The Curry-Howard Isomorphism

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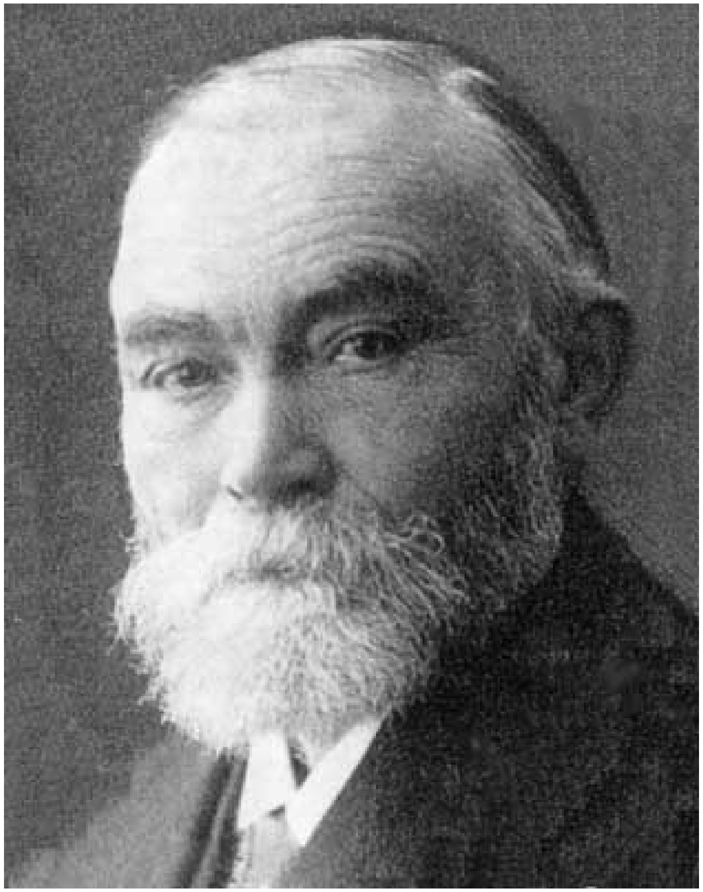
Part 0

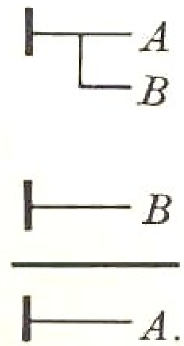
Frege's *Begriffsschrift*

Gotlob Frege (1848–1925)



Frege 1879 — *modus ponens*

We could write this inference perhaps as follows:



This would become awkward if long expressions were to take the places of A and B, since each of them would have to be written twice. That is why I use the following

Frege 1879 — *modus ponens*

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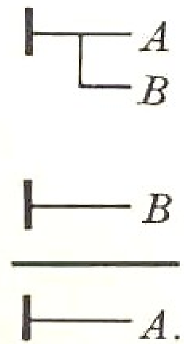
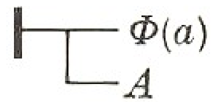


This would become awkward if long expressions were to take the places of A and B, since each of them would have to be written twice. That is why I use the following

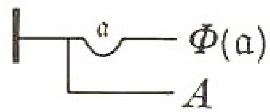
$$\frac{B \supset A \qquad B}{A}$$

Frege 1879 — quantification

It is clear also that from



we can derive



if A is an expression in which a does not occur and if a stands only in the argument places of $\Phi(a)$.[14] If $-\overset{\mathfrak{a}}{\smile}-\Phi(\mathfrak{a})$ is denied, we must be able to specify a meaning for a such that $\Phi(a)$ will be denied. If, therefore, $-\overset{\mathfrak{a}}{\smile}-\Phi(\mathfrak{a})$ were to be denied and

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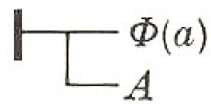

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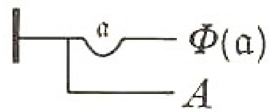
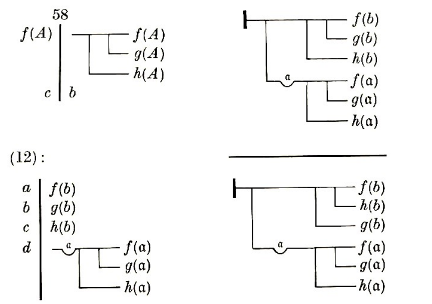


if A is an expression in which a does not occur and if a stands only in the argument places of $\Phi(a)$.[14] If $\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$ is denied, we must be able to specify a meaning for a such that $\Phi(a)$ will be denied. If, therefore, $\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$ were to be denied and

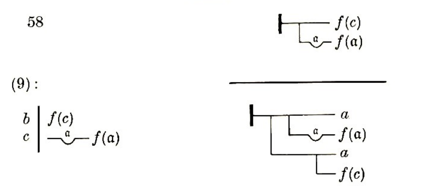
$$\frac{A \supset \Phi(a)}{A \supset \forall a.\, \Phi(a)}$$

Frege 1879

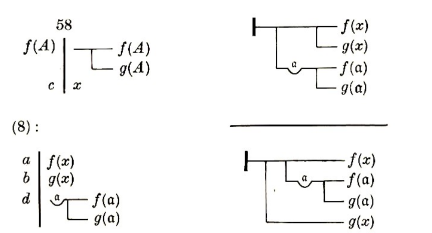
We see how this judgment replaces one mode of inference, namely, Felapton or Fesapo, between which we do not distinguish here since no subject has been singled out.



(60).

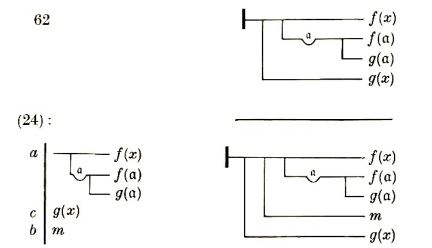


(61).

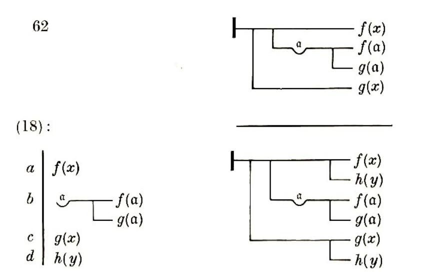


(62).

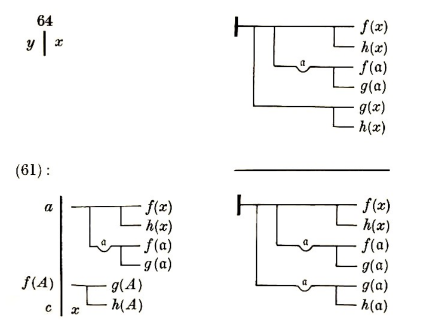
This judgment replaces the mode of inference Barbara when the minor premiss, $g(x)$, has a particular content.



(63).



(64).



(65).

Frege in modern notation

$$\frac{B \supset A \qquad B}{A}$$

$$A \supset (B \supset A)$$

$$(C \supset (B \supset A)) \supset ((C \supset B) \supset (C \supset A))$$

$$(C \supset (B \supset A)) \supset (B \supset (C \supset A))$$

Part 1

Gentzen's Natural Deduction

Gerhard Gentzen (1909–1945)

Gentzen 1934: Natural Deduction

&–*I*

$$\frac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}}$$

&–*E*

$$\frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \qquad \frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$$

∨–*I*

$$\frac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}}$$

∨–*E*

$$\frac{\mathfrak{A} \vee \mathfrak{B} \quad \begin{matrix}[\mathfrak{A}] \\ \mathfrak{C}\end{matrix} \quad \begin{matrix}[\mathfrak{B}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

∀–*I*

$$\frac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

∀–*E*

$$\frac{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$$

∃–*I*

$$\frac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

∃–*E*

$$\frac{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \quad \begin{matrix}[\mathfrak{F}\mathfrak{a}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

⊃–*I*

$$\frac{\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\mathfrak{A} \supset \mathfrak{B}}$$

⊃–*E*

$$\frac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$$

¬–*I*

$$\frac{\begin{matrix}[\mathfrak{A}] \\ \curlywedge\end{matrix}}{\neg \mathfrak{A}}$$

¬–*E*

$$\frac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge} \qquad \frac{\curlywedge}{\mathfrak{D}} .$$

Gentzen 1934: Natural Deduction

$$\dfrac{\begin{matrix}[A]^x \\ \vdots \\ B\end{matrix}}{A \supset B}\ \supset\text{-I}^x \qquad \dfrac{A \supset B \qquad A}{B}\ \supset\text{-E}$$

$$\dfrac{A \qquad B}{A \,\&\, B}\ \&\text{-I} \qquad \dfrac{A \,\&\, B}{A}\ \&\text{-E}_0 \qquad \dfrac{A \,\&\, B}{B}\ \&\text{-E}_1$$

A proof

$$\dfrac{\dfrac{[B \mathbin{\&} A]^z}{A}\ \&\text{-E}_1 \qquad \dfrac{[B \mathbin{\&} A]^z}{B}\ \&\text{-E}_0}{\dfrac{A \mathbin{\&} B}{(B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\ \supset\text{-I}^z}\ \&\text{-I}$$

Simplifying proofs

$$\dfrac{\dfrac{\begin{matrix}[A]^x\\ \vdots\\ B\end{matrix}}{A \supset B}\;\supset\text{-I}^x \qquad \begin{matrix}\vdots\\ A\end{matrix}}{B}\;\supset\text{-E} \qquad \Rightarrow \qquad \begin{matrix}\vdots\\ A\\ \vdots\\ B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad \begin{matrix}\vdots\\ B\end{matrix}}{A \mathbin{\&} B}\;\&\text{-I}}{A}\;\&\text{-E}_0 \qquad \Rightarrow \qquad \begin{matrix}\vdots\\ A\end{matrix}$$

Simplifying a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \,\&\, A]^z}{A}\,\&\text{-E}_1 \qquad \dfrac{[B \,\&\, A]^z}{B}\,\&\text{-E}_0}{A \,\&\, B}\,\&\text{-I}}{(B \,\&\, A) \supset (A \,\&\, B)}\,\supset\text{-I}^z \qquad \dfrac{[B]^y \quad [A]^x}{B \,\&\, A}\,\&\text{-I}}{A \,\&\, B}\,\supset\text{-E}$$

Simplifying a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \mathbin{\&} A]^z}{A}\,\&\text{-E}_1 \qquad \dfrac{[B \mathbin{\&} A]^z}{B}\,\&\text{-E}_0}{A \mathbin{\&} B}\,\&\text{-I}}{(B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\,\supset\text{-I}^z \qquad \dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{A \mathbin{\&} B}\,\supset\text{-E}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{B}\,\&\text{-E}_0}{A \mathbin{\&} B}\,\&\text{-I}$$

Simplifying a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \mathbin{\&} A]^z}{A}\,\&\text{-E}_1 \qquad \dfrac{[B \mathbin{\&} A]^z}{B}\,\&\text{-E}_0}{A \mathbin{\&} B}\,\&\text{-I}}{(B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\,\supset\text{-I}^z \qquad \dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{A \mathbin{\&} B}\,\supset\text{-E}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{B}\,\&\text{-E}_0}{A \mathbin{\&} B}\,\&\text{-I}$$

$$\Downarrow$$

$$\dfrac{[A]^x \quad [B]^y}{A \mathbin{\&} B}\,\&\text{-I}$$

Part 2

Church's Lambda Calculus

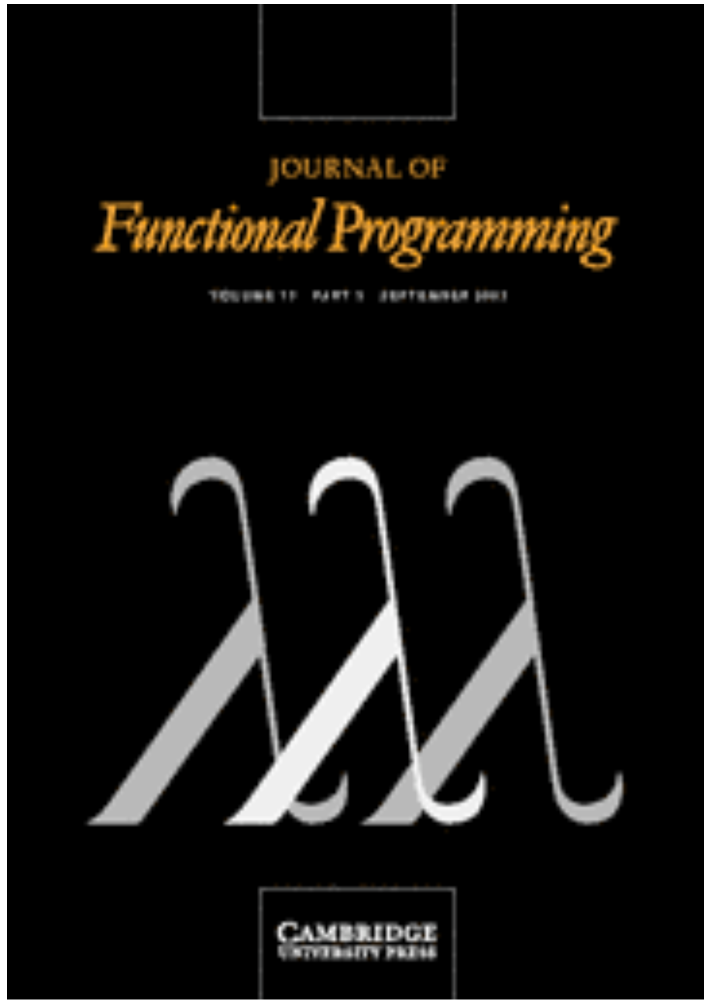
Alonzo Church (1903–1995)

Church 1932: Lambda Calculus

An occurrence of a variable **x** in a given formula is called an occurrence of **x** as a *bound variable* in the given formula if it is an occurrence of **x** in a part of the formula of the form λ**x**[**M**]; that is, if there is a formula **M** such that λ**x**[**M**] occurs in the given formula and the occurrence of **x** in question is an occurrence in λ**x**[**M**]. All other occurrences of a variable in a formula are called occurrences as a *free variable*.

A formula is said to be *well-formed* if it is a variable, or if it is one

Lambda



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Lambda

Lambda

Reduction rules

$$(\lambda x.\, u)\, t \quad \Rightarrow \quad u[t/x]$$

$$\text{fst}\ (t, u) \quad \Rightarrow \quad t$$

$$\text{snd}\ (t, u) \quad \Rightarrow \quad u$$

Simplifying a term

$$(\lambda z.\, (\mathrm{snd}\; z, \mathrm{fst}\; z))\; (y, x)$$

Simplifying a term

$$(\lambda z.\,(\mathrm{snd}\ z, \mathrm{fst}\ z))\ (y, x)$$

$$\Downarrow$$

$$(\mathrm{snd}\ (y, x), \mathrm{fst}\ (y, x))$$

Simplifying a term

$$(\lambda z.\, (\mathrm{snd}\; z, \mathrm{fst}\; z))\; (y, x)$$

$$\Downarrow$$

$$(\mathrm{snd}\; (y, x), \mathrm{fst}\; (y, x))$$

$$\Downarrow$$

$$(x, y)$$

Church 1940: Typed Lambda Calculus

$$\dfrac{\begin{array}{c}[x : A]^x \\ \vdots \\ u : B\end{array}}{\lambda x.\, u : A \supset B}\ \supset\text{-I}^x \qquad \dfrac{s : A \supset B \qquad t : A}{s\, t : B}\ \supset\text{-E}$$

$$\dfrac{t : A \qquad u : B}{(t, u) : A \,\&\, B}\ \&\text{-I} \qquad \dfrac{s : A \,\&\, B}{\text{fst}\ s : A}\ \&\text{-E}_0 \qquad \dfrac{s : A \,\&\, B}{\text{snd}\ s : B}\ \&\text{-E}_1$$

A program

$$\dfrac{\dfrac{\dfrac{[z : B \,\&\, A]^{z}}{\mathrm{snd}\ z : A}\ \&\text{-E}_1 \qquad \dfrac{[z : B \,\&\, A]^{z}}{\mathrm{fst}\ z : B}\ \&\text{-E}_0}{(\mathrm{snd}\ z, \mathrm{fst}\ z) : A \,\&\, B}\ \&\text{-I}}{\lambda z.\ (\mathrm{snd}\ z, \mathrm{fst}\ z) : (B \,\&\, A) \supset (A \,\&\, B)}\ \supset\text{-I}^{z}$$

Simplifying programs

$$\dfrac{\dfrac{\begin{array}{c}[x : A]^x \\ \vdots \\ u : B\end{array}}{\lambda x.\, u : A \supset B}\ \supset\text{-I}^x \qquad \begin{array}{c}\vdots \\ t : A\end{array}}{(\lambda x.\, u)\, t : B}\ \supset\text{-E} \quad \Rightarrow \quad \begin{array}{c}\vdots \\ t : A \\ \vdots \\ u[t/x] : B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ t : A\end{array} \qquad \begin{array}{c}\vdots\\ u : B\end{array}}{(t, u) : A \,\&\, B}\ \&\text{-I}}{\mathrm{fst}\ (t, u) : A}\ \&\text{-E}_0 \quad\Rightarrow\quad \begin{array}{c}\vdots\\ t : A\end{array}$$

Simplifying a program

$$
\dfrac{\dfrac{\dfrac{\dfrac{[z : B \,\&\, A]^z}{\mathrm{snd}\ z : A}\,\&\text{-E}_1 \qquad \dfrac{[z : B \,\&\, A]^z}{\mathrm{fst}\ z : B}\,\&\text{-E}_0}{(\mathrm{snd}\ z, \mathrm{fst}\ z) : A \,\&\, B}\,\&\text{-I}}{\lambda z.\,(\mathrm{snd}\ z, \mathrm{fst}\ z) : (B \,\&\, A) \supset (A \,\&\, B)}\supset\text{-I}^z \qquad \dfrac{[y : B]^y \quad [x : A]^x}{(y, x) : B \,\&\, A}\,\&\text{-I}}{(\lambda z.\,(\mathrm{snd}\ z, \mathrm{fst}\ z))\,(y, x) : A \,\&\, B}\supset\text{-E}
$$

Simplifying a program

$$
\dfrac{\dfrac{\dfrac{[z : B \mathbin{\&} A]^z}{\mathrm{snd}\ z : A}\ \&\text{-E}_1 \qquad \dfrac{[z : B \mathbin{\&} A]^z}{\mathrm{fst}\ z : B}\ \&\text{-E}_0}{(\mathrm{snd}\ z, \mathrm{fst}\ z) : A \mathbin{\&} B}\ \&\text{-I}}{\lambda z.\,(\mathrm{snd}\ z, \mathrm{fst}\ z) : (B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\ \supset\text{-I}^z \qquad \dfrac{[y : B]^y \quad [x : A]^x}{(y, x) : B \mathbin{\&} A}\ \&\text{-I}
$$

$$
\overline{(\lambda z.\,(\mathrm{snd}\ z, \mathrm{fst}\ z))\ (y, x) : A \mathbin{\&} B}\ \supset\text{-E}
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\dfrac{[y : B]^y \quad [x : A]^x}{(y, x) : B \mathbin{\&} A}\ \&\text{-I}}{\mathrm{snd}\ (y, x) : A}\ \&\text{-E}_1 \qquad \dfrac{\dfrac{[y : B]^y \quad [x : A]^x}{(y, x) : B \mathbin{\&} A}\ \&\text{-I}}{\mathrm{fst}\ (y, x) : B}\ \&\text{-E}_0}{(\mathrm{snd}\ (y, x), \mathrm{fst}\ (y, x)) : A \mathbin{\&} B}\ \&\text{-I}
$$

Simplifying a program

$$\dfrac{\dfrac{\dfrac{\dfrac{[z : B \mathbin{\&} A]^z}{\text{snd}\ z : A}\ \&\text{-E}_1 \qquad \dfrac{[z : B \mathbin{\&} A]^z}{\text{fst}\ z : B}\ \&\text{-E}_0}{(\text{snd}\ z, \text{fst}\ z) : A \mathbin{\&} B}\ \&\text{-I}}{\lambda z.\,(\text{snd}\ z, \text{fst}\ z) : (B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\ \supset\text{-I}^z \qquad \dfrac{[y : B]^y \quad [x : A]^x}{(y, x) : B \mathbin{\&} A}\ \&\text{-I}}{(\lambda z.\,(\text{snd}\ z, \text{fst}\ z))\,(y, x) : A \mathbin{\&} B}\ \supset\text{-E}$$

$$\Downarrow$$

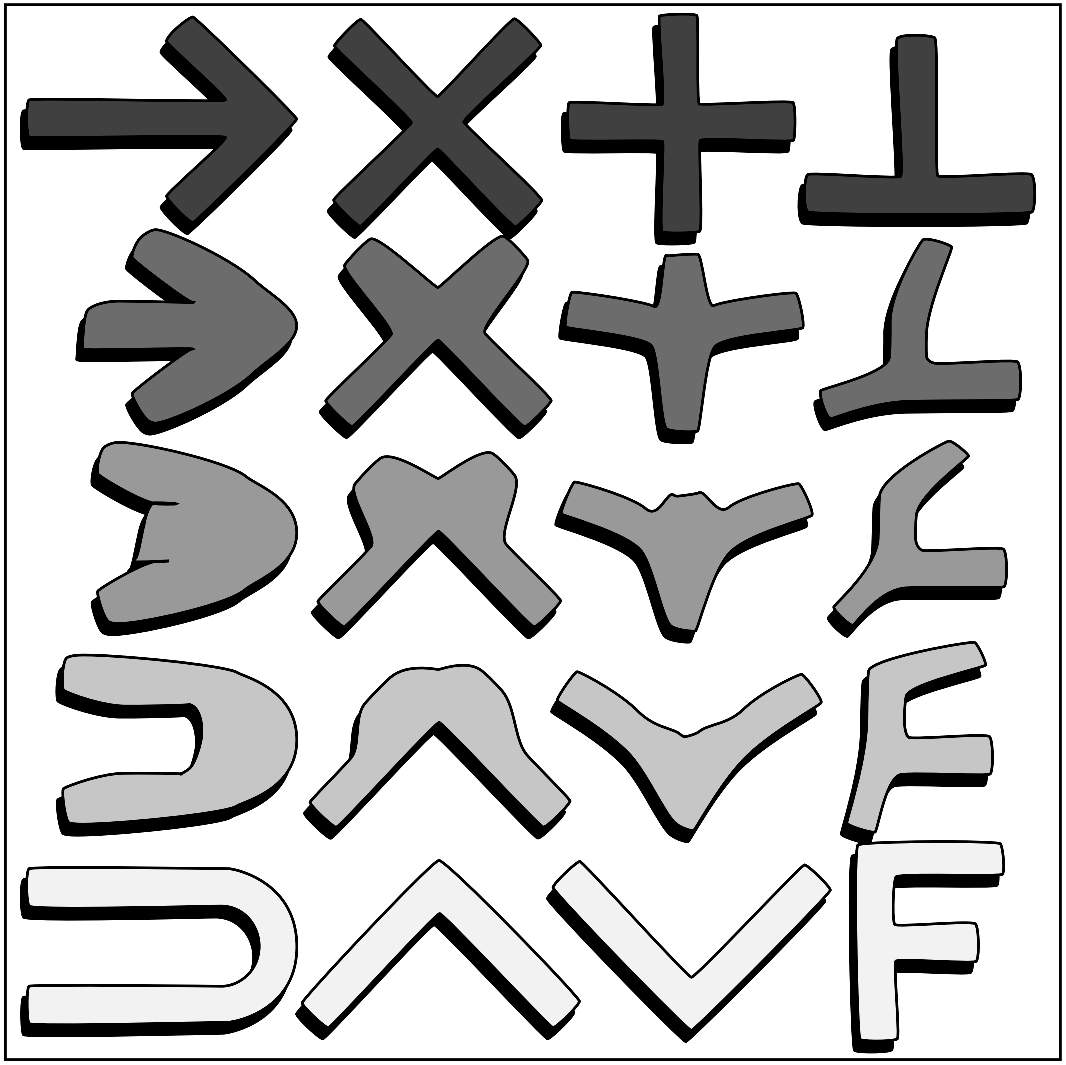
$$\dfrac{\dfrac{\dfrac{[y : B]^y \quad [x : A]^x}{(y, x) : B \mathbin{\&} A}\ \&\text{-I}}{\text{snd}\ (y, x) : A}\ \&\text{-E}_1 \qquad \dfrac{\dfrac{[y : B]^y \quad [x : A]^x}{(y, x) : B \mathbin{\&} A}\ \&\text{-I}}{\text{fst}\ (y, x) : B}\ \&\text{-E}_0}{(\text{snd}\ (y, x), \text{fst}\ (y, x)) : A \mathbin{\&} B}\ \&\text{-I}$$

$$\Downarrow$$

$$\dfrac{[x : A]^x \quad [y : B]^y}{(x, y) : A \mathbin{\&} B}\ \&\text{-I}$$

Part 3

The Curry-Howard Isomorphism



LC'90

The Curry-Howard homeomorphism

Haskell Curry (1900–1982) / William Howard

Howard 1980

THE FORMULAE-AS-TYPES NOTION OF CONSTRUCTION

W. A. Howard

Department of Mathematics, University of Illinois at Chicago Circle, Chicago, Illinois 60680, U.S.A.

Dedicated to H. B. Curry on the occasion of his 80th birthday.

The following consists of notes which were privately circulated in 1969. Since they have been referred to a few times in the literature, it seems worth while to publish them. They have been rearranged for easier reading, and some inessential corrections have been made.

Howard 1980

1. Formulation of the sequent calculus

Let P(⊃) denote positive implicational propositional logic. The prime formulae of P(⊃) are propositional variables. If α and β are formulae, so is $\alpha \supset \beta$. A *sequent* has the form $\Gamma \to \beta$, where Γ is a (possibly empty) finite sequence of formulae and β is a formula. The axioms and rules of inference of P(⊃) are as follows.

(1.1) Axioms: all sequents of the form
$$\alpha \to \alpha$$

(1.2)
$$\frac{\Gamma, \alpha \to \beta}{\Gamma \to \alpha \supset \beta}$$

(1.3)
$$\frac{\Gamma \to \alpha \qquad \Delta \to \alpha \supset \beta}{\Gamma, \Delta \to \beta}$$

(1.4) Thinning, permutation and contraction rules

Howard 1980

2. *Type symbols, terms and constructions*

By a type symbol is meant a formula of $P(\supset)$. We will consider a λ-formalism in which each term has a type symbol α as a superscript (which we may not always write); the term is said to be of type α. The rules of term formation are as follows.

(2.1) Variables X^{α}, Y^{β},... are terms

(2.2) λ-abstraction: from F^{β} get $(\lambda X^{\alpha}.F^{\beta})^{\alpha \supset \beta}$.

(2.3) Application: from $G^{\alpha \supset \beta}$ and H^{α} get $(G^{\alpha \supset \beta} H^{\alpha})^{\beta}$.

Part 4

Programs and Proofs

Programs

- Lisp (McCarthy, 1960)
- Iswim (Landin, 1966)
- Scheme (Steele and Sussman, 1975)
- ML (Milner, Gordon, Wadsworth, 1979)
- Hope (Burstall, MacQueen, Sannella, 1980)
- Miranda (Turner, 1985)
- Haskell (Hudak, Peyton Jones, and Wadler, 1987)
- O'Caml (Leroy, 1996)
- Links (Wadler et al, 2005)

Proofs

- Automath (de Bruijn, 1970)
- Type Theory (Martin Löf, 1975)
- ML/LCF (Milner, Gordon, and Wadsworth, 1979)
- HOL (Gordon and Melham, 1988)
- CoQ (Huet and Coquand, 1988)
- Isabelle (Paulson, 1993)

Proofs/Programs

- Hindley/Milner (1969/1975)
- Girard/Reynolds (1972/1975)
- Linear Logic/Syntactic Control of Interference (1987/1978)
- Classical Logic/Continuation-Passing Style (1990)
- And dual to Or/Call-by-value dual to Call-by-name (2000)