Some Types of Types

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Part I

Propositions as Types

Gerhard Gentzen (1909–1945)

Gerhard Gentzen (1935) — Natural Deduction

$\&$–I

$$\frac{\mathfrak{A} \qquad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}}$$

$\&$–E

$$\frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \qquad \frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$$

$\lor$–I

$$\frac{\mathfrak{A}}{\mathfrak{A} \lor \mathfrak{B}} \qquad \frac{\mathfrak{B}}{\mathfrak{A} \lor \mathfrak{B}}$$

$\lor$–E

$$\frac{\mathfrak{A} \lor \mathfrak{B} \quad \begin{matrix}[\mathfrak{A}] \\ \mathfrak{C}\end{matrix} \quad \begin{matrix}[\mathfrak{B}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

$\forall$–I

$$\frac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

$\forall$–E

$$\frac{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$$

$\exists$–I

$$\frac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

$\exists$–E

$$\frac{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \quad \begin{matrix}[\mathfrak{F}\mathfrak{a}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

$\supset$–I

$$\frac{\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\mathfrak{A} \supset \mathfrak{B}}$$

$\supset$–E

$$\frac{\mathfrak{A} \qquad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$$

$\neg$–I

$$\frac{\begin{matrix}[\mathfrak{A}] \\ \curlywedge\end{matrix}}{\neg \mathfrak{A}}$$

$\neg$–E

$$\frac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge} \qquad \frac{\curlywedge}{\mathfrak{D}} \, .$$

Gerhard Gentzen (1935) — Natural Deduction

$$\dfrac{\begin{array}{c}[A]^x \\ \vdots \\ B\end{array}}{A \supset B}\,\supset\text{-I}^x \qquad \dfrac{A \supset B \qquad A}{B}\,\supset\text{-E}$$

$$\dfrac{A \qquad B}{A \mathbin{\&} B}\,\&\text{-I} \qquad \dfrac{A \mathbin{\&} B}{A}\,\&\text{-E}_0 \qquad \dfrac{A \mathbin{\&} B}{B}\,\&\text{-E}_1$$

A proof

$$
\dfrac{\dfrac{\dfrac{[B \mathbin{\&} A]^z}{A}\;\&\text{-E}_1 \qquad \dfrac{[B \mathbin{\&} A]^z}{B}\;\&\text{-E}_0}{A \mathbin{\&} B}\;\&\text{-I}}{(B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\;\supset\text{-I}^z
$$

Simplifying proofs

$$
\dfrac{\dfrac{\begin{matrix}[A]^x \\ \vdots \\ B\end{matrix}}{A \supset B}\,\supset\text{-I}^x \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B}\,\supset\text{-E} \quad \Rightarrow \quad \begin{matrix}\vdots \\ A \\ \vdots \\ B\end{matrix}
$$

$$
\dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \mathbin{\&} B}\,\&\text{-I}}{A}\,\&\text{-E}_0 \quad \Rightarrow \quad \begin{matrix}\vdots \\ A\end{matrix}
$$

Simplifying a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \,\&\, A]^z}{A}\,\&\text{-E}_1 \qquad \dfrac{[B \,\&\, A]^z}{B}\,\&\text{-E}_0}{A \,\&\, B}\,\&\text{-I}}{(B \,\&\, A) \supset (A \,\&\, B)}\,\supset\text{-I}^z \qquad \dfrac{[B]^y \quad [A]^x}{B \,\&\, A}\,\&\text{-I}}{A \,\&\, B}\,\supset\text{-E}$$

Simplifying a proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{[B \mathbin{\&} A]^z}{A}\,\&\text{-E}_1 \qquad \dfrac{[B \mathbin{\&} A]^z}{B}\,\&\text{-E}_0}{A \mathbin{\&} B}\,\&\text{-I}}{(B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\,\supset\text{-I}^z \qquad \dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{A \mathbin{\&} B}\,\supset\text{-E}
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{[B]^y \quad [A]^x}{B \mathbin{\&} A}\,\&\text{-I}}{B}\,\&\text{-E}_0}{A \mathbin{\&} B}\,\&\text{-I}
$$

Simplifying a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \,\&\, A]^z}{A}\,\&\text{-E}_1 \qquad \dfrac{[B \,\&\, A]^z}{B}\,\&\text{-E}_0}{A \,\&\, B}\,\&\text{-I}}{(B \,\&\, A) \supset (A \,\&\, B)}\supset\text{-I}^z \qquad \dfrac{[B]^y \quad [A]^x}{B \,\&\, A}\,\&\text{-I}}{A \,\&\, B}\supset\text{-E}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{[B]^y \quad [A]^x}{B \,\&\, A}\,\&\text{-I}}{A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{[B]^y \quad [A]^x}{B \,\&\, A}\,\&\text{-I}}{B}\,\&\text{-E}_0}{A \,\&\, B}\,\&\text{-I}$$

$$\Downarrow$$

$$\dfrac{[A]^x \quad [B]^y}{A \,\&\, B}\,\&\text{-I}$$

Alonzo Church (1903–1995)

Alonzo Church (1940) — Typed λ-calculus

$$\dfrac{\begin{array}{c}[x : A]^x \\ \vdots \\ N : B\end{array}}{\lambda x.\, N : A \supset B}\ \supset\text{-I}^x \qquad \dfrac{L : A \supset B \qquad M : A}{L\, M : B}\ \supset\text{-E}$$

$$\dfrac{M : A \qquad N : B}{(M,\, N) : A \,\&\, B}\ \&\text{-I} \qquad \dfrac{L : A \,\&\, B}{\mathsf{fst}\ L : A}\ \&\text{-E}_0 \qquad \dfrac{L : A \,\&\, B}{\mathsf{snd}\ L : B}\ \&\text{-E}_1$$

A program

$$
\dfrac{\dfrac{\dfrac{[z : B \mathbin{\&} A]^z}{\mathsf{snd}\, z : A}\ \&\text{-E}_1 \qquad \dfrac{[z : B \mathbin{\&} A]^z}{\mathsf{fst}\, z : B}\ \&\text{-E}_0}{(\mathsf{snd}\, z,\ \mathsf{fst}\, z) : A \mathbin{\&} B}\ \&\text{-I}}{\lambda z.\, (\mathsf{snd}\, z,\ \mathsf{fst}\, z) : (B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\ \supset\text{-I}^z
$$

Evaluating programs

$$
\dfrac{\dfrac{\begin{matrix}[x : A]^x \\ \vdots \\ N : B\end{matrix}}{\lambda x.\, N : A \supset B}\ \supset\text{-I}^x \qquad \begin{matrix}\vdots \\ M : A\end{matrix}}{(\lambda x.\, N)\, M : B}\ \supset\text{-E} \quad \Rightarrow \quad \begin{matrix}\vdots \\ M : A \\ \vdots \\ N\{M/x\} : B\end{matrix}
$$

$$
\dfrac{\dfrac{\begin{matrix}\vdots \\ M : A\end{matrix} \qquad \begin{matrix}\vdots \\ N : B\end{matrix}}{(M,\, N) : A \,\&\, B}\ \&\text{-I}}{\mathsf{fst}\,(M,\, N) : A}\ \&\text{-E}_0 \quad \Rightarrow \quad \begin{matrix}\vdots \\ M : A\end{matrix}
$$

Alan Turing (1942)

AN EARLY PROOF OF NORMALIZATION
BY A.M. TURING

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Dedicated to H.B. Curry on the occasion of his 80th birthday

In the extract printed below, Turing shows that every formula of Church's simple type theory has a normal form. The extract is the first page of an unpublished (and incomplete) typescript entitled 'Some theorems about Church's system'. (Turing left his manuscripts to me; they are deposited in the library of King's College, Cambridge). An account of this system was published by Church in 'A formulation of the simple theory of types' (J. Symbolic Logic 5 (1940), pp. 56-68).

Evaluating a program

$$\dfrac{\dfrac{\dfrac{[z : B \mathbin{\&} A]^z}{\mathsf{snd}\ z : A}\ \&\text{-E}_1 \qquad \dfrac{[z : B \mathbin{\&} A]^z}{\mathsf{fst}\ z : B}\ \&\text{-E}_0}{(\mathsf{snd}\ z,\ \mathsf{fst}\ z) : A \mathbin{\&} B}\ \&\text{-I}}{\lambda z.\,(\mathsf{snd}\ z,\ \mathsf{fst}\ z) : (B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\ \supset\text{-I}^z \qquad \dfrac{[y : B]^y \quad [x : A]^x}{(y,\ x) : B \mathbin{\&} A}\ \&\text{-I}}{(\lambda z.\,(\mathsf{snd}\ z,\ \mathsf{fst}\ z))\ (y,\ x) : A \mathbin{\&} B}\ \supset\text{-E}$$

Evaluating a program

$$\dfrac{\dfrac{\dfrac{[z : B \,\&\, A]^z}{\mathsf{snd}\ z : A}\ \&\text{-E}_1 \qquad \dfrac{[z : B \,\&\, A]^z}{\mathsf{fst}\ z : B}\ \&\text{-E}_0}{(\mathsf{snd}\ z,\ \mathsf{fst}\ z) : A \,\&\, B}\ \&\text{-I}}{\lambda z.\,(\mathsf{snd}\ z,\ \mathsf{fst}\ z) : (B \,\&\, A) \supset (A \,\&\, B)}\ \supset\text{-I}^z \qquad \dfrac{[y : B]^y \quad [x : A]^x}{(y,\ x) : B \,\&\, A}\ \&\text{-I}}{(\lambda z.\,(\mathsf{snd}\ z,\ \mathsf{fst}\ z))\,(y,\ x) : A \,\&\, B}\ \supset\text{-E}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{[y : B]^y \quad [x : A]^x}{(y,\ x) : B \,\&\, A}\ \&\text{-I}}{\mathsf{snd}\ (y,\ x) : A}\ \&\text{-E}_1 \qquad \dfrac{\dfrac{[y : B]^y \quad [x : A]^x}{(y,\ x) : B \,\&\, A}\ \&\text{-I}}{\mathsf{fst}\ (y,\ x) : B}\ \&\text{-E}_0}{(\mathsf{snd}\ (y,\ x),\ \mathsf{fst}\ (y,\ x)) : A \,\&\, B}\ \&\text{-I}$$

Evaluating a program

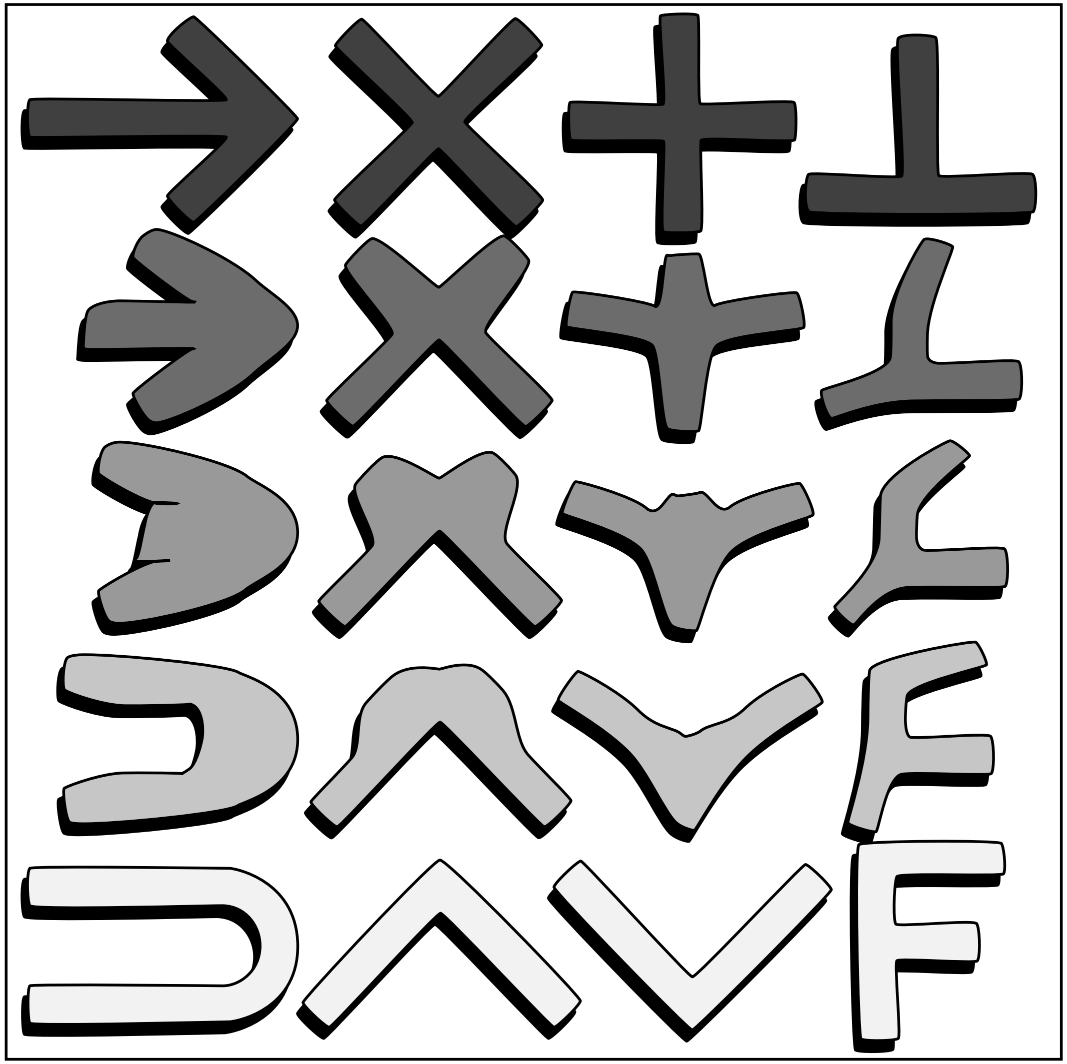
$$
\dfrac{\dfrac{\dfrac{[z : B \mathbin{\&} A]^z}{\mathsf{snd}\ z : A}\ \&\text{-E}_1 \qquad \dfrac{[z : B \mathbin{\&} A]^z}{\mathsf{fst}\ z : B}\ \&\text{-E}_0}{(\mathsf{snd}\ z,\ \mathsf{fst}\ z) : A \mathbin{\&} B}\ \&\text{-I}}{\lambda z.\,(\mathsf{snd}\ z,\ \mathsf{fst}\ z) : (B \mathbin{\&} A) \supset (A \mathbin{\&} B)}\ \supset\text{-I}^z \qquad \dfrac{[y : B]^y \quad [x : A]^x}{(y,\ x) : B \mathbin{\&} A}\ \&\text{-I}}{(\lambda z.\,(\mathsf{snd}\ z,\ \mathsf{fst}\ z))\ (y,\ x) : A \mathbin{\&} B}\ \supset\text{-E}
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\dfrac{[y : B]^y \quad [x : A]^x}{(y,\ x) : B \mathbin{\&} A}\ \&\text{-I}}{\mathsf{snd}\ (y,\ x) : A}\ \&\text{-E}_1 \qquad \dfrac{\dfrac{[y : B]^y \quad [x : A]^x}{(y,\ x) : B \mathbin{\&} A}\ \&\text{-I}}{\mathsf{fst}\ (y,\ x) : B}\ \&\text{-E}_0}{(\mathsf{snd}\ (y,\ x),\ \mathsf{fst}\ (y,\ x)) : A \mathbin{\&} B}\ \&\text{-I}
$$

$$\Downarrow$$

$$
\dfrac{[x : A]^x \quad [y : B]^y}{(x,\ y) : A \mathbin{\&} B}\ \&\text{-I}
$$



LC'90 *The Curry-Howard homeomorphism*

Haskell Curry (1900–1982) / William Howard (1926–)

Howard 1980

THE FORMULAE-AS-TYPES NOTION OF CONSTRUCTION

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Dedicated to H. B. Curry on the occasion of his 80th birthday.

The following consists of notes which were privately circulated in 1969. Since they have been referred to a few times in the literature, it seems worth while to publish them. They have been rearranged for easier reading, and some inessential corrections have been made.

Curry-Howard correspondence

propositions	*as*	types
proofs	*as*	programs
normalisation of proofs	*as*	evaluation of programs

Curry-Howard correspondence

Natural Deduction	$\leftrightarrow$	Typed Lambda Calculus
Gentzen (1935)		Church (1940)
Type Schemes	$\leftrightarrow$	ML Type System
Hindley (1969)		Milner (1975)
System F	$\leftrightarrow$	Polymorphic Lambda Calculus
Girard (1972)		Reynolds (1974)
Modal Logic	$\leftrightarrow$	Monads (state, exceptions)
Lewis (1910)		Kleisli (1965), Moggi (1987)
Classical-Intuitionistic Embedding	$\leftrightarrow$	Continuation Passing Style
Gödel (1933)		Reynolds (1972)
Linear Logic	$\leftrightarrow$	Session Types
Girard (1987)		Honda (1993)

Functional Languages

- Lisp (McCarthy, 1960)
- Iswim (Landin, 1966)
- Scheme (Steele and Sussman, 1975)
- ML (Milner, Gordon, Wadsworth, 1979)
- Haskell (Hudak, Hughes, Peyton Jones, and Wadler, 1987)
- O'Caml (Leroy, 1996)
- Erlang (Armstrong, Virding, Williams, 1996)
- Scala (Odersky, 2004)
- $F^{\#}$ (Syme, 2006)
- Clojure (Hickey, 2007)
- Elm (Czaplicki, 2012)

Proof assistants

- Automath (de Bruijn, 1970)
- Type Theory (Martin Löf, 1975)
- Mizar (Trybulec, 1975)
- ML/LCF (Milner, Gordon, and Wadsworth, 1979)
- NuPrl (Constable, 1985)
- HOL (Gordon and Melham, 1988)
- Coq (Huet and Coquand, 1988)
- Isabelle (Paulson, 1993)
- Epigram (McBride and McKinna, 2004)
- Agda (Norell, 2005)

Two styles

How I do it

How everyone else does it

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\ \mathrm{Id}$$

$$\frac{\begin{array}{c}[x : A]^x \\ \vdots \\ N : B\end{array}}{\lambda x.\, N : A \supset B}\ \supset\text{-}\mathrm{I}^x$$

$$\frac{\Gamma, x : A \vdash N : B}{\Gamma \vdash \lambda x.\, N : A \supset B}\ \supset\text{-}\mathrm{I}$$

$$\frac{L : A \supset B \qquad M : A}{L\,M : B}\ \supset\text{-}\mathrm{E}$$

$$\frac{\Gamma \vdash L : A \supset B \qquad M : A}{\Gamma \vdash L\,M : B}\ \supset\text{-}\mathrm{E}$$

Part II

The Girard-Reynolds Isomorphism

A tale of Two Theorems

Girard’s Representation Theorem

Reynolds’s Abstraction Theorem

A tale of Two Theorems

Girard's Representation Theorem

projection : proofs $\longrightarrow$ terms

Reynolds's Abstraction Theorem

embedding : terms $\longrightarrow$ proofs

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \rightarrow \quad \times \quad + \quad \perp}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \rightarrow \quad \times \quad + \quad \bot}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Girard-Reynolds Isomorphism

$$\frac{\forall \quad \forall^2 \quad \forall^1 \quad \longrightarrow}{\forall \qquad\qquad\qquad \longrightarrow}$$

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \rightarrow \quad \times \quad + \quad \perp}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Girard-Reynolds Isomorphism

$$\frac{\forall \quad \forall^2 \quad \forall^1 \quad \longrightarrow}{\forall \qquad\qquad\qquad \longrightarrow}$$

Rather than enriching the type systems to match logic,
we impoverish logic to match the type structure.

— Daniel Leivant

Second-order lambda calculus (F2)

Type variables	X, Y, Z		
Types	A, B, C	$::=$	X
		$\mid$	$A \to B$
		$\mid$	$\forall X.\, B$
Individual variables	x, y, z		
Terms	s, t, u	$::=$	x^A
		$\mid$	$\lambda x^A.\, u$
		$\mid$	$s\ t$
		$\mid$	$\Lambda X.\, u$
		$\mid$	$s\ A$

Second-order lambda calculus (F2)

$$\dfrac{\begin{matrix}[x^A] \\ \vdots \\ u^B\end{matrix}}{(\lambda x^A.\, u)^{A \to B}} \to\text{-I}^x \qquad \dfrac{s^{A\to B} \qquad t^A}{(s\ t)^B} \to\text{-E}$$

$$\dfrac{u^B}{(\Lambda X.\, u)^{\forall X.\, B}} \forall\text{-I} \quad X \text{ does not escape} \qquad \dfrac{s^{\forall X.\, B}}{(s\ A)^{B[A/X]}} \forall\text{-E}$$

Second-order propositional logic (P2)

Predicate variables	$\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$		
Propositions	$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$	$::=$	$t^C \in \boldsymbol{\mathcal{A}}^C$
		$\mid$	$\boldsymbol{A} \rightarrow \boldsymbol{B}$
		$\mid$	$\forall \boldsymbol{\mathcal{X}}^C. \boldsymbol{B}$
		$\mid$	$\forall x^C. \boldsymbol{B}$
		$\mid$	$\forall X. \boldsymbol{B}$
Predicates	$\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}$	$::=$	$\boldsymbol{\mathcal{X}}^C$
		$\mid$	$\{x^C \mid \boldsymbol{A}\}$
Hypothesis labels	$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$		
Proofs	$\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}$		

Second-order propositional logic (P2)

$$\dfrac{\begin{array}{c}[A]^{x}\\ \vdots\\ B\end{array}}{A \to B}\ \to\text{-I}^{x} \qquad \dfrac{A \to B \qquad A}{B}\ \to\text{-E}$$

$$\dfrac{B}{\forall \mathcal{X}^{C}.\, B}\ \forall\text{-I} \quad \mathcal{X} \text{ does not escape} \qquad \dfrac{\forall \mathcal{X}^{C}.\, B}{B[\mathcal{A}^{C}/\mathcal{X}]}\ \forall\text{-E}$$

$$\dfrac{B}{\forall x^{C}.\, B}\ \forall^{1}\text{-I} \quad x \text{ does not escape} \qquad \dfrac{\forall x^{C}.\, B}{B[t^{C}/x]}\ \forall^{1}\text{-E}$$

$$\dfrac{B}{\forall X.\, B}\ \forall^{2}\text{-I} \quad X \text{ does not escape} \qquad \dfrac{\forall X.\, B}{B[A/X]}\ \forall^{2}\text{-E}$$

β rules

$$
\begin{array}{lll}
(\lambda x^T.\, u)\; t & =_\beta & u[t/x] \\
(\Lambda X.\, u)\; A & =_\beta & u[A/X] \\
t^C \in \{x^C \mid \boldsymbol{A}\} & =_\beta & \boldsymbol{A}[t/x]
\end{array}
$$

$$
\frac{\boldsymbol{A}}{\boldsymbol{B}}\,\beta \qquad \boldsymbol{A} =_\beta \boldsymbol{B}
$$

Part III

Girard projection

Girard projection

Propositions

$$
\begin{array}{lcl}
(t^C \in \boldsymbol{\mathcal{A}}^C)^\circ & \equiv & \boldsymbol{\mathcal{A}}^\circ \\
(\boldsymbol{A} \longrightarrow \boldsymbol{B})^\circ & \equiv & \boldsymbol{A}^\circ \rightarrow \boldsymbol{B}^\circ \\
(\forall \boldsymbol{\mathcal{X}}^C \boldsymbol{.}\, \boldsymbol{B})^\circ & \equiv & \forall X.\, \boldsymbol{B}^\circ \\
(\forall x^C.\, \boldsymbol{B})^\circ & \equiv & \boldsymbol{B}^\circ \\
(\forall X.\, \boldsymbol{B})^\circ & \equiv & \boldsymbol{B}^\circ
\end{array}
$$

Predicates

$$
\begin{array}{lcl}
(\boldsymbol{\mathcal{X}}^C)^\circ & \equiv & X \\
(\{x^C \mid \boldsymbol{A}\})^\circ & \equiv & \boldsymbol{A}^\circ
\end{array}
$$

Girard projection

$$\left(\dfrac{\begin{array}{c}[\boldsymbol{A}]^{\boldsymbol{x}}\\ \vdots\ \boldsymbol{u}\\ \boldsymbol{B}\end{array}}{\boldsymbol{A}\rightarrow\boldsymbol{B}}\ \rightarrow\text{-}\mathbf{I}^{\boldsymbol{x}}\right)^{\circ}\equiv\dfrac{\begin{array}{c}[x^{\boldsymbol{A}^{\circ}}]\\ \vdots\\ \boldsymbol{u}^{\circ\,\boldsymbol{B}^{\circ}}\end{array}}{(\lambda x^{\boldsymbol{A}^{\circ}}.\,\boldsymbol{u}^{\circ})^{\boldsymbol{A}^{\circ}\rightarrow\boldsymbol{B}^{\circ}}}\ \rightarrow\text{-I}^{x}$$

$$\left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{s}\\ \boldsymbol{A}\rightarrow\boldsymbol{B}\end{array}\qquad\begin{array}{c}\vdots\ \boldsymbol{t}\\ \boldsymbol{A}\end{array}}{\boldsymbol{B}}\ \rightarrow\text{-}\mathbf{E}\right)^{\circ}\equiv\dfrac{\begin{array}{c}\vdots\\ \boldsymbol{s}^{\circ\,\boldsymbol{A}^{\circ}\rightarrow\boldsymbol{B}^{\circ}}\end{array}\qquad\begin{array}{c}\vdots\\ \boldsymbol{t}^{\circ\,\boldsymbol{A}^{\circ}}\end{array}}{(\boldsymbol{s}^{\circ}\ \boldsymbol{t}^{\circ})^{\boldsymbol{B}^{\circ}}}\ \rightarrow\text{-E}$$

Girard projection

$$\left(\dfrac{\begin{array}{c}\vdots\, \boldsymbol{u} \\ \boldsymbol{B}\end{array}}{\forall \boldsymbol{\mathcal{X}}^{C}\,.\,\boldsymbol{B}}\ \forall\textbf{-I}\right)^{\circ} \equiv \dfrac{\begin{array}{c}\vdots \\ \boldsymbol{u}^{\circ\,\boldsymbol{B}^{\circ}}\end{array}}{(\Lambda X.\,\boldsymbol{u}^{\circ})^{\forall X.\,\boldsymbol{B}^{\circ}}}\ \forall\text{-I}$$

$$\left(\dfrac{\begin{array}{c}\vdots\, \boldsymbol{s} \\ \forall \boldsymbol{\mathcal{X}}^{C}\,.\,\boldsymbol{B}\end{array}}{\boldsymbol{B}[\boldsymbol{\mathcal{A}}^{C}/\boldsymbol{\mathcal{X}}]}\ \forall\textbf{-E}\right)^{\circ} \equiv \dfrac{\begin{array}{c}\vdots \\ \boldsymbol{s}^{\circ\,\forall X.\,\boldsymbol{B}^{\circ}}\end{array}}{(\boldsymbol{s}^{\circ}\ \boldsymbol{\mathcal{A}}^{\circ})^{\boldsymbol{B}^{\circ}[\boldsymbol{\mathcal{A}}^{\circ}/X]}}\ \forall\text{-E}$$

Girard projection

$$\left(\dfrac{\begin{matrix}\vdots\, u \\ B\end{matrix}}{\forall x^{C}.\, B}\ \forall^1\text{-I}\right)^{\circ} \equiv \begin{matrix}\vdots \\ u^{\circ}\, B^{\circ}\end{matrix} \qquad \left(\dfrac{\begin{matrix}\vdots\, s \\ \forall x^{C}.\, B\end{matrix}}{B[t^{C}/x]}\ \forall^1\text{-E}\right)^{\circ} \equiv \begin{matrix}\vdots \\ s^{\circ}\, B^{\circ}\end{matrix}$$

$$\left(\dfrac{\begin{matrix}\vdots\, u \\ B\end{matrix}}{\forall X.\, B}\ \forall^2\text{-I}\right)^{\circ} \equiv \begin{matrix}\vdots \\ u^{\circ}\, B^{\circ}\end{matrix} \qquad \left(\dfrac{\begin{matrix}\vdots\, s \\ \forall X.\, B\end{matrix}}{B[A/X]}\ \forall^2\text{-E}\right)^{\circ} \equiv \begin{matrix}\vdots \\ s^{\circ}\, B^{\circ}\end{matrix}$$

$$\left(\dfrac{\begin{matrix}\vdots\, t \\ A\end{matrix}}{B}\ \beta\right)^{\circ} \equiv \begin{matrix}\vdots \\ t^{\circ}\, A^{\circ}\end{matrix}$$

Part IV

Reynolds embedding

Reynolds embedding

Types

$$
\begin{array}{lcl}
(X)^* & \equiv & \boldsymbol{\mathcal{X}}^X \\
(A \to B)^* & \equiv & \{z^{A \to B} \mid \forall x^A.\, x \in A^* \longrightarrow z\, x \in B^*\} \\
(\forall X.\, B)^* & \equiv & \{z^{\forall X.\, B} \mid \forall X.\, \forall \boldsymbol{\mathcal{X}}^X \boldsymbol{.}\, z\, X \in B^*\}
\end{array}
$$

Reynolds embedding

$$\left(\frac{\begin{array}{c}[x^A] \\ \vdots \\ u^B\end{array}}{(\lambda x^A.\, u)^{A \to B}} \to\text{-I}^x \right)^* \equiv \cfrac{\cfrac{\cfrac{\begin{array}{c}[x \in A^*]^{\boldsymbol{x}} \\ \vdots\; u^* \\ u \in B^*\end{array}}{(\lambda x^A.\, u)\; x \in B^*}\,\beta}{x \in A^* \to (\lambda x^A.\, u)\; x \in B^*}\,\to\text{-}\mathbf{I}^{\boldsymbol{x}}}{\forall x^A.\, x \in A^* \to (\lambda x^A.\, u)\; x \in B^*}\,\forall^1\text{-}\mathbf{I}$$

$$\left(\frac{\begin{array}{c}\vdots \\ s^{A \to B}\end{array} \qquad \begin{array}{c}\vdots \\ t^A\end{array}}{(s\; t)^B} \to\text{-E} \right)^* \equiv \cfrac{\cfrac{\begin{array}{c}\vdots\; s^* \\ \forall x^A.\, x \in A^* \to s\; x \in B^*\end{array}}{t \in A^* \to s\; t \in B^*}\,\forall^1\text{-E} \qquad \begin{array}{c}\vdots\; t^* \\ t \in A^*\end{array}}{s\; t \in B^*}\,\to\text{-}\mathbf{E}$$

Reynolds embedding

$$\left(\dfrac{\begin{matrix}\vdots \\ u^{B}\end{matrix}}{(\Lambda X.\, u)^{\forall X.\, B}}\ \forall\text{-I}\right)^{*} \equiv \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ u^{*} \\ u \in B^{*}\end{matrix}}{(\Lambda X.\, u)\ X \in B^{*}}\ \beta}{\forall \mathcal{X}^{X}.\, (\Lambda X.\, u)\ X \in B^{*}}\ \forall\text{-I}}{\forall X.\, \forall \mathcal{X}^{X}.\, (\Lambda X.\, u)\ X \in B^{*}}\ \forall^{2}\text{-I}$$

$$\left(\dfrac{\begin{matrix}\vdots \\ s^{\forall X.\, B}\end{matrix}}{(s\ A)^{B[A/X]}}\ \forall\text{-E}\right)^{*} \equiv \dfrac{\dfrac{\begin{matrix}\vdots\ s^{*} \\ \forall X.\, \forall \mathcal{X}^{X}.\, s\ X \in B^{*}\end{matrix}}{\forall \mathcal{X}^{A}.\, s\ A \in B^{*}[A/X]}\ \forall^{2}\text{-E}}{s\ A \in B^{*}[A/X, A^{*}/\mathcal{X}]}\ \forall\text{-E}$$

Part V

Doubling

Doubling

Propositions

$$
\begin{array}{lcl}
(t^C \in \boldsymbol{\mathcal{A}}^C)^\ddagger & \equiv & (t^C, t'^{C'}) \in \boldsymbol{\mathcal{A}}^{\ddagger\, C\times C'} \\
(\boldsymbol{A} \rightarrow \boldsymbol{B})^\ddagger & \equiv & \boldsymbol{A}^\ddagger \rightarrow \boldsymbol{B}^\ddagger \\
(\forall \boldsymbol{\mathcal{X}}^C.\, \boldsymbol{B})^\ddagger & \equiv & \forall \boldsymbol{\mathcal{X}}^{C\times C'}.\, \boldsymbol{B}^\ddagger \\
(\forall x^C.\, \boldsymbol{B})^\ddagger & \equiv & \forall x^C, x'^{C'}.\, \boldsymbol{B}^\ddagger \\
(\forall X.\, \boldsymbol{B})^\ddagger & \equiv & \forall X, X'.\, \boldsymbol{B}^\ddagger
\end{array}
$$

Predicates

$$
\begin{array}{lcl}
(\boldsymbol{\mathcal{X}}^C)^\ddagger & \equiv & \boldsymbol{\mathcal{X}}^{C\times C'} \\
(\{x^C \mid \boldsymbol{A}\})^\ddagger & \equiv & \{(x^C, x'^{C'}) \mid \boldsymbol{A}^\ddagger\}
\end{array}
$$

Doubling

$$\left(\dfrac{\begin{array}{c}[A]^x \\ \vdots\, u \\ B\end{array}}{A \rightarrow B} \rightarrow\text{-I}^x \right)^{\ddagger} \equiv \dfrac{\begin{array}{c}[A^{\ddagger}]^x \\ \vdots\, u^{\ddagger} \\ B^{\ddagger}\end{array}}{A^{\ddagger} \rightarrow B^{\ddagger}} \rightarrow\text{-I}^x$$

$$\left(\dfrac{\begin{array}{c}\vdots\, s \\ A \rightarrow B\end{array} \quad \begin{array}{c}\vdots\, t \\ A\end{array}}{B} \rightarrow\text{-E} \right)^{\ddagger} \equiv \dfrac{\begin{array}{c}\vdots\, s^{\ddagger} \\ A^{\ddagger} \rightarrow B^{\ddagger}\end{array} \quad \begin{array}{c}\vdots\, t^{\ddagger} \\ A^{\ddagger}\end{array}}{B^{\ddagger}} \rightarrow\text{-E}$$

Doubling

$$\left(\dfrac{\begin{array}{c} \vdots\, u \\ B \end{array}}{\forall \mathcal{X}^{C} .\, B}\ \forall\text{-I} \right)^{\ddagger} \equiv \dfrac{\begin{array}{c} \vdots\, u^{\ddagger} \\ B^{\ddagger} \end{array}}{\forall \mathcal{X}^{C \times C'} .\, B^{\ddagger}}\ \forall\text{-I}$$

$$\left(\dfrac{\begin{array}{c} \vdots\, s \\ \forall \mathcal{X}^{C} .\, B \end{array}}{B[\mathcal{A}^{C} / \mathcal{X}]}\ \forall\text{-E} \right)^{\ddagger} \equiv \dfrac{\begin{array}{c} \vdots\, s^{\ddagger} \\ \forall \mathcal{X}^{C \times C'} .\, B^{\ddagger} \end{array}}{B^{\ddagger}[\mathcal{A}^{\ddagger\, C \times C'} / \mathcal{X}]}\ \forall\text{-E}$$

Doubling

$$\left(\dfrac{\begin{matrix} \vdots\; \boldsymbol{u} \\ \boldsymbol{B} \end{matrix}}{\forall x^C .\, \boldsymbol{B}}\ \forall^1\text{-I} \right)^{\ddagger} \equiv \dfrac{\begin{matrix} \vdots\; \boldsymbol{u}^{\ddagger} \\ \boldsymbol{B}^{\ddagger} \end{matrix}}{\forall x^C, x'^{C'} .\, \boldsymbol{B}^{\ddagger}}\ \forall^1\text{-I twice}$$

$$\left(\dfrac{\begin{matrix} \vdots\; \boldsymbol{s} \\ \forall x^C .\, \boldsymbol{B} \end{matrix}}{\boldsymbol{B}[t^C/x]}\ \forall^1\text{-E} \right)^{\ddagger} \equiv \dfrac{\begin{matrix} \vdots\; \boldsymbol{s}^{\ddagger} \\ \forall x^C, x'^{C'} .\, \boldsymbol{B}^{\ddagger} \end{matrix}}{\boldsymbol{B}^{\ddagger}[t^C/x, t'^{C'}/x']}\ \forall^1\text{-E twice}$$

Doubling

$$\left(\dfrac{\begin{array}{c} \vdots\, \boldsymbol{u} \\ \boldsymbol{B} \end{array}}{\forall X.\, \boldsymbol{B}}\ \forall^2\text{-I} \right)^{\ddagger} \equiv \dfrac{\begin{array}{c} \vdots\, \boldsymbol{u}^{\ddagger} \\ \boldsymbol{B}^{\ddagger} \end{array}}{\forall X, X'.\, \boldsymbol{B}^{\ddagger}}\ \forall^2\text{-I twice}$$

$$\left(\dfrac{\begin{array}{c} \vdots\, \boldsymbol{s} \\ \forall X.\, \boldsymbol{B} \end{array}}{\boldsymbol{B}[A/X]}\ \forall^2\text{-E} \right)^{\ddagger} \equiv \dfrac{\begin{array}{c} \vdots\, \boldsymbol{s}^{\ddagger} \\ \forall X, X'.\, \boldsymbol{B}^{\ddagger} \end{array}}{\boldsymbol{B}^{\ddagger}[A/X, A'/X']}\ \forall^2\text{-E twice}$$

Doubling

$$\left(\dfrac{\begin{array}{c} \vdots\, \boldsymbol{t} \\ \boldsymbol{A} \end{array}}{\boldsymbol{B}}\,\beta \right)^{\ddagger} \equiv \dfrac{\begin{array}{c} \vdots\, \boldsymbol{t}^{\ddagger} \\ \boldsymbol{A}^{\ddagger} \end{array}}{\boldsymbol{B}^{\ddagger}}\,\beta \text{ twice}$$

Part VI

Induction and Parametricity

Induction and Parametricity

Proposition 0.1

$$\forall n.\, n \in \mathbf{N} \longrightarrow n \in \mathsf{N}^*$$

Proposition 0.2

$$\forall n.\, n \in \mathbf{N} \longrightarrow n \in \mathsf{N}^{*\ddagger}$$

Proposition 0.3

$$\forall n, n'.\, (n, n') \in \mathsf{N}^{*\ddagger} \longrightarrow n = n' \land n \in \mathbf{N}$$

Corrolary 0.4

$$(\forall n.\, n \in \mathsf{N}^* \longrightarrow (n, n) \in \mathsf{N}^{*\ddagger}) \leftrightarrow (\forall n.\, n \in \mathsf{N}^* \longrightarrow n \in \mathbf{N})$$

Part VII

Successor

$$\boldsymbol{A}_{\mathsf{s}} \equiv \forall m^{\mathsf{N}}.\, m \in \boldsymbol{\mathcal{X}} \to \mathsf{s}\, m \in \boldsymbol{\mathcal{X}} \qquad\qquad \boldsymbol{A}_{\mathsf{z}} \equiv \mathsf{z} \in \boldsymbol{\mathcal{X}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\boldsymbol{A}_{\mathsf{s}}]^{s}}{n \in \boldsymbol{\mathcal{X}} \to \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\,\forall^1\text{-E} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[n \in \mathbf{N}]^{n}}{\forall \boldsymbol{\mathcal{X}}^{\mathsf{N}}.\, \boldsymbol{A}_{\mathsf{s}} \to \boldsymbol{A}_{\mathsf{z}} \to n \in \boldsymbol{\mathcal{X}}}\,\beta}{\boldsymbol{A}_{\mathsf{s}} \to \boldsymbol{A}_{\mathsf{z}} \to n \in \boldsymbol{\mathcal{X}}}\,\forall\text{-}\mathbf{E} \quad [\boldsymbol{A}_{\mathsf{s}}]^{s}}{\boldsymbol{A}_{\mathsf{z}} \to n \in \boldsymbol{\mathcal{X}}}\,\to\text{-}\mathbf{E} \quad [\boldsymbol{A}_{\mathsf{z}}]^{z}}{n \in \boldsymbol{\mathcal{X}}}\,\to\text{-}\mathbf{E}}{\mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\,\to\text{-}\mathbf{E}}{\boldsymbol{A}_{\mathsf{z}} \to \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\,\to\text{-}\mathbf{I}^{z}}{\boldsymbol{A}_{\mathsf{s}} \to \boldsymbol{A}_{\mathsf{z}} \to \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\,\to\text{-}\mathbf{I}^{s}}{\forall \boldsymbol{\mathcal{X}}^{\mathsf{N}}.\, \boldsymbol{A}_{\mathsf{s}} \to \boldsymbol{A}_{\mathsf{z}} \to \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\,\forall\text{-}\mathbf{I}}{\mathsf{s}\, n \in \mathbf{N}}\,\beta}{n \in \mathbf{N} \to \mathsf{s}\, n \in \mathbf{N}}\,\to\text{-}\mathbf{I}^{n}}{\forall n^{\mathsf{N}}.\, n \in \mathbf{N} \to \mathsf{s}\, n \in \mathbf{N}}\,\forall^1\text{-I}$$

$$\dfrac{[n^{\mathsf{N}}]}{(n\ X)^{(X\to X)\to X\to X}}\ \forall\text{-E}$$

$$\dfrac{(n\ X)^{(X\to X)\to X\to X} \qquad [s^{X\to X}]}{(n\ X\ s)^{X\to X}}\ \to\text{-E}$$

$$\dfrac{(n\ X\ s)^{X\to X} \qquad [z^{X}]}{(n\ X\ s\ z)^{X}}\ \to\text{-E}$$

$$\dfrac{[s^{X\to X}] \qquad (n\ X\ s\ z)^{X}}{(s\ (n\ X\ s\ z))^{X}}\ \to\text{-E}$$

$$\dfrac{(s\ (n\ X\ s\ z))^{X}}{(\lambda z^{X}.\, s\ (n\ X\ s\ z))^{X\to X}}\ \to\text{-I}^{z}$$

$$\dfrac{(\lambda z^{X}.\, s\ (n\ X\ s\ z))^{X\to X}}{(\lambda s^{X\to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z))^{(X\to X)\to X\to X}}\ \to\text{-I}^{s}$$

$$\dfrac{(\lambda s^{X\to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z))^{(X\to X)\to X\to X}}{(\Lambda X.\, \lambda s^{X\to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z))^{\mathsf{N}}}\ \forall\text{-I}$$

$$\dfrac{(\Lambda X.\, \lambda s^{X\to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z))^{\mathsf{N}}}{(\lambda n^{\mathsf{N}}.\, \Lambda X.\, \lambda s^{X\to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z))^{\mathsf{N}\to\mathsf{N}}}\ \to\text{-I}^{n}$$

Part VIII

Conclusions

Propositions as Types

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