Call-by-name is dual to call-by-value

Philip Wadler

Avaya Labs

wadler@avaya.com

Duals

$$A \mathbin{\&} \neg A = \bot$$

$$A \vee \neg A = \top$$

$$A \mathbin{\&} (B \vee C) = (A \mathbin{\&} B) \vee (A \mathbin{\&} C)$$

$$A \vee (B \mathbin{\&} C) = (A \vee B) \mathbin{\&} (A \vee C)$$

Related work

- Filinski (1989)
  Symmetric λ-calculus
- Griffin (1990)
  Computational interpretation of classical logic
- Parigot (1992)
  $\lambda\mu$-calculus
- Barbanera and Berardi (1996)
  Symmetric λ-calculus
- Selinger (1999)
  Control categories and duality
- Curien and Herbelin (2000)
  The Duality of Computation

Gerhard Gentzen (1909-1945)

Gentzen 1934: Natural Deduction

&–I

$$\frac{\mathfrak{A} \qquad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}}$$

&–E

$$\frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \qquad \frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$$

∨–I

$$\frac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}}$$

∨–E

$$\frac{\mathfrak{A} \vee \mathfrak{B} \qquad \begin{matrix}[\mathfrak{A}] \\ \mathfrak{C}\end{matrix} \qquad \begin{matrix}[\mathfrak{B}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

∀–I

$$\frac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

∀–E

$$\frac{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$$

∃–I

$$\frac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

∃–E

$$\frac{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \qquad \begin{matrix}[\mathfrak{F}\mathfrak{a}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

⊃–I

$$\frac{\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\mathfrak{A} \supset \mathfrak{B}}$$

⊃–E

$$\frac{\mathfrak{A} \qquad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$$

¬–I

$$\frac{\begin{matrix}[\mathfrak{A}] \\ \curlywedge\end{matrix}}{\neg \mathfrak{A}}$$

¬–E

$$\frac{\mathfrak{A} \qquad \neg \mathfrak{A}}{\curlywedge} \qquad \frac{\curlywedge}{\mathfrak{D}} \; .$$

Gentzen 1934: Sequent Calculus

$$\&\text{–}IS: \quad \frac{\Gamma \to \Theta, \mathfrak{A} \qquad \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \mathbin{\&} \mathfrak{B}},$$

$$\&\text{–}IA: \quad \frac{\mathfrak{A}, \Gamma \to \Theta}{\mathfrak{A} \mathbin{\&} \mathfrak{B}, \Gamma \to \Theta} \qquad \frac{\mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \mathbin{\&} \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IA: \quad \frac{\mathfrak{A}, \Gamma \to \Theta \qquad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IS: \quad \frac{\Gamma \to \Theta, \mathfrak{A}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}},$$

$$\forall\text{–}IS: \quad \frac{\Gamma \to \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \to \Theta, \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}},$$

$$\exists\text{–}IA: \quad \frac{\mathfrak{F}\mathfrak{a}, \Gamma \to \Theta}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}, \Gamma \to \Theta}.$$

Part 1

The sequent calculus

Logical rules

$$\frac{\Gamma \longrightarrow \Theta, A \qquad \Gamma \longrightarrow \Theta, B}{\Gamma \longrightarrow \Theta, A \,\&\, B}\ \&\text{R}$$

$$\frac{A, \Gamma \longrightarrow \Theta}{A \,\&\, B, \Gamma \longrightarrow \Theta} \qquad \frac{B, \Gamma \longrightarrow \Theta}{A \,\&\, B, \Gamma \longrightarrow \Theta}\ \&\text{L}$$

$$\frac{\Gamma \longrightarrow \Theta, A}{\Gamma \longrightarrow \Theta, A \vee B} \qquad \frac{\Gamma \longrightarrow \Theta, B}{\Gamma \longrightarrow \Theta, A \vee B}\ \vee\text{R}$$

$$\frac{A, \Gamma \longrightarrow \Theta \qquad B, \Gamma \longrightarrow \Theta}{A \vee B, \Gamma \longrightarrow \Theta}\ \vee\text{L}$$

$$\frac{A, \Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \neg A}\ \neg\text{R} \qquad \frac{\Gamma \longrightarrow \Theta, A}{\neg A, \Gamma \longrightarrow \Theta}\ \neg\text{L}$$

Structural rules

$$\frac{}{A, \Gamma \longrightarrow \Theta, A}\ \text{Id}$$

$$\frac{\Gamma \longrightarrow \Theta, A \qquad A, \Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta}\ \text{Cut}$$

Part 2

The dual calculus

Syntax and Judgements

Type	A, B	$::=$	$X \mid A \mathbin{\&} B \mid A \vee B \mid \neg A$
Term	M, N	$::=$	$x \mid \langle M, N\rangle \mid \langle M\rangle\mathrm{inl} \mid \langle N\rangle\mathrm{inr} \mid [K]\mathrm{not} \mid (S).\bar{x}$
Co-term	K, L	$::=$	$\bar{x} \mid [K, L] \mid \mathrm{fst}[K] \mid \mathrm{snd}[L] \mid \mathrm{not}\langle M\rangle \mid x.(S)$
Statement	S, T	$::=$	$M \bullet K$
Environment	Γ	$::=$	$x_1 : A_1, \ldots, x_n : A_n$
Coenvironment	Θ	$::=$	$\bar{x}_1 : A_1, \ldots, \bar{x}_n : A_n$

Term judgment	$\Gamma \triangleright \Theta \mid M : A$
Coterm judgment	$K : A \mid \Gamma \triangleleft \Theta$
Statement judgment	$\Gamma \triangleright S \triangleleft \Theta$

Logical rules

$$\frac{\Gamma \triangleright \Theta \mid M : A \qquad \Gamma \triangleright \Theta \mid N : B}{\Gamma \triangleright \Theta \mid \langle M, N \rangle : A \mathbin{\&} B} \;\&\mathrm{R}$$

$$\frac{K : A \mid \Gamma \triangleleft \Theta}{\mathrm{fst}[K] : A \mathbin{\&} B \mid \Gamma \triangleleft \Theta} \qquad \frac{L : B \mid \Gamma \triangleleft \Theta}{\mathrm{snd}[L] : A \mathbin{\&} B \mid \Gamma \triangleleft \Theta} \;\&\mathrm{L}$$

$$\frac{\Gamma \triangleright \Theta \mid M : A}{\Gamma \triangleright \Theta \mid \langle M \rangle \mathrm{inl} : A \vee B} \qquad \frac{\Gamma \triangleright \Theta \mid N : B}{\Gamma \triangleright \Theta \mid \langle N \rangle \mathrm{inr} : A \vee B} \;\vee\mathrm{R}$$

$$\frac{K : A \mid \Gamma \triangleleft \Theta \qquad L : B \mid \Gamma \triangleleft \Theta}{[K, L] : A \vee B \mid \Gamma \triangleleft \Theta} \;\vee\mathrm{L}$$

$$\frac{K : A \mid \Gamma \triangleleft \Theta}{\Gamma \triangleright \Theta \mid [K]\mathrm{not} : \neg A} \;\neg\mathrm{R} \qquad \frac{\Gamma \triangleright \Theta \mid M : A}{\mathrm{not}\langle M \rangle : \neg A \mid \Gamma \triangleleft \Theta} \;\neg\mathrm{L}$$

Structural rules

$$\frac{}{x : A, \Gamma \triangleright \Theta \mid x : A}\ \mathrm{IdR} \qquad \frac{}{\bar{x} : A \mid \Gamma \triangleleft \Theta, \bar{x} : A}\ \mathrm{IdL}$$

$$\frac{\Gamma \triangleright S \triangleleft \Theta, \bar{x} : A}{\Gamma \triangleright \Theta \mid (S).\bar{x} : A}\ \mathrm{RI} \qquad \frac{x : A, \Gamma \triangleright S \triangleleft \Theta}{x.(S) : A \mid \Gamma \triangleleft \Theta}\ \mathrm{LI}$$

$$\frac{\Gamma \triangleright \Theta \mid M : A \qquad K : A \mid \Gamma \triangleleft \Theta}{\Gamma \triangleright M \bullet K \triangleleft \Theta}\ \mathrm{Cut}$$

Derived rule

$$\dfrac{}{x : A, \Gamma \triangleright x \bullet \bar{x} \triangleleft \Theta, \bar{x} : A}\ \text{Id}$$

$$\equiv$$

$$\dfrac{\dfrac{}{x : A, \Gamma \triangleright \Theta, \bar{x} : A \mid x : A}\ \text{IdR} \qquad \dfrac{}{\bar{x} : A \mid x : A, \Gamma \triangleleft \Theta, \bar{x} : A}\ \text{IdL}}{x : A, \Gamma \triangleright x \bullet \bar{x} \triangleleft \Theta, \bar{x} : A}\ \text{Cut}$$

Derived rules

$$\frac{\Gamma \triangleright \Theta, \bar{x} : A \mid M : A}{\Gamma \triangleright M \bullet \bar{x} \triangleleft \Theta, \bar{x} : A}\ \text{RE}$$

$$\equiv$$

$$\frac{\Gamma \triangleright \Theta, \bar{x} : A \mid M : A \qquad \dfrac{}{\bar{x} : A \mid \Gamma \triangleleft \Theta, \bar{x} : A}\ \text{IdL}}{\Gamma \triangleright M \bullet \bar{x} \triangleleft \Theta, \bar{x} : A}\ \text{Cut}$$

$$\frac{K : A \mid x : A, \Gamma \triangleleft \Theta}{x : A, \Gamma \triangleright x \bullet K \triangleleft \Theta}\ \text{LE}$$

$$\equiv$$

$$\frac{\dfrac{}{x : A, \Gamma \triangleright \Theta \mid x : A}\ \text{IdR} \qquad K : A \mid x : A, \Gamma \triangleleft \Theta}{x : A, \Gamma \triangleright x \bullet K \triangleleft \Theta}\ \text{Cut}$$

Example: Excluded middle

$$\dfrac{}{x : A \triangleright \bar{z} : A \vee \neg A \mid x : A}\ \text{IdR}$$

$$\dfrac{x : A \triangleright \bar{z} : A \vee \neg A \mid x : A}{x : A \triangleright \bar{z} : A \vee \neg A \mid \langle x \rangle \mathrm{inl} : A \vee \neg A}\ \vee\text{R}$$

$$\dfrac{x : A \triangleright \bar{z} : A \vee \neg A \mid \langle x \rangle \mathrm{inl} : A \vee \neg A}{x : A \triangleright \langle x \rangle \mathrm{inl} \bullet \bar{z} \triangleleft \bar{z} : A \vee \neg A}\ \text{RE}$$

$$\dfrac{x : A \triangleright \langle x \rangle \mathrm{inl} \bullet \bar{z} \triangleleft \bar{z} : A \vee \neg A}{x.(\langle x \rangle \mathrm{inl} \bullet \bar{z}) : A \mid \ \triangleleft \bar{z} : A \vee \neg A}\ \text{LI}$$

$$\dfrac{x.(\langle x \rangle \mathrm{inl} \bullet \bar{z}) : A \mid \ \triangleleft \bar{z} : A \vee \neg A}{\triangleright \bar{z} : A \vee \neg A \mid [x.(\langle x \rangle \mathrm{inl} \bullet \bar{z})]\mathrm{not} : \neg A}\ \neg\text{R}$$

$$\dfrac{\triangleright \bar{z} : A \vee \neg A \mid [x.(\langle x \rangle \mathrm{inl} \bullet \bar{z})]\mathrm{not} : \neg A}{\triangleright \bar{z} : A \vee \neg A \mid \langle [x.(\langle x \rangle \mathrm{inl} \bullet \bar{z})]\mathrm{not} \rangle \mathrm{inr} : A \vee \neg A}\ \vee\text{R}$$

$$\dfrac{\triangleright \bar{z} : A \vee \neg A \mid \langle [x.(\langle x \rangle \mathrm{inl} \bullet \bar{z})]\mathrm{not} \rangle \mathrm{inr} : A \vee \neg A}{\triangleright \langle [x.(\langle x \rangle \mathrm{inl} \bullet \bar{z})]\mathrm{not} \rangle \mathrm{inr} \bullet \bar{z} \triangleleft \bar{z} : A \vee \neg A}\ \text{RE}$$

$$\dfrac{\triangleright \langle [x.(\langle x \rangle \mathrm{inl} \bullet \bar{z})]\mathrm{not} \rangle \mathrm{inr} \bullet \bar{z} \triangleleft \bar{z} : A \vee \neg A}{\triangleright \ \mid (\langle [x.(\langle x \rangle \mathrm{inl} \bullet \bar{z})]\mathrm{not} \rangle \mathrm{inr} \bullet \bar{z}).\bar{z} : A \vee \neg A}\ \text{RI}$$

Example: Confluence

$$(w \bullet \bar{x}).\bar{x} \bullet y.(y \bullet \bar{z})$$

$$\swarrow \qquad\qquad \searrow$$

$$w \bullet y.(y \bullet \bar{z}) \qquad\qquad (w \bullet \bar{x}).\bar{x} \bullet \bar{z}$$

$$\searrow \qquad\qquad \swarrow$$

$$w \bullet \bar{z}$$

Example: Non-confluence

$$(u \bullet \bar{v}).\bar{w} \bullet x.(y \bullet \bar{z})$$

$$\swarrow \qquad\qquad \searrow$$

$$u \bullet \bar{v} \qquad\qquad\qquad y \bullet \bar{z}$$

Part 3

Call-by-value

Call-by-value reductions

$$\text{Values} \quad V, W \quad ::= \quad x \mid \langle V, W\rangle \mid \langle V\rangle\text{inl} \mid \langle W\rangle\text{inr} \mid [K]\text{not}$$

$$
\begin{array}{lll}
(\beta\&)_v & \langle V, W\rangle \bullet \text{fst}[K] & \Rightarrow_v \quad V \bullet K \\
(\beta\&)_v & \langle V, W\rangle \bullet \text{snd}[L] & \Rightarrow_v \quad W \bullet L \\
(\beta\vee)_v & \langle V\rangle\text{inl} \bullet [K, L] & \Rightarrow_v \quad V \bullet K \\
(\beta\vee)_v & \langle W\rangle\text{inr} \bullet [K, L] & \Rightarrow_v \quad W \bullet L \\
(\eta\&)_v & \langle (V \bullet \text{fst}[\bar{x}]).\bar{x}, (V \bullet \text{snd}[\bar{y}]).\bar{y}\rangle & \Rightarrow_v \quad V \\
(\eta\vee)_v & [x.(\langle x\rangle\text{inl} \bullet K), y.(\langle y\rangle\text{inr} \bullet K)] & \Rightarrow_v \quad K
\end{array}
$$

Call-by-value reductions, continued

Evaluation context $E \quad ::= \quad \{\,\} \mid \langle E, M\rangle \mid \langle V, E\rangle \mid \langle E\rangle\mathrm{inl} \mid \langle E\rangle\mathrm{inr}$

$$
\begin{array}{lllll}
(\beta\neg)_v & [K]\mathrm{not} \bullet \mathrm{not}\langle M\rangle & \Rightarrow_v & M \bullet K & \\
(\eta\neg)_v & [x.(V \bullet \mathrm{not}\langle x\rangle)]\mathrm{not} & \Rightarrow_v & V & \\
& & & & \\
(\beta\mathrm{L})_v & V \bullet x.(S) & \Rightarrow_v & S\{V/x\} & \\
(\eta\mathrm{L})_v & x.(x \bullet K) & \Rightarrow_v & K & \\
& & & & \\
(\beta\mathrm{R})_v & (S).\bar{x} \bullet K & \Rightarrow_v & S\{K/\bar{x}\} & \\
(\eta\mathrm{R})_v & (M \bullet \bar{x}).\bar{x} & \Rightarrow_v & M & \\
& & & & \\
(\varsigma)_v & E\{M\} & \Rightarrow_v & (M \bullet x.(E\{x\} \bullet \bar{z})).\bar{z}, & \text{if } M \neq V
\end{array}
$$

Part 4

Call-by-name

Call-by-name reductions

$$\text{Covalues} \quad P, Q \quad ::= \quad \bar{x} \mid [P, Q] \mid \text{fst}[P] \mid \text{snd}[Q] \mid \text{not}\langle M \rangle$$

$$
\begin{array}{lll}
(\beta\&)_n & \langle M, N \rangle \bullet \text{fst}[P] & \Rightarrow_n \quad M \bullet P \\
(\beta\&)_n & \langle M, N \rangle \bullet \text{snd}[Q] & \Rightarrow_n \quad N \bullet Q \\
 & & \\
(\beta\vee)_n & \langle M \rangle \text{inl} \bullet [P, Q] & \Rightarrow_n \quad M \bullet P \\
(\beta\vee)_n & \langle N \rangle \text{inr} \bullet [P, Q] & \Rightarrow_n \quad N \bullet Q \\
 & & \\
(\eta\&)_n & \langle (M \bullet \text{fst}[\bar{x}]).\bar{x}, (M \bullet \text{snd}[\bar{y}]).\bar{y} \rangle & \Rightarrow_n \quad M \\
(\eta\vee)_n & [x.(\langle x \rangle \text{inl} \bullet P), y.(\langle y \rangle \text{inr} \bullet P)] & \Rightarrow_n \quad P
\end{array}
$$

Call-by-name reductions, continued

Evaluation cocontext $G \quad ::= \quad \{\,\} \mid [G, K] \mid [P, G] \mid \mathrm{fst}[G] \mid \mathrm{snd}[G]$

$$
\begin{array}{lll}
(\beta\neg)_n & [K]\mathrm{not} \bullet \mathrm{not}\langle M \rangle & \Rightarrow_n \quad M \bullet K \\
(\eta\neg)_n & \mathrm{not}\langle([\bar{x}]\mathrm{not} \bullet P).\bar{x} \rangle & \Rightarrow_n \quad P \\
\\
(\beta \mathrm{L})_n & M \bullet x.(S) & \Rightarrow_n \quad S\{M/x\} \\
(\eta \mathrm{L})_n & x.(x \bullet K) & \Rightarrow_n \quad K \\
\\
(\beta \mathrm{R})_n & (S).\bar{x} \bullet P & \Rightarrow_n \quad S\{P/\bar{x}\} \\
(\eta \mathrm{R})_n & (M \bullet \bar{x}).\bar{x} & \Rightarrow_n \quad M \\
\\
(\varsigma)_n & G\{K\} & \Rightarrow_n \quad z.((z \bullet G\{\bar{x}\}).\bar{x} \bullet K), \quad \text{if } K \neq P
\end{array}
$$

Part 5

Duality

Duality

$$
\begin{aligned}
(X)^\circ &\equiv X \\
(A \mathbin{\&} B)^\circ &\equiv A^\circ \vee B^\circ \\
(A \vee B)^\circ &\equiv A^\circ \mathbin{\&} B^\circ \\
(\neg A)^\circ &\equiv \neg A^\circ
\end{aligned}
$$

$$
\begin{aligned}
(x)^\circ &\equiv \bar{x} \\
(\langle M, N\rangle)^\circ &\equiv [M^\circ, N^\circ] \\
(\langle M\rangle \mathrm{inl})^\circ &\equiv \mathrm{fst}[M^\circ] \\
(\langle N\rangle \mathrm{inr})^\circ &\equiv \mathrm{snd}[M^\circ] \\
([K]\mathrm{not})^\circ &\equiv \mathrm{not}\langle K^\circ\rangle \\
((S).\bar{x})^\circ &\equiv x.(S^\circ)
\end{aligned}
$$

$$
\begin{aligned}
(\bar{x})^\circ &\equiv x \\
([K, L])^\circ &\equiv \langle K^\circ, L^\circ\rangle \\
(\mathrm{fst}[K])^\circ &\equiv \langle K^\circ\rangle \mathrm{inl} \\
(\mathrm{snd}[L])^\circ &\equiv \langle K^\circ\rangle \mathrm{inr} \\
(\mathrm{not}\langle M\rangle)^\circ &\equiv [M^\circ]\mathrm{not} \\
(x.(S))^\circ &\equiv (S^\circ).\bar{x}
\end{aligned}
$$

$$
(M \bullet K)^\circ \equiv K^\circ \bullet M^\circ
$$

Call-by-value is dual to call-by-name

$$\left.\begin{array}{r} \Gamma \rhd \Theta \mid M : A \\ K : A \mid \Gamma \lhd \Theta \\ \Gamma \rhd S \lhd \Theta \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} M^\circ : A^\circ \mid \Theta^\circ \lhd \Gamma^\circ \\ \Theta^\circ \rhd \Gamma^\circ \mid K^\circ : A^\circ \\ \Theta^\circ \rhd S^\circ \lhd \Gamma^\circ \end{array}\right.$$

$$\left.\begin{array}{r} M^{\circ\circ} \\ K^{\circ\circ} \\ S^{\circ\circ} \end{array}\right\} \quad \equiv \quad \left\{\begin{array}{l} M \\ K \\ S \end{array}\right.$$

$$\left.\begin{array}{r} M \Rightarrow_v N \\ K \Rightarrow_v L \\ S \Rightarrow_v T \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} M^\circ \Rightarrow_n N^\circ \\ K^\circ \Rightarrow_n L^\circ \\ S^\circ \Rightarrow_n T^\circ \end{array}\right.$$

Part 6

Call-by-value Continuation-passing style

Continuation-passing style

$$
\begin{aligned}
(X)^V &\equiv X \\
(A \mathbin{\&} B)^V &\equiv A^V \times B^V \\
(A \vee B)^V &\equiv A^V + B^V \\
(\neg A)^V &\equiv A^V \to R
\end{aligned}
$$

$$
\begin{aligned}
(x)^V &\equiv x \\
(\langle V, W \rangle)^V &\equiv \langle V^V, W^V \rangle \\
(\langle V \rangle \mathrm{inl})^V &\equiv \langle V^V \rangle \mathrm{inl} \\
(\langle W \rangle \mathrm{inr})^V &\equiv \langle W^V \rangle \mathrm{inr} \\
([K]\mathrm{not})^V &\equiv K^v
\end{aligned}
$$

$$
\begin{array}{lcl}
(x)^v & \equiv & \hat{\lambda}\bar{z}.\, \bar{z}\, x \\
(\langle M, N\rangle)^v & \equiv & \hat{\lambda}\bar{z}.\, M^v\, (\hat{\lambda}x.\, N^v\, (\hat{\lambda}y.\, \bar{z}\, \langle x, y\rangle)) \\
(\langle M\rangle \mathrm{inl})^v & \equiv & \hat{\lambda}\bar{z}.\, M^v\, (\hat{\lambda}x.\, \bar{z}\, (\langle x\rangle \mathrm{inl})) \\
(\langle N\rangle \mathrm{inr})^v & \equiv & \hat{\lambda}\bar{z}.\, N^v\, (\hat{\lambda}y.\, \bar{z}\, (\langle y\rangle \mathrm{inr})) \\
([K]\mathrm{not})^v & \equiv & \hat{\lambda}\bar{z}.\, \bar{z}\, (\lambda x.\, K^v\, x) \\
((S).\bar{x})^v & \equiv & \hat{\lambda}\bar{x}.\, S^v
\end{array}
$$

$$
\begin{array}{lcl}
(\bar{x})^v & \equiv & \lambda z.\, \bar{x}\, z \\
([K, L])^v & \equiv & \lambda z.\, \mathsf{case}\ z\ \mathsf{of}\ \langle x\rangle \mathrm{inl} \to K^v\, x,\ \langle y\rangle \mathrm{inr} \to L^v\, y \\
(\mathrm{fst}[K])^v & \equiv & \lambda z.\, \mathsf{case}\ z\ \mathsf{of}\ \langle x, -\rangle \to K^v\, x \\
(\mathrm{snd}[L])^v & \equiv & \lambda z.\, \mathsf{case}\ z\ \mathsf{of}\ \langle -, y\rangle \to L^v\, y \\
(\mathrm{not}\langle M\rangle)^v & \equiv & \lambda z.\, M^v\, z \\
(x.(S))^v & \equiv & \lambda x.\, S^v
\end{array}
$$

$$
(M \bullet K)^v \equiv M^v\, K^v
$$

Inverse CPS translation

$$
\begin{array}{lcl}
(X)_V & \equiv & X \\
(A \times B)_V & \equiv & A_V \,\&\, B_V \\
(A + B)_V & \equiv & A_V \vee B_V \\
(A \to R)_V & \equiv & \neg A_V \\
\\
(x)_V & \equiv & x \\
(\langle V, W \rangle)_V & \equiv & \langle V_V, W_V \rangle \\
(\langle V \rangle \mathrm{inl})_V & \equiv & \langle V_V \rangle \mathrm{inl} \\
(\langle W \rangle \mathrm{inr})_V & \equiv & \langle W_V \rangle \mathrm{inr} \\
(K)_V & \equiv & [K_v]\mathrm{not} \\
\\
(\lambda \bar{z}.\, \bar{z}\, V)_v & \equiv & V_V \\
(\lambda \bar{x}.\, S)_v & \equiv & (S_v).\bar{x}
\end{array}
$$

$$
\begin{aligned}
(\lambda z.\, \bar{x}\, z)_v &\equiv \bar{x} \\
(\lambda z.\, \mathrm{case}\ z\ \mathrm{of}\ \langle x\rangle\mathrm{inl} \to K\,x,\ \langle y\rangle\mathrm{inr} \to L\,y)_v &\equiv [K_v, L_v] \\
(\lambda z.\, \mathrm{case}\ z\ \mathrm{of}\ \langle x, -\rangle \to K\,x)_v &\equiv \mathrm{fst}[K_v] \\
(\lambda z.\, \mathrm{case}\ z\ \mathrm{of}\ \langle -, y\rangle \to L\,y)_v &\equiv \mathrm{snd}[L_v] \\
(\lambda z.\, z\,V)_v &\equiv \mathrm{not}\langle V_V\rangle \\
(\lambda z.\, (\lambda \bar{x}.\, S)\, z)_v &\equiv \mathrm{not}\langle (S_v).\bar{x}\rangle \\
(\lambda x.\, S)_v &\equiv x.(S_v)
\end{aligned}
$$

$$
\begin{aligned}
(\bar{x}\,V)_v &\equiv V_V \bullet \bar{x} \\
(\mathrm{case}\ V\ \mathrm{of}\ \langle x\rangle\mathrm{inl} \to K\,x,\ \langle y\rangle\mathrm{inr} \to L\,y)_v &\equiv V_V \bullet [K_v, L_v] \\
(\mathrm{case}\ V\ \mathrm{of}\ \langle x, -\rangle \to K\,x)_v &\equiv V_V \bullet \mathrm{fst}[K_v] \\
(\mathrm{case}\ V\ \mathrm{of}\ \langle -, y\rangle \to L\,y)_v &\equiv V_V \bullet \mathrm{snd}[L_v] \\
(K\,V)_v &\equiv V_V \bullet K_v \\
((\lambda \bar{x}.\, S)\, K)_v &\equiv (S_v).\bar{x} \bullet K_v
\end{aligned}
$$

CPS preserves types and reductions

$$\left.\begin{array}{r} \Gamma \triangleright \Theta \mid M : A \\ K : A \mid \Gamma \triangleleft \Theta \\ \Gamma \triangleright S \triangleleft \Theta \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} (\Gamma)^V, (\neg\Theta)^V \triangleright M^v : (\neg\neg A)^V \\ (\Gamma)^V, (\neg\Theta)^V \triangleright K^v : (\neg A)^V \\ (\Gamma)^V, (\neg\Theta)^V \triangleright S^v : R \end{array}\right.$$

$$\left.\begin{array}{r} M \Rightarrow_v N_v \\ K \Rightarrow_v L_v \\ S \Rightarrow_v T_v \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} M^v \Rightarrow N \\ K^v \Rightarrow L \\ S^v \Rightarrow T \end{array}\right.$$

A reflection on CPS

$$\begin{aligned} ((M')_v)^v &\equiv M' \\ ((K')_v)^v &\equiv K' \\ ((S')_v)^v &\equiv S' \end{aligned}$$

$$M \Rightarrow_v ((M)^v)_v$$

$$K \Rightarrow_v ((K)^v)_v$$

$$S \Rightarrow_v ((S)^v)_v$$

$$\left.\begin{aligned} M &\Rightarrow_v N \\ K &\Rightarrow_v L \\ S &\Rightarrow_v T \end{aligned}\right\} \quad \text{implies} \quad \left\{\begin{aligned} M^v &\Rightarrow N^v \\ K^v &\Rightarrow L^v \\ S^v &\Rightarrow T^v \end{aligned}\right.$$

$$\left.\begin{aligned} M' &\Rightarrow N' \\ K' &\Rightarrow L' \\ S' &\Rightarrow T' \end{aligned}\right\} \quad \text{implies} \quad \left\{\begin{aligned} M'_v &\Rightarrow_v N'_v \\ K'_v &\Rightarrow_v L'_v \\ S'_v &\Rightarrow_v T'_v \end{aligned}\right.$$

Part 7

Call-by-name
Continuation-passing style

Continuation-passing style

$$
\begin{aligned}
(X)^N &\equiv X \\
(A \mathbin{\&} B)^N &\equiv A^N + B^N \\
(A \vee B)^N &\equiv A^N \times B^N \\
(\neg A)^N &\equiv A^N \to R
\end{aligned}
$$

$$
\begin{aligned}
(\bar{x})^N &\equiv \bar{x} \\
([P, Q])^N &\equiv \langle P^N, Q^N \rangle \\
(\mathrm{fst}[P])^N &\equiv \mathrm{fst}[P^N] \\
(\mathrm{snd}[Q])^N &\equiv \mathrm{snd}[Q^N] \\
(\mathrm{not}\langle M \rangle)^N &\equiv M^n
\end{aligned}
$$

$$
\begin{aligned}
(x)^n &\equiv \lambda\bar{z}.\, x\,\bar{z} \\
(\langle M, N\rangle)^n &\equiv \lambda\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ \mathrm{fst}[\bar{x}] \longrightarrow M^n\,\bar{x},\ \mathrm{snd}[\bar{y}] \longrightarrow N^n\,\bar{y} \\
(\langle M\rangle\mathrm{inl})^n &\equiv \lambda\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ \langle\bar{x}, -\rangle \longrightarrow M^n\,\bar{x} \\
(\langle N\rangle\mathrm{inr})^n &\equiv \lambda\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ \langle -, \bar{y}\rangle \longrightarrow N^n\,\bar{y} \\
([K]\mathrm{not})^n &\equiv \lambda\bar{z}.\, K^n\,\bar{z} \\
((S).\bar{x})^n &\equiv \lambda\bar{x}.\, S^n
\end{aligned}
$$

$$
\begin{aligned}
(\bar{x})^n &\equiv \hat{\lambda}z.\, z\,\bar{x} \\
([K, L])^n &\equiv \hat{\lambda}z.\, K^n\,(\hat{\lambda}\bar{x}.\, L^n\,(\hat{\lambda}\bar{y}.\, z\,\langle\bar{x}, \bar{y}\rangle)) \\
(\mathrm{fst}[K])^n &\equiv \hat{\lambda}z.\, K^n\,(\hat{\lambda}\bar{x}.\, z\,(\mathrm{fst}[\bar{x}])) \\
(\mathrm{snd}[L])^n &\equiv \hat{\lambda}z.\, L^n\,(\hat{\lambda}\bar{y}.\, z\,(\mathrm{snd}[\bar{y}])) \\
(\mathrm{not}\langle M\rangle)^n &\equiv \hat{\lambda}z.\, z\,(\lambda\bar{x}.\, M^n\,\bar{x}) \\
(x.(S))^n &\equiv \hat{\lambda}x.\, S^n
\end{aligned}
$$

$$
(M \bullet K)^n \equiv K^n\,M^n
$$

Inverse CPS translation

$$
\begin{array}{lcl}
(X)_N & \equiv & X \\
(A \times B)_N & \equiv & A_N \vee B_N \\
(A + B)_N & \equiv & A_N \mathbin{\&} B_N \\
(A \to R)_N & \equiv & \neg A_N \\
\\
(\bar{x})_N & \equiv & \bar{x} \\
(\langle P, Q \rangle)_N & \equiv & [P_N, Q_N] \\
(\mathrm{fst}[P])_N & \equiv & \mathrm{fst}[P_N] \\
(\mathrm{snd}[Q])_N & \equiv & \mathrm{snd}[Q_N] \\
(M)_N & \equiv & \mathrm{not}\langle M_n \rangle \\
\\
(\lambda z.\, z\, P)_n & \equiv & P_N \\
(\lambda x.\, S)_n & \equiv & x.(S_n)
\end{array}
$$

$$
\begin{aligned}
(\lambda\bar{z}.\, x\ \bar{z})_n &\equiv x \\
(\lambda\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ \mathrm{fst}[\bar{x}] \to M\ \bar{x},\ \mathrm{snd}[\bar{y}] \to N\ \bar{y})_n &\equiv \langle M_n, N_n\rangle \\
(\lambda\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ \langle\bar{x}, -\rangle \to M\ \bar{x})_n &\equiv \langle M_n\rangle\mathrm{inl} \\
(\lambda\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ \langle -, \bar{y}\rangle \to N\ \bar{y})_n &\equiv \langle N_n\rangle\mathrm{inr} \\
(\lambda\bar{z}.\, \bar{z}\ P)_n &\equiv [P_N]\mathrm{not} \\
(\lambda\bar{z}.\, (\lambda x.\, S)\ \bar{z})_n &\equiv [x.(S_n)]\mathrm{not} \\
(\lambda\bar{x}.\, S)_n &\equiv (S_n).\bar{x}
\end{aligned}
$$

$$
\begin{aligned}
(x\ P)_n &\equiv x \bullet P_N \\
(\mathrm{case}\ P\ \mathrm{of}\ \mathrm{fst}[\bar{x}] \to M\ \bar{x},\ \mathrm{snd}[\bar{y}] \to N\ \bar{y})_n &\equiv \langle M_n, N_n\rangle \bullet P_N \\
(\mathrm{case}\ P\ \mathrm{of}\ \langle\bar{x}, -\rangle \to M\ \bar{x})_n &\equiv \langle M_n\rangle\mathrm{inl} \bullet P_N \\
(\mathrm{case}\ P\ \mathrm{of}\ \langle -, \bar{y}\rangle \to N\ \bar{y})_n &\equiv \langle N_n\rangle\mathrm{inr} \bullet P_N \\
(M\ P)_n &\equiv M_n \bullet P_N \\
((\lambda x.\, S)\ M)_n &\equiv M_n \bullet x.(S_n)
\end{aligned}
$$

CPS preserves types and reductions

$$\left.\begin{array}{r} \Gamma \triangleright \Theta \mid M : A \\ K : A \mid \Gamma \triangleleft \Theta \\ \Gamma \triangleright S \triangleleft \Theta \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} (\Theta)^N, (\neg\Gamma)^N \triangleright M^n : (\neg A)^N \\ (\Theta)^N, (\neg\Gamma)^N \triangleright K^n : (\neg\neg A)^N \\ (\Theta)^N, (\neg\Gamma)^N \triangleright S^n : R \end{array}\right.$$

$$\left.\begin{array}{r} M \Rightarrow_n N_n \\ K \Rightarrow_n L_n \\ S \Rightarrow_n T_n \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} M^n \Rightarrow N \\ K^n \Rightarrow L \\ S^n \Rightarrow T \end{array}\right.$$

A reflection on CPS

$$
\begin{aligned}
((M')_n)^n &\equiv M' \\
((K')_n)^n &\equiv K' \\
((S')_n)^n &\equiv S'
\end{aligned}
$$

$$
\begin{aligned}
M &\Rightarrow_n ((M)^n)_n \\
K &\Rightarrow_n ((K)^n)_n \\
S &\Rightarrow_n ((S)^n)_n
\end{aligned}
$$

$$
\left.\begin{aligned}
M &\Rightarrow_n N \\
K &\Rightarrow_n L \\
S &\Rightarrow_n T
\end{aligned}\right\} \quad \text{implies} \quad \left\{\begin{aligned}
M^n &\Rightarrow N^n \\
K^n &\Rightarrow L^n \\
S^n &\Rightarrow T^n
\end{aligned}\right.
$$

$$
\left.\begin{aligned}
M' &\Rightarrow N' \\
K' &\Rightarrow L' \\
S' &\Rightarrow T'
\end{aligned}\right\} \quad \text{implies} \quad \left\{\begin{aligned}
M'_n &\Rightarrow_n N'_n \\
K'_n &\Rightarrow_n L'_n \\
S'_n &\Rightarrow_n T'_n
\end{aligned}\right.
$$

Part 8

Functions

Call-by-value Functions

$$\frac{\Gamma, x : A \rhd \Theta \mid N : B}{\Gamma \rhd \Theta \mid \lambda x.\, N : A \to B} \to \mathrm{R} \qquad \frac{\Gamma \rhd \Theta \mid M : A \qquad L : B \mid \Gamma \lhd \Theta}{M \mathbin{@} L : A \to B \mid \Gamma \lhd \Theta} \to \mathrm{L}$$

$$(\beta \to)_v \qquad (\lambda x.\, N) \bullet (M \mathbin{@} L) \quad \Rightarrow_v \quad M \bullet x.(N \bullet L)$$

$$(\eta \to)_v \qquad \lambda x.\, (V \bullet x \mathbin{@} \bar{y}).\bar{y} \quad \Rightarrow_v \quad V$$

$$A \to B \quad \equiv \quad \neg A \vee B$$

$$\lambda x.\, N \quad \equiv \quad (\langle [x.(\langle N \rangle \mathrm{inr} \bullet \bar{z})]\mathrm{not} \rangle \mathrm{inl} \bullet \bar{z}).\bar{z}$$

$$M \mathbin{@} L \quad \equiv \quad [\mathrm{not}\langle M \rangle, L]$$

Call-by-name Functions

$$\frac{L : B \mid \Gamma \triangleleft \Theta, \bar{x} : A}{\lambda \bar{x}.\, L : A \to B \mid \Gamma \triangleleft \Theta} \to \mathrm{L} \qquad \frac{K : A \mid \Gamma \triangleleft \Theta \qquad \Gamma \triangleright \Theta \mid N : B}{\Gamma \triangleright \Theta \mid K \mathbin{@} N : A \to B} \to \mathrm{R}$$

$$\begin{array}{lccc} (\beta \to)_n & (K \mathbin{@} N) \bullet (\lambda \bar{x}.\, L) & \Rightarrow_n & (N \bullet L).\bar{x} \bullet K \\ (\eta \to)_n & \lambda \bar{x}.\, y.(\bar{x} \mathbin{@} y \bullet P) & \Rightarrow_n & P \end{array}$$

$$\begin{array}{lcl} A \to B & \equiv & \neg A \,\&\, B \\ \lambda \bar{x}.\, L & \equiv & z.(z \bullet \mathrm{fst}[\mathrm{not}\langle (z \bullet \mathrm{snd}[L]).\bar{x} \rangle]) \\ K \mathbin{@} N & \equiv & \langle [K]\mathrm{not}, N \rangle \end{array}$$

Part 9

Conclusions

Related work

- Filinski (1989)
  Symmetric λ-calculus
- Griffin (1990)
  Computational interpretation of classical logic
- Parigot (1992)
  $\lambda\mu$-calculus
- Barbanera and Berardi (1996)
  Symmetric λ-calculus
- Selinger (1999)
  Control categories and duality
- Curien and Herbelin (2000)
  The Duality of Computation