\text{Sequent Calculus}}$$

(Intuitonistic) (Classical)

$$\frac{\text{Lambda Calculus}}{\text{Natural Deduction}} = \frac{\text{Dual Calculus}}{\text{Sequent Calculus}}$$

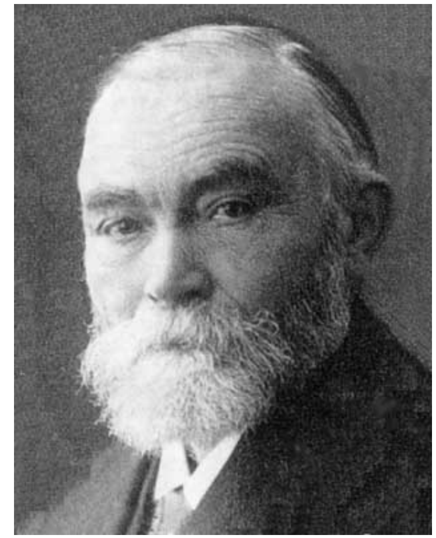
$$\text{(Intuitonistic)} \qquad \text{(Classical)}$$

Part 3

The History

Logic

George Boole (1815–1864)



Gottlob Frege (1848–1925)

Boole (1847): *Laws of Thought*

Frege (1879): *Begriffsschrift*

Duality

$$A \mathbin{\&} \neg A = \bot$$

$$A \vee \neg A = \top$$

$$A \mathbin{\&} (B \vee C) = (A \mathbin{\&} B) \vee (A \mathbin{\&} C)$$

$$A \vee (B \mathbin{\&} C) = (A \vee B) \mathbin{\&} (A \vee C)$$

Poncelet (1818), Gergonne (1826): duals in projective geometry

Boole (1847), Frege (1879): no duality!

Schröder (1890): duals in logic

Natural Deduction and Lambda Calculus

Alonzo Church (1903–1995)

Gerhard Gentzen (1909–1945)

Church (1932): λ-calculus

Gentzen (1935): natural deduction

Church (1940): simply-typed λ-calculus

The Curry-Howard Isomorphism

Haskell Curry (1900–1982)

William Howard

Curry and Feys (1958): combinatory logic

Prawitz (1965): proof reduction

de Bruijn (1968): encoding of proofs

Howard (1980): natural deduction $\simeq$ λ-calculus

Curry-Howard for Classical Logic

Gentzen (1935): sequent calculus

Filinski (1989): symmetric λ-calculus

Griffin (1990): Curry-Howard for classical logic

Parigot (1992): $\lambda\mu$-calculus

Danos, Joinet, and Schellinx (1995): dual encodings in linear logic

Barbanera and Berardi (1996): symmetric λ-calculus

Streicher and Reus (1998): dual cps transforms

Selinger (1998): dual control categories

Curien and Herbelin (2000): Curry-Howard for classical sequent calculus

Wadler (2003): dual calculus

Part 4

Natural Deduction

(Intuitionistic)

Gentzen 1935: Natural Deduction

&–I
$$\frac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}}$$

&–E
$$\frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \qquad \frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$$

∨–I
$$\frac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}}$$

∨–E
$$\frac{\mathfrak{A} \vee \mathfrak{B} \quad \begin{matrix}[\mathfrak{A}]\\ \mathfrak{C}\end{matrix} \quad \begin{matrix}[\mathfrak{B}]\\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

∀–I
$$\frac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

∀–E
$$\frac{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$$

∃–I
$$\frac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

∃–E
$$\frac{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \quad \begin{matrix}[\mathfrak{F}\mathfrak{a}]\\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

⊃–I
$$\frac{\begin{matrix}[\mathfrak{A}]\\ \mathfrak{B}\end{matrix}}{\mathfrak{A} \supset \mathfrak{B}}$$

⊃–E
$$\frac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$$

¬–I
$$\frac{\begin{matrix}[\mathfrak{A}]\\ \curlywedge\end{matrix}}{\neg \mathfrak{A}}$$

¬–E
$$\frac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge} \qquad \frac{\curlywedge}{\mathfrak{D}} .$$

Gentzen 1935: Natural Deduction

$$\frac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \,\&\, B}\,\&\mathrm{I} \qquad\qquad \frac{\begin{matrix}\vdots \\ A \,\&\, B\end{matrix}}{A} \qquad \frac{\begin{matrix}\vdots \\ A \,\&\, B\end{matrix}}{B}\,\&\mathrm{E}$$

$$\frac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B} \qquad \frac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B}\,\vee\mathrm{I} \qquad\qquad \frac{\begin{matrix}\vdots \\ A \vee B\end{matrix} \qquad \begin{matrix}\{A\} \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\{B\} \\ \vdots \\ C\end{matrix}}{C}\,\vee\mathrm{E}$$

$$\frac{\begin{matrix}\{A\} \\ \vdots \\ B\end{matrix}}{A \supset B}\,\supset\mathrm{I} \qquad\qquad \frac{\begin{matrix}\vdots \\ A \supset B\end{matrix} \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B}\,\supset\mathrm{E}$$

Prawitz 1965: Simplifying proofs

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}\qquad\begin{matrix}\vdots\\ B\end{matrix}}{A\,\&\,B}\,\&\mathrm{I}}{A}\,\&\mathrm{E}\quad\longrightarrow\quad\begin{matrix}\vdots\\ A\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{A\vee B}\,\vee\mathrm{I}\qquad\begin{matrix}\{A\}\\ \vdots\\ C\end{matrix}\qquad\begin{matrix}\{B\}\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathrm{E}\quad\longrightarrow\quad\begin{matrix}\vdots\\ A\\ \vdots\\ C\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\{A\}\\ \vdots\\ B\end{matrix}}{A\supset B}\,\supset\mathrm{I}\qquad\begin{matrix}\vdots\\ A\end{matrix}}{B}\,\supset\mathrm{E}\quad\longrightarrow\quad\begin{matrix}\vdots\\ A\\ \vdots\\ B\end{matrix}$$

Part 5

The Lambda Calculus

Church 1932: Lambda Calculus

An occurrence of a variable **x** in a given formula is called an occurrence of **x** as a *bound variable* in the given formula if it is an occurrence of **x** in a part of the formula of the form λ**x**[**M**]; that is, if there is a formula **M** such that λ**x**[**M**] occurs in the given formula and the occurrence of **x** in question is an occurrence in λ**x**[**M**]. All other occurrences of a variable in a formula are called occurrences as a *free variable.*

A formula is said to be *well-formed* if it is a variable, or if it is one

Church 1940: Simply-typed Lambda Calculus

$$\frac{\begin{array}{c}\vdots\\ M : A\end{array} \qquad \begin{array}{c}\vdots\\ N : B\end{array}}{(M, N) : A \,\&\, B}\,\&\mathrm{I} \qquad \frac{\begin{array}{c}\vdots\\ O : A \,\&\, B\end{array}}{\mathrm{fst}\,O : A} \qquad \frac{\begin{array}{c}\vdots\\ O : A \,\&\, B\end{array}}{\mathrm{snd}\,O : B}\,\&\mathrm{E}$$

$$\frac{\begin{array}{c}\vdots\\ M : A\end{array}}{\mathrm{inl}\,M : A \vee B} \qquad \frac{\begin{array}{c}\vdots\\ N : B\end{array}}{\mathrm{inr}\,N : A \vee B}\,\vee\mathrm{I} \qquad \frac{\begin{array}{c}\vdots\\ O : A \vee B\end{array} \qquad \begin{array}{c}\{x : A\}\\ \vdots\\ P : C\end{array} \qquad \begin{array}{c}\{y : B\}\\ \vdots\\ Q : C\end{array}}{\mathrm{case}\ O\ \mathrm{of}\ \mathrm{inl}\,x \Rightarrow P,\ \mathrm{inr}\,y \Rightarrow Q : C}\,\vee\mathrm{E}$$

$$\frac{\begin{array}{c}\{x : A\}\\ \vdots\\ N : B\end{array}}{\lambda x.\, N : A \supset B}\,\supset\!\mathrm{I} \qquad \frac{\begin{array}{c}\vdots\\ O : A \supset B\end{array} \qquad \begin{array}{c}\vdots\\ M : A\end{array}}{O\,M : B}\,\supset\!\mathrm{E}$$

Church 1932, 1940: Reducing terms

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ M : A\end{matrix} \qquad \begin{matrix}\vdots\\ N : B\end{matrix}}{(M, N) : A \mathbin{\&} B}\,\&\mathrm{I}}{\mathrm{fst}\,(M, N) : A}\,\&\mathrm{E} \quad \longrightarrow \quad \begin{matrix}\vdots\\ M : A\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ M : A\end{matrix}}{\mathrm{inl}\,M : A \vee B}\,\vee\mathrm{I} \qquad \begin{matrix}\{x : A\}\\ \vdots\\ P : C\end{matrix} \qquad \begin{matrix}\{y : B\}\\ \vdots\\ Q : C\end{matrix}}{\mathrm{case}\ (\mathrm{inl}\,M)\ \mathrm{of}\ \mathrm{inl}\,x \Rightarrow P,\ \mathrm{inr}\,y \Rightarrow Q : C}\,\vee\mathrm{E} \quad \longrightarrow \quad \begin{matrix}\vdots\\ M : A\\ \vdots\\ P\{M/x\} : C\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\{x : A\}\\ \vdots\\ N : B\end{matrix}}{\lambda x.\, N : A \supset B}\,{\supset}\mathrm{I} \qquad \begin{matrix}\vdots\\ M : A\end{matrix}}{(\lambda x.\, N)\ M : B}\,{\supset}\mathrm{E} \quad \longrightarrow \quad \begin{matrix}\vdots\\ M : A\\ \vdots\\ N\{M/x\} : B\end{matrix}$$

Church 1932: Call-by-name

$$
\begin{array}{lll}
(\beta\&) & \mathrm{fst}\,(M, N) & \longrightarrow_n \quad M \\
(\beta\&) & \mathrm{snd}\,(M, N) & \longrightarrow_n \quad N \\
(\beta\vee) & \mathrm{case}\;(\mathrm{inl}\,M)\;\mathrm{of}\;\mathrm{inl}\,x \Rightarrow P,\;\mathrm{inr}\,y \Rightarrow Q & \longrightarrow_n \quad P\{M/x\} \\
(\beta\vee) & \mathrm{case}\;(\mathrm{inr}\,N)\;\mathrm{of}\;\mathrm{inl}\,x \Rightarrow P,\;\mathrm{inr}\,y \Rightarrow Q & \longrightarrow_n \quad Q\{N/y\} \\
(\beta\supset) & (\lambda x.\,N)\,M & \longrightarrow_n \quad N\{M/x\}
\end{array}
$$

Rosser 1936, Plotkin 1975: Call-by-value

$$\text{Value} \quad V, W \quad ::= \quad x \mid (V, W) \mid \text{inl}\, V \mid \text{inr}\, W \mid \lambda x.\, N$$

$$
\begin{array}{lll}
(\beta\&) & \text{fst}\,(V, W) & \longrightarrow_v \quad V \\
(\beta\&) & \text{snd}\,(V, W) & \longrightarrow_v \quad W \\
(\beta\vee) & \text{case inl}\, V \text{ of inl}\, x \Rightarrow P,\ \text{inr}\, y \Rightarrow Q & \longrightarrow_v \quad P\{V/x\} \\
(\beta\vee) & \text{case inr}\, W \text{ of inl}\, x \Rightarrow P,\ \text{inr}\, y \Rightarrow Q & \longrightarrow_v \quad Q\{W/y\} \\
(\beta\supset) & (\lambda x.\, N)\, V & \longrightarrow_v \quad N\{V/x\}
\end{array}
$$

Part 6

Sequent Calculus

(Classical)

Gentzen 1935: Sequent Calculus

$$\&\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{A} \qquad \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \mathbin{\&} \mathfrak{B}},$$

$$\&\text{–}IA\colon\ \frac{\mathfrak{A}, \Gamma \to \Theta}{\mathfrak{A} \mathbin{\&} \mathfrak{B}, \Gamma \to \Theta} \qquad \frac{\mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \mathbin{\&} \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IA\colon\ \frac{\mathfrak{A}, \Gamma \to \Theta \qquad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{A}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}},$$

$$\forall\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \to \Theta, \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}},$$

$$\exists\text{–}IA\colon\ \frac{\mathfrak{F}\mathfrak{a}, \Gamma \to \Theta}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}, \Gamma \to \Theta}.$$

Gentzen 1935: Logical rules

$$\frac{\Gamma \rightarrow \Theta,\ A \qquad \Gamma \rightarrow \Theta,\ B}{\Gamma \rightarrow \Theta,\ A \,\&\, B}\ \&\text{R}$$

$$\frac{A, \Gamma \rightarrow \Theta}{A \,\&\, B, \Gamma \rightarrow \Theta} \qquad \frac{B, \Gamma \rightarrow \Theta}{A \,\&\, B, \Gamma \rightarrow \Theta}\ \&\text{L}$$

$$\frac{\Gamma \rightarrow \Theta,\ A}{\Gamma \rightarrow \Theta,\ A \vee B} \qquad \frac{\Gamma \rightarrow \Theta,\ B}{\Gamma \rightarrow \Theta,\ A \vee B}\ \vee\text{R}$$

$$\frac{A, \Gamma \rightarrow \Theta \qquad B, \Gamma \rightarrow \Theta}{A \vee B, \Gamma \rightarrow \Theta}\ \vee\text{L}$$

$$\frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta,\ \neg A}\ \neg\text{R} \qquad \frac{\Gamma \rightarrow \Theta,\ A}{\neg A, \Gamma \rightarrow \Theta}\ \neg\text{L}$$

Gentzen 1935: Structural rules

$$\frac{}{A \rightarrow A}\ \text{Id}$$

$$\frac{\Gamma \rightarrow \Theta,\ A \qquad A,\ \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda}\ \text{Cut}$$

Gentzen 1935: Duality

$$
\begin{array}{lcl}
(X)^\circ & \equiv & X \\
(A \,\&\, B)^\circ & \equiv & A^\circ \vee B^\circ \\
(A \vee B)^\circ & \equiv & A^\circ \,\&\, B^\circ \\
(\neg A)^\circ & \equiv & \neg A^\circ
\end{array}
$$

$$(A_1, \ldots, A_n)^\circ \equiv A_n^\circ, \ldots, A_1^\circ$$

Proposition 1 *A sequent is derivable if and only if its dual is derivable,*

$$\Gamma \rightarrow \Theta \quad \textit{iff} \quad \Theta^\circ \rightarrow \Gamma^\circ.$$

Gentzen 1935: Cut Elimination

$$\dfrac{\dfrac{\Gamma \rightarrow \Theta, A \qquad \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \,\&\, B}\,\&\text{R} \qquad \dfrac{A, \Gamma \rightarrow \Theta}{A \,\&\, B, \Gamma \rightarrow \Theta}\,\&\text{L}}{\Gamma \rightarrow \Theta}\,\text{Cut}$$

$$\longrightarrow \qquad \dfrac{\Gamma \rightarrow \Theta, A \qquad A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta}\,\text{Cut}$$

$$\dfrac{\dfrac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A}\,\neg\text{R} \qquad \dfrac{\Gamma \rightarrow \Theta, A}{\neg A, \Gamma \rightarrow \Theta}\,\neg\text{L}}{\Gamma \rightarrow \Theta}\,\text{Cut}$$

$$\longrightarrow \qquad \dfrac{\Gamma \rightarrow \Theta, A \qquad A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta}\,\text{Cut}$$

Part 7

The dual calculus

Intuitionistic natural deduction

Term $\quad x_1 : A_1, \ldots, x_m : A_m \rightarrow M : A$

Classical sequent calculus

Term $\quad x_1 : A_1, \ldots, x_m : A_m \rightarrow \alpha_1 : B_1, \ldots, \alpha_n : B_n \mid M : A$

Coterm $\quad K : A \mid x_1 : A_1, \ldots, x_m : A_m \rightarrow \alpha_1 : B_1, \ldots, \alpha_n : B_n$

Statement $\quad x_1 : A_1, \ldots, x_m : A_m \mid S \mapsto \alpha_1 : B_1, \ldots, \alpha_n : B_n$

Terms, Coterms, Statements

Term	M, N	$::=$	$x \mid \langle M, N\rangle \mid \langle M\rangle\mathrm{inl} \mid \langle N\rangle\mathrm{inr} \mid [K]\mathrm{not} \mid (S).\alpha$
Coterm	K, L	$::=$	$\alpha \mid [K, L] \mid \mathrm{fst}[K] \mid \mathrm{snd}[L] \mid \mathrm{not}\langle M\rangle \mid x.(S)$
Statement	S, T	$::=$	$M \bullet K$

Right sequent	$\Gamma \rightarrow \Theta \mid M : A$
Left sequent	$K : A \mid \Gamma \rightarrow \Theta$
Center sequent	$\Gamma \mid S \mapsto \Theta$

Logical rules

$$\frac{\Gamma \rightarrow \Theta \mid M : A \qquad \Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \langle M, N \rangle : A \,\&\, B}\ \&\text{R}$$

$$\frac{K : A \mid \Gamma \rightarrow \Theta}{\text{fst}[K] : A \,\&\, B \mid \Gamma \rightarrow \Theta} \qquad \frac{L : B \mid \Gamma \rightarrow \Theta}{\text{snd}[L] : A \,\&\, B \mid \Gamma \rightarrow \Theta}\ \&\text{L}$$

$$\frac{\Gamma \rightarrow \Theta \mid M : A}{\Gamma \rightarrow \Theta \mid \langle M \rangle\text{inl} : A \vee B} \qquad \frac{\Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \langle N \rangle\text{inr} : A \vee B}\ \vee\text{R}$$

$$\frac{K : A \mid \Gamma \rightarrow \Theta \qquad L : B \mid \Gamma \rightarrow \Theta}{[K, L] : A \vee B \mid \Gamma \rightarrow \Theta}\ \vee\text{L}$$

$$\frac{K : A \mid \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta \mid [K]\text{not} : \neg A}\ \neg\text{R} \qquad \frac{\Gamma \rightarrow \Theta \mid M : A}{\text{not}\langle M \rangle : \neg A \mid \Gamma \rightarrow \Theta}\ \neg\text{L}$$

Structural rules

$$\frac{}{x : A \rightarrow \;\mid x : A}\;\text{IdR} \qquad \frac{}{\alpha : A \mid\; \rightarrow \alpha : A}\;\text{IdL}$$

$$\frac{\Gamma \mid S \mapsto \Theta, \alpha : A}{\Gamma \rightarrow \Theta \mid (S).\alpha : A}\;\text{RI} \qquad \frac{x : A, \Gamma \mid S \mapsto \Theta}{x.(S) : A \mid \Gamma \rightarrow \Theta}\;\text{LI}$$

$$\frac{\Gamma \rightarrow \Theta \mid M : A \qquad K : A \mid \Delta \rightarrow \Lambda}{\Gamma, \Delta \mid M \bullet K \mapsto \Theta, \Lambda}\;\text{Cut}$$

Duality

$$
\begin{array}{lcl}
(X)^\circ & \equiv & X \\
(A \mathbin{\&} B)^\circ & \equiv & A^\circ \vee B^\circ \\
(A \vee B)^\circ & \equiv & A^\circ \mathbin{\&} B^\circ \\
(\neg A)^\circ & \equiv & \neg A^\circ
\end{array}
$$

$$
\begin{array}{lcl@{\qquad}lcl}
(x)^\circ & \equiv & x^\circ & (\alpha)^\circ & \equiv & \alpha^\circ \\
(\langle M, N \rangle)^\circ & \equiv & [M^\circ, N^\circ] & ([K, L])^\circ & \equiv & \langle K^\circ, L^\circ \rangle \\
(\langle M \rangle \mathrm{inl})^\circ & \equiv & \mathrm{fst}[M^\circ] & (\mathrm{fst}[K])^\circ & \equiv & \langle K^\circ \rangle \mathrm{inl} \\
(\langle N \rangle \mathrm{inr})^\circ & \equiv & \mathrm{snd}[M^\circ] & (\mathrm{snd}[L])^\circ & \equiv & \langle K^\circ \rangle \mathrm{inr} \\
([K]\mathrm{not})^\circ & \equiv & \mathrm{not}\langle K^\circ \rangle & (\mathrm{not}\langle M \rangle)^\circ & \equiv & [M^\circ]\mathrm{not} \\
((S).\alpha)^\circ & \equiv & \alpha^\circ.(S^\circ) & (x.(S))^\circ & \equiv & (S^\circ).x^\circ
\end{array}
$$

$$
(M \bullet K)^\circ \equiv K^\circ \bullet M^\circ
$$

Duality

Proposition 2 *A sequent is derivable if and only if its dual is derivable,*

$$\left.\begin{array}{r} \Gamma \rightarrow \Theta \mid M : A \\ K : A \mid \Gamma \rightarrow \Theta \\ \Gamma \mid S \mapsto \Theta \end{array}\right\} \textit{ iff } \left\{\begin{array}{l} M^\circ : A^\circ \mid \Theta^\circ \rightarrow \Gamma^\circ \\ \Theta^\circ \rightarrow \Gamma^\circ \mid K^\circ : A^\circ \\ \Theta^\circ \mid S^\circ \mapsto \Gamma^\circ. \end{array}\right.$$

Gentzen (1935): Cut Elimination

$$\dfrac{\dfrac{\Gamma \rightarrow \Theta \mid M : A \qquad \Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \langle M, N \rangle : A \,\&\, B}\,\&\text{R} \qquad \dfrac{K : A \mid \Gamma \rightarrow \Theta}{\text{fst}[K] : A \,\&\, B \mid \Gamma \rightarrow \Theta}\,\&\text{L}}{\Gamma \mid \langle M, N \rangle \bullet \text{fst}[K] \mapsto \Theta}\,\text{Cut}$$

$$\longrightarrow \qquad \dfrac{\Gamma \rightarrow \Theta \mid M : A \qquad K : A \mid \Gamma \rightarrow \Theta}{\Gamma \mid M \bullet K \mapsto \Theta}\,\text{Cut}$$

$$\dfrac{\dfrac{K : A \mid \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta \mid [K]\text{not} : \neg A}\,\neg\text{R} \qquad \dfrac{\Gamma \rightarrow \Theta \mid M : A}{\text{not}\langle M \rangle : \neg A \mid \Gamma \rightarrow \Theta}\,\neg\text{L}}{\Gamma \mid [K]\text{not} \bullet \text{not}\langle M \rangle \mapsto \Theta}\,\text{Cut}$$

$$\longrightarrow \qquad \dfrac{\Gamma \rightarrow \Theta \mid M : A \qquad K : A \mid \Gamma \rightarrow \Theta}{\Gamma \mid M \bullet K \mapsto \Theta}\,\text{Cut}$$

Part 8

Call-by-value is Dual to Call-by-name

Critical pair

$$
\begin{array}{llcl}
(\beta\mathrm{L}) & M \bullet x.(S) & \longrightarrow & S\{M/x\} \\
(\beta\mathrm{R}) & (S).\alpha \bullet K & \longrightarrow & S\{K/\alpha\}
\end{array}
$$

Sometimes confluent.

$$
\begin{array}{ccc}
 & (x \bullet \alpha).\alpha \bullet y.(y \bullet \beta) & \\
\swarrow & & \searrow \\
x \bullet y.(y \bullet \beta) & & (x \bullet \alpha).\alpha \bullet \beta \\
\searrow & & \swarrow \\
 & x \bullet \beta &
\end{array}
$$

Sometimes not.

$$
\begin{array}{ccc}
 & (x \bullet \alpha).\beta \bullet y.(z \bullet \gamma) & \\
\swarrow & & \searrow \\
x \bullet \alpha & & z \bullet \gamma
\end{array}
$$

Call-by-value

$$
\begin{array}{llcl}
(\beta\mathrm{L}) & V \bullet x.(S) & \longrightarrow & S\{V/x\} \\
(\beta\mathrm{R}) & (S).\alpha \bullet K & \longrightarrow & S\{K/\alpha\}
\end{array}
$$

Sometimes confluent.

$$(x \bullet \alpha).\alpha \bullet y.(y \bullet \beta)$$

$$\swarrow$$

$$x \bullet y.(y \bullet \beta)$$

$$\searrow$$

$$x \bullet \beta$$

Sometimes not.

$$(x \bullet \alpha).\beta \bullet y.(z \bullet \gamma)$$

$$\swarrow$$

$$x \bullet \alpha$$

Call-by-name

$$
\begin{array}{lccc}
(\beta\mathrm{L}) & M \bullet x.(S) & \longrightarrow & S\{M/x\} \\
(\beta\mathrm{R}) & (S).\alpha \bullet P & \longrightarrow & S\{P/\alpha\}
\end{array}
$$

Sometimes confluent.

$$(x \bullet \alpha).\alpha \bullet y.(y \bullet \beta)$$

$$\searrow$$

$$(x \bullet \alpha).\alpha \bullet \beta$$

$$\swarrow$$

$$x \bullet \beta$$

Sometimes not.

$$(x \bullet \alpha).\beta \bullet y.(z \bullet \gamma)$$

$$\searrow$$

$$z \bullet \gamma$$

Call-by-value reductions

$$\text{Value} \quad V, W \quad ::= \quad x \mid \langle V, W\rangle \mid \langle V\rangle\text{inl} \mid \langle W\rangle\text{inr} \mid [K]\text{not}$$

$$
\begin{array}{llll}
(\beta\&) & \langle V, W\rangle \bullet \text{fst}[K] & \longrightarrow_v & V \bullet K \\
(\beta\&) & \langle V, W\rangle \bullet \text{snd}[L] & \longrightarrow_v & W \bullet L \\
(\beta\vee) & \langle V\rangle\text{inl} \bullet [K, L] & \longrightarrow_v & V \bullet K \\
(\beta\vee) & \langle W\rangle\text{inr} \bullet [K, L] & \longrightarrow_v & W \bullet L \\
(\beta\neg) & [K]\text{not} \bullet \text{not}\langle M\rangle & \longrightarrow_v & M \bullet K \\
(\beta\text{L}) & V \bullet x.(S) & \longrightarrow_v & S\{V/x\} \\
(\beta\text{R}) & (S).\alpha \bullet K & \longrightarrow_v & S\{K/\alpha\}
\end{array}
$$

Call-by-name reductions

$$\text{Covalue} \quad P, Q \quad ::= \quad \alpha \mid [P, Q] \mid \text{fst}[P] \mid \text{snd}[Q] \mid \text{not}\langle M \rangle$$

$$
\begin{array}{llll}
(\beta\lor) & \langle M \rangle \text{inl} \bullet [P, Q] & \longrightarrow_n & M \bullet P \\
(\beta\lor) & \langle N \rangle \text{inr} \bullet [P, Q] & \longrightarrow_n & N \bullet Q \\
(\beta\&) & \langle M, N \rangle \bullet \text{fst}[P] & \longrightarrow_n & M \bullet P \\
(\beta\&) & \langle M, N \rangle \bullet \text{snd}[Q] & \longrightarrow_n & N \bullet Q \\
(\beta\neg) & [K]\text{not} \bullet \text{not}\langle M \rangle & \longrightarrow_n & M \bullet K \\
(\beta\text{R}) & (S).\alpha \bullet P & \longrightarrow_n & S\{P/\alpha\} \\
(\beta\text{L}) & M \bullet x.(S) & \longrightarrow_n & S\{M/x\}
\end{array}
$$

Call-by-value, additional reductions

Term context $E \quad ::= \quad \langle \{\,\}, M\rangle \mid \langle V, \{\,\}\rangle \mid \langle \{\,\}\rangle \mathrm{inl} \mid \langle \{\,\}\rangle \mathrm{inr}$

$$
\begin{array}{lll}
(\eta\mathrm{L}) & K & \longrightarrow_v \quad x.(x \bullet K) \\
(\eta\mathrm{R}) & M & \longrightarrow_v \quad (M \bullet \alpha).\alpha \\
(\varsigma) & E\{M\} & \longrightarrow_v \quad (M \bullet x.(E\{x\} \bullet \beta)).\beta
\end{array}
$$

Call-by-name, additional reductions

Coterm context $F ::= [\{\ \}, K] \mid [P, \{\ \}] \mid \mathrm{fst}[\{\ \}] \mid \mathrm{snd}[\{\ \}]$

$$
\begin{array}{llll}
(\eta R) & M & \longrightarrow_n & (M \bullet \alpha).\alpha \\
(\eta L) & K & \longrightarrow_n & x.(x \bullet K) \\
(\varsigma) & F\{K\} & \longrightarrow_n & y.((y \bullet F\{\alpha\}).\alpha \bullet K)
\end{array}
$$

Call-by-value is dual to call-by-name

Proposition 3 *Call-by-value is dual to call-by-name,*

$$\left.\begin{array}{r} M \longrightarrow_v N \\ K \longrightarrow_v L \\ S \longrightarrow_v T \end{array}\right\} \textit{ iff } \left\{\begin{array}{l} M^\circ \longrightarrow_n N^\circ \\ K^\circ \longrightarrow_n L^\circ \\ S^\circ \longrightarrow_n T^\circ. \end{array}\right.$$

Part 9

Excluded middle

Excluded middle

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{x : A \rightarrow \blacksquare\ x : A}\ \mathrm{IdR}}{x : A \rightarrow \blacksquare\ \langle x \rangle \mathrm{inl} : A \vee \neg A}\ \vee\mathrm{R}}{x : A\ \blacksquare\ \langle x \rangle \mathrm{inl} \bullet \gamma \mapsto \gamma : A \vee \neg A}\ \mathrm{RE}}{x.(\langle x \rangle \mathrm{inl} \bullet \gamma) : A\ \blacksquare \rightarrow \gamma : A \vee \neg A}\ \mathrm{LI}}{\rightarrow \gamma : A \vee \neg A\ \blacksquare\ [x.(\langle x \rangle \mathrm{inl} \bullet \gamma)]\mathrm{not} : \neg A}\ \neg\mathrm{R}}{\rightarrow \gamma : A \vee \neg A\ \blacksquare\ \langle [x.(\langle x \rangle \mathrm{inl} \bullet \gamma)]\mathrm{not} \rangle \mathrm{inr} : A \vee \neg A}\ \vee\mathrm{R}}{\blacksquare\ \langle [x.(\langle x \rangle \mathrm{inl} \bullet \gamma)]\mathrm{not} \rangle \mathrm{inr} \bullet \delta \mapsto \gamma : A \vee \neg A, \delta : A \vee \neg A}\ \mathrm{RE}}{\blacksquare\ \langle [x.(\langle x \rangle \mathrm{inl} \bullet \gamma)]\mathrm{not} \rangle \mathrm{inr} \bullet \gamma \mapsto \gamma : A \vee \neg A}\ \mathrm{Cont}}{\rightarrow \blacksquare\ (\langle [x.(\langle x \rangle \mathrm{inl} \bullet \gamma)]\mathrm{not} \rangle \mathrm{inr} \bullet \gamma).\gamma : A \vee \neg A}\ \mathrm{RI}
$$

Part 10

Functions

Encoding functions in call-by-value

$$\frac{x : A, \Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \lambda x.\, N : A \supset B} \supset\!\mathrm{R} \qquad \frac{\Gamma \rightarrow \Theta \mid M : A \qquad L : B \mid \Gamma \rightarrow \Theta}{M \mathbin{@} L : A \supset B \mid \Gamma \rightarrow \Theta} \supset\!\mathrm{L}$$

$$(\beta\supset) \qquad \lambda x.\, N \bullet V \mathbin{@} L \quad \longrightarrow_v \quad V \bullet x.(N \bullet L)$$

Proposition 4 *Under call-by-value, implication can be defined by*

$$\begin{aligned} A \supset B &\equiv \neg(A \mathbin{\&} \neg B) \\ \lambda x.\, N &\equiv [z.(z \bullet \mathrm{fst}[x.(z \bullet \mathrm{snd}[\mathrm{not}\langle N \rangle])])]\mathrm{not} \\ M \mathbin{@} L &\equiv \mathrm{not}\langle\langle M, [L]\mathrm{not}\rangle\rangle. \end{aligned}$$

Note translation of a function abstraction is a value.

Encoding functions in call-by-name

$$\frac{x : A, \Gamma \to \Theta \mid N : B}{\Gamma \to \Theta \mid \lambda x.\, N : A \supset B} \supset\text{R} \qquad \frac{\Gamma \to \Theta \mid M : A \qquad L : B \mid \Gamma \to \Theta}{M \,@\, L : A \supset B \mid \Gamma \to \Theta} \supset\text{L}$$

$$(\beta\supset) \qquad \lambda x.\, N \bullet M \,@\, Q \quad \longrightarrow_v \quad M \bullet x.(N \bullet Q)$$

Proposition 5 *Under call-by-name, implication can be defined by*

$$\begin{array}{lcl} A \supset B & \equiv & \neg A \vee B \\ \lambda x.\, N & \equiv & (\langle[x.(\langle N\rangle\text{inr} \bullet \gamma)]\text{not}\rangle\text{inl} \bullet \gamma).\gamma \\ M \,@\, L & \equiv & [\text{not}\langle M\rangle, L]. \end{array}$$

The translation of a function application is a covalue.

Part 11

Continuation-passing style

Call-by-value CPS

$$
\begin{aligned}
(X)^V &\equiv X \\
(A \,\&\, B)^V &\equiv (A)^V \times (B)^V \\
(A \vee B)^V &\equiv (A)^V + (B)^V \\
(\neg A)^V &\equiv (A)^V \rightarrow R
\end{aligned}
$$

Call-by-value CPS

$$
\begin{array}{lcl}
(x)^v & \equiv & \boldsymbol{\lambda}\gamma.\, \gamma\, x \\
(\langle M, N\rangle)^v & \equiv & \boldsymbol{\lambda}\gamma.\, (M)^v\, (\boldsymbol{\lambda}x.\, (N)^v\, (\boldsymbol{\lambda}y.\, \gamma\, \langle x, y\rangle)) \\
(\langle M\rangle \mathrm{inl})^v & \equiv & \boldsymbol{\lambda}\gamma.\, (M)^v\, (\boldsymbol{\lambda}x.\, \gamma\, (\mathrm{inl}\, x)) \\
(\langle N\rangle \mathrm{inr})^v & \equiv & \boldsymbol{\lambda}\gamma.\, (N)^v\, (\boldsymbol{\lambda}y.\, \gamma\, (\mathrm{inr}\, y)) \\
([K]\mathrm{not})^v & \equiv & \boldsymbol{\lambda}\gamma.\, \gamma\, (\lambda z.\, (K)^v\, z) \\
((S).\alpha)^v & \equiv & \boldsymbol{\lambda}\alpha.\, (S)^v \\
 & & \\
(\alpha)^v & \equiv & \boldsymbol{\lambda}z.\, \alpha\, z \\
([K, L])^v & \equiv & \boldsymbol{\lambda}z.\, \mathrm{case}\ z\ \mathrm{of}\ \mathrm{inl}\, x \Rightarrow (K)^v\, x,\ \mathrm{inr}\, y \Rightarrow (L)^v\, y \\
(\mathrm{fst}[K])^v & \equiv & \boldsymbol{\lambda}z.\, \mathrm{case}\ z\ \mathrm{of}\ \langle x, -\rangle \Rightarrow (K)^v\, x \\
(\mathrm{snd}[L])^v & \equiv & \boldsymbol{\lambda}z.\, \mathrm{case}\ z\ \mathrm{of}\ \langle -, y\rangle \Rightarrow (L)^v\, y \\
(\mathrm{not}\langle M\rangle)^v & \equiv & \boldsymbol{\lambda}z.\, (\lambda\gamma.\, (M)^v\, \gamma)\, z \\
(x.(S))^v & \equiv & \boldsymbol{\lambda}x.\, (S)^v \\
 & & \\
(M \bullet K)^v & \equiv & (M)^v\, (K)^v
\end{array}
$$

CPS preserves types

Proposition 6 *The call-by-value CPS translation preserves types.*

$$\left.\begin{array}{r} \Gamma \rightarrow \Theta \mid V : A \\ \Gamma \rightarrow \Theta \mid M : A \\ K : A \mid \Gamma \rightarrow \Theta \\ \Gamma \mid S \mapsto \Theta \end{array}\right\} \textit{ iff } \left\{\begin{array}{l} (\Gamma)^V, (\neg\Theta)^V \rightarrow (V)^V : (A)^V \\ (\Gamma)^V, (\neg\Theta)^V \rightarrow (M)^v : (\neg\neg A)^V \\ (\Gamma)^V, (\neg\Theta)^V \rightarrow (K)^v : (\neg A)^V \\ (\Gamma)^V, (\neg\Theta)^V \rightarrow (S)^v : R \end{array}\right.$$

CPS preserves and reflect reductions

Proposition 7 *Let M, K, S be in the dual calculus, and N, L, T be in the CPS target calculus. Then*

$$\left.\begin{array}{r} M \longrightarrow_v (N)_v \\ K \longrightarrow_v (L)_v \\ S \longrightarrow_v (T)_v \end{array}\right\} \text{ iff } \left\{\begin{array}{lcl} (M)^v \longrightarrow N & & ((N)_v)^v \equiv N \\ (K)^v \longrightarrow L & \text{and} & ((L)_v)^v \equiv L \\ (S)^v \longrightarrow T, & & ((T)_v)^v \equiv T. \end{array}\right.$$

Call-by-name CPS

$$
\begin{aligned}
(X)^N &\equiv X \\
(A \mathbin{\&} B)^N &\equiv (A)^N + (B)^N \\
(A \vee B)^N &\equiv (A)^N \times (B)^N \\
(\neg A)^N &\equiv (A)^N \to R
\end{aligned}
$$

Call-by-name CPS

$$
\begin{array}{lcl}
(\alpha)^n & \equiv & \boldsymbol{\lambda} z.\, z\, \alpha \\
([K, L])^n & \equiv & \boldsymbol{\lambda} z.\, (K)^n\, (\boldsymbol{\lambda} \alpha.\, (L)^n\, (\boldsymbol{\lambda} \beta.\, z\, \langle \alpha, \beta \rangle)) \\
(\mathrm{fst}[K])^n & \equiv & \boldsymbol{\lambda} z.\, (K)^n\, (\boldsymbol{\lambda} \alpha.\, z\, (\mathrm{inl}\, \alpha)) \\
(\mathrm{snd}[L])^n & \equiv & \boldsymbol{\lambda} z.\, (L)^n\, (\boldsymbol{\lambda} \beta.\, z\, (\mathrm{inr}\, \beta)) \\
(\mathrm{not}\langle M \rangle)^n & \equiv & \boldsymbol{\lambda} z.\, z\, (\lambda \gamma.\, (M)^n\, \gamma) \\
(x.(S))^n & \equiv & \boldsymbol{\lambda} x.\, (S)^n \\
\\
(x)^n & \equiv & \boldsymbol{\lambda} \gamma.\, x\, \gamma \\
(\langle M, N \rangle)^n & \equiv & \boldsymbol{\lambda} \gamma.\, \mathrm{case}\ \gamma\ \mathrm{of}\ \mathrm{inl}\, \alpha \Rightarrow (M)^n\, \alpha,\ \mathrm{inr}\, \beta \Rightarrow (N)^n\, \beta \\
(\langle M \rangle \mathrm{inl})^n & \equiv & \boldsymbol{\lambda} \gamma.\, \mathrm{case}\ \gamma\ \mathrm{of}\ \langle \alpha, - \rangle \Rightarrow (M)^n\, \alpha \\
(\langle N \rangle \mathrm{inr})^n & \equiv & \boldsymbol{\lambda} \gamma.\, \mathrm{case}\ \gamma\ \mathrm{of}\ \langle -, \beta \rangle \Rightarrow (N)^n\, \beta \\
([K]\mathrm{not})^n & \equiv & \boldsymbol{\lambda} \gamma.\, (\lambda z.\, (K)^n\, z)\, \gamma \\
((S).\alpha)^n & \equiv & \boldsymbol{\lambda} \alpha.\, (S)^n \\
\\
(M \bullet K)^n & \equiv & (K)^n\, (M)^n
\end{array}
$$

CPS preserves types

Proposition 8 *The call-by-name CPS translation preserves types.*

$$\left.\begin{array}{r} P : A \mid \Gamma \to \Theta \\ \Gamma \to \Theta \mid M : A \\ K : A \mid \Gamma \to \Theta \\ \Gamma \mid S \mapsto \Theta \end{array}\right\} \textit{ iff } \left\{\begin{array}{l} (\neg\Gamma)^N, (\Theta)^N \to (P)^N : (A)^N \\ (\neg\Gamma)^N, (\Theta)^N \to (M)^n : (\neg A)^N \\ (\neg\Gamma)^N, (\Theta)^N \to (K)^n : (\neg\neg A)^N \\ (\neg\Gamma)^N, (\Theta)^N \to (S)^n : R \end{array}\right.$$

CPS preserves and reflect reductions

Proposition 9 *Let K, M, S be in the dual calculus, and L, N, T be in the CPS target calculus. Then*

$$\left.\begin{array}{r} M \longrightarrow_n (N)_n \\ K \longrightarrow_n (L)_n \\ S \longrightarrow_n (T)_n \end{array}\right\} \textit{ iff } \left\{\begin{array}{lcl} (M)^n \longrightarrow N & & ((N)_n)^n \equiv N \\ (K)^n \longrightarrow L & \textit{and} & ((L)_n)^n \equiv L \\ (S)^n \longrightarrow T, & & ((T)_n)^n \equiv T. \end{array}\right.$$

Part 12

Conclusions

Contributions of this work

- Replace one connective by three:

$$A \supset B \quad \Longrightarrow \quad \begin{array}{c} A \,\&\, B \\ A \vee B \\ \neg A. \end{array}$$

- Encodings:

$$\begin{array}{rcll} A \supset B & \equiv & (\neg A) \vee B, & \text{call-by-name,} \\ A \supset B & \equiv & \neg(A \,\&\, (\neg B)), & \text{call-by-value.} \end{array}$$

- Role of values and covalues clarified.

- Improved CPS results, following Sabry and Wadler (1997).

Implementation

Implemented by Kate Moore, Geoffrey Washburn, Stephanie Weirich, Steve Zdancewic at U Penn.

Another question

What more will we discover by the centenary of the birth of λ-calculus, natural deduction, and sequent calculus?