# A Reflection on Call-by-Value

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## Abstract

A number of compilers exploit the following strategy: translate a term to continuation-passing style (CPS) and optimize the resulting term using a sequence of reductions. Recent work suggests that an alternative strategy is superior: optimize directly in an extended source calculus. We suggest that the appropriate relation between the source and target calculi may be captured by a special case of a *Galois connection* known as a *reflection*. Previous work has focused on the weaker notion of an *equational correspondence*, which is based on equality rather than reduction. We show that Moggi's monad translation and Plotkin's CPS translation can both be regarded as reflections, and thereby strengthen a number of results in the literature.

## 1 Introduction

Like the wheel, continuation-passing style, or CPS, is such an excellent idea that it was reinvented many times (Reynolds 1993). Subsequently it was formalized by Plotkin (1975), and utilized in compilers for higher-order call-by-value languages written by Steele (1978), Kranz *et al.* (1986), Appel (1992), and others.

The front end for all these compilers is similar: translate a term to CPS and optimize. In symbols,

$$M^* \longrightarrow\!\!\!\!\!\rightarrow_T P, \tag{1}$$

where $M$ is a term of the source language $S$, $P$ is a term of the target language $T$ (terms in CPS), $* : S \to T$ is the compiling transform (the CPS translation), and $\longrightarrow\!\!\!\!\!\rightarrow_T$ denotes reduction in $T$.

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Recently a number of researchers have suggested an alternative way to build a compiler: rather than translate the term to CPS and optimize, perform the optimizations in an extended source calculus. In symbols,

$$M \longrightarrow\!\!\!\!\!\rightarrow_S P^{\#}, \tag{2}$$

where $\# : T \to S$ is the decompiling transform, and $\longrightarrow\!\!\!\!\!\rightarrow_S$ denotes reduction in the (extended) source calculus.

If equation (1) holds exactly when equation (2) holds, we say that maps $*$ and $\#$ form a *Galois connection* from $S$ to $T$. Galois connections guarantee that for each optimization in the target language there is some corresponding optimization in the source language. We might also reasonably require that compiling is the inverse of decompiling, so $P^{*\#}$ is identical to $P$. A Galois connection satisfying this additional constraint is called a *reflection* in $S$ of $T$. Whenever a reflection exists, there must be a subset of $S$ that is isomorphic to all of $T$; we call this subset the *kernel*.

Our claim is that a *reflection* describes precisely the situation of interest to compiler writers: it relates the intermediate languages of direct and CPS compilers and enables direct compilers to work with a representation that is isomorphic to CPS terms. Previous work, such as Sabry and Felleisen (1993) and Hatcliff and Danvy (1994), has focused on the weaker notion of *equational correspondence*, which is based on equality rather than reduction.

We show that two widely-studied translations may be regarded as reflections. Our source calculus is the computational lambda calculus, $\lambda_c$, of Moggi (1988), which extends the call-by-value calculus $\lambda_v$ of Plotkin (1975). Our first translation is into the monadic meta-language, $\lambda_{ml}$, also of Moggi (1988); and our second translation is the CPS translation, also of Plotkin (1975). The monad translation may be regarded as a generalization of the CPS translation, and indeed the latter translation factors through the former.

*The computational lambda calculus* $\lambda_c$. Moggi (1988) introduced monads as a general notion of computation. The monadic meta-language $\lambda_{ml}$ was

designed to express any semantics based on monads, and the computation lambda calculus $\lambda_c$ was designed as an extension of the call-by-value lambda calculus that is sound and complete for any monadic semantics. So, by design, two terms are equal in $\lambda_c$ if and only if their translations are equal in $\lambda_{ml}$.

What is surprising is that very little attention was paid to reductions. The original technical report (Moggi 1988) specified theories of reduction and of equality for $\lambda_c$, but only the equality theory of $\lambda_c$ appears in the conference paper (Moggi 1989), and $\lambda_c$ rates barely a line in the journal version (Moggi 1991). None of these contains a reduction theory for $\lambda_{ml}$, but this was considered by Hatcliff and Danvy (1994). However, ours is the first work we know of to relate the reductions of $\lambda_c$ and $\lambda_{ml}$.

Moggi (1988) presents $\lambda_c$ as an untyped calculus of reductions, and presents $\lambda_{ml}$ as a typed calculus of equalities. Here we uniformly use untyped calculi of reductions. Everything works equally well for typed calculi of reductions, such as those considered by Hatcliff and Danvy (1994), since our translations preserve types.

*Related work.* As noted above, a number of researchers have considered ways to reflect the benefits of CPS within an extended source calculus. Sabry and Felleisen (1993) proposed such a calculus, and noted that the kernel arising by normalization with regard to the so-called $A$ reductions is closely related to the target of the CPS translation. Some practical ramifications of this were explored by Flanagan, Sabry, Duba, and Felleisen (1993).

Lawall and Danvy (1993) make much of relating forward and inverse translations via two Galois connections and an isomorphism. Their Galois connections are based on artificial orders, and as they note their isomorphism does not preserve these orders. Our choice of order solves these problems and is more natural: we use the usual reduction relation on terms.

Administrative reductions play a central role in many CPS translations, from Plotkin's work onwards. Recent work explains administrative reductions of the target in terms of reductions in the source (Sabry and Felleisen 1993, Flanagan *et al.* 1993, Hatcliff and Danvy 1994). Each of our translations will incorporate administrative reductions, and each of our reflections will have a kernel that can be explained as source terms reduced to administrative normal form.

*Outline.* The remainder of this paper is structured as follows. Section 2 summarizes our results. Section 3 introduces Galois connections and reflections, and explains why they embody a property every compiler writer seeks. Section 4 reviews the traditional translation of $\lambda_v$ into $\lambda_{ml}$ and why it fails to be a reflection. Section 5 shows that there is a reflection between $\lambda_c$ and a variant $\lambda_{ml*}$ of $\lambda_{ml}$. Section 6 factors this reflection through an intermediate calculus $\lambda_{c*}$ corresponding to the isomorphic image of $\lambda_{ml*}$ in $\lambda_c$. Section 7 extends the results to translations into CPS. Section 8 describes related work. Section 9 concludes.

$$
\begin{array}{ccccc}
\lambda_v & \to & \lambda_{ml} & \to & \lambda_v \\
 & & & & \\
\lambda_c & & & & \\
\cup & & & & \\
\lambda_{c*} & \cong & \lambda_{ml*} & & \\
\cup & & \cup & & \\
\lambda_{c**} & \cong & \lambda_{ml**} & \cong & \lambda_{cps}
\end{array}
$$

Figure 1: Summary of results

## 2 Summary

If you like to see a summary in advance, read this section now; if you like to see a summary on completion, save it until the end. Depending on your preference, you may thereby read this section once, twice, or never.

Figure 1 illustrates a road-map of the terrain we cover. First we consider the traditional monad translation from $\lambda_v$ into $\lambda_{ml}$. This translation is not a reflection; but expanding $\lambda_v$ into $\lambda_c$, shrinking $\lambda_{ml}$ into $\lambda_{ml*}$, and fine-tuning the translation, finally yields a reflection in $\lambda_c$ of $\lambda_{ml*}$. The existence of a reflection guarantees that there is a kernel $\lambda_{c*}$ of $\lambda_c$ that is isomorphic to $\lambda_{ml*}$. Calculus $\lambda_c$ has seven reduction rules, and $\lambda_{c*}$ arises by normalizing with respect to two of these rules, $(let.1)$ and $(let.2)$.

Second we consider the traditional CPS translation from $\lambda_v$ back into $\lambda_v$. Again, this translation is not a reflection; but expanding (source) $\lambda_v$ into $\lambda_c$, shrinking (target) $\lambda_v$ into $\lambda_{cps}$, and fine-tuning the translation, finally yields a reflection in $\lambda_c$ of $\lambda_{cps}$. Again, the existence of a reflection guarantees that there is a kernel $\lambda_{c**}$ of $\lambda_c$ that is isomorphic to $\lambda_{cps}$. Calculus $\lambda_{c**}$ arises by normalizing $\lambda_{c*}$ with regard to one further reduction rule, $(assoc)$. Furthermore, the second translation factors through the first; and so there is also a kernel $\lambda_{ml**}$ of $\lambda_{ml*}$ that is also isomorphic to $\lambda_{cps}$.

## 3 Galois Connections and Reflections

Let's review the standard results about Galois connections and reflections (Mac Lane 1971, Davey and Priestley 1990). The standard results need to be adapted slightly, as reduction is a preorder (it is reflexive and transitive) but not a partial order (it is not anti-symmetric).

We write $\longrightarrow$ for a single-step of reduction; $\longrightarrow\!\!\!\!\!\rightarrow$ for the reflexive and transitive closure of reduction; $=$ for the reflexive, transitive, and symmetric closure of reduction; and $\equiv$ for syntactic identity up

to renaming.

Assume a source calculus $S$ with reduction relation $\longrightarrow\!\!\!\!\!\rightarrow_S$, a target calculus $T$ with reduction relation $\longrightarrow\!\!\!\!\!\rightarrow_T$. Reductions are directed in such a way that they naturally correspond to evaluation steps or optimizations. Let the maps $* : S \to T$ and $\# : T \to S$ correspond to compiling and decompiling, respectively. Finally, let $M, N$ range over terms of $S$, and $P, Q$ range over terms of $T$.

**Definition 3.1** *Maps $*$ and $\#$ form a* Galois connection *from $S$ to $T$ whenever*

$$M \longrightarrow\!\!\!\!\!\rightarrow_S P^{\#} \text{ if and only if } M^* \longrightarrow\!\!\!\!\!\rightarrow_T P.$$

There is an alternative characterization of a Galois connection.

**Proposition 3.1** *Maps $*$ and $\#$ form a Galois connection from $S$ to $T$ if and only if the following four conditions hold.*

(i) $M \longrightarrow\!\!\!\!\!\rightarrow_S M^{*\#}$,
(ii) $P^{\#*} \longrightarrow\!\!\!\!\!\rightarrow_T P$,
(iii) $M \longrightarrow\!\!\!\!\!\rightarrow_S N$ *implies* $M^* \longrightarrow\!\!\!\!\!\rightarrow_T N^*$,
(iv) $P \longrightarrow\!\!\!\!\!\rightarrow_T Q$ *implies* $P^{\#} \longrightarrow\!\!\!\!\!\rightarrow_S Q^{\#}$.

If the same four conditions hold with $\longrightarrow\!\!\!\!\!\rightarrow_S$ and $\longrightarrow\!\!\!\!\!\rightarrow_T$ replaced by $=_S$ and $=_T$, then one has an *equational correspondence*, as defined by Sabry and Felleisen (1993). Hence, every Galois connection implies an equational correspondence, though not conversely.

The alternative characterization reveals a difficulty, as sketched in Figure 2. Consider a source term $M$ compiling to a target term $M^*$, and consider an optimization $M^* \longrightarrow\!\!\!\!\!\rightarrow_T P$ in the target. A Galois connection guarantees the existence of a corresponding optimization $M \longrightarrow\!\!\!\!\!\rightarrow_S P^{\#}$ in the source. However the Galois connection does *not* guarantee that this optimization is as good as the target optimization. In fact, as revealed by law (ii) of Proposition 3.1, the source optimization is necessarily *worse* than the original optimization, since $P^{\#*} \longrightarrow\!\!\!\!\!\rightarrow_T P$.

We therefore reasonably insist on having not just a Galois connection, but a *reflection*.

**Definition 3.2** *Maps $*$ and $\#$ form a* reflection *in $S$ of $T$ if they form a Galois connection and $P \equiv P^{\#*}$.*

For a reflection, $\#$ is necessarily injective.

Galois connections and reflections compose.

**Proposition 3.2** *Let $*_1$ and $\#_1$ form a Galois connection (reflection) from $S$ to $T$, and $*_2$ and $\#_2$ form a Galois connection (reflection) from $T$ to $U$. Then $*_1*_2$ and $\#_2\#_1$ form a Galois connection (reflection) from $S$ to $U$.*

Every reflection factors into an *inclusion* and an *order isomorphism*. Write $S^{*\#}$ for the subset of $S$ containing just those terms of the form $M^{*\#}$, and write $Id : S^{*\#} \to S$ for the trivial inclusion function. A reflection $*$ and $\#$ in $S$ of $T$ is an *inclusion* if $\#$ is the identity (and hence $S \supseteq T$), and is an *order isomorphism* if $*$ and $\#$ are inverses (and hence $S \cong T$).

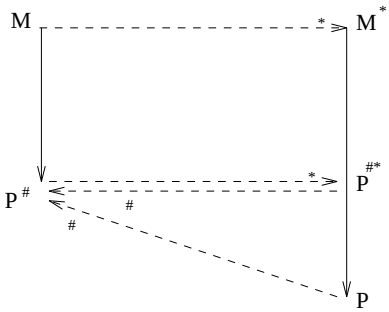


Figure 2: Motivating reflections

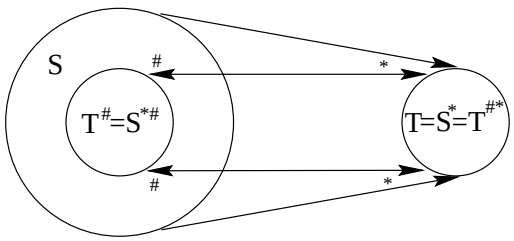


Figure 3: A reflection and its kernel isomorphism

**Proposition 3.3** *Let $*$ and $\#$ form a reflection in $S$ of $T$.*

1. *Translations $*\#$ and Id form an inclusion in $S$ of $S^{*\#}$.*
2. *Translations $*$ and $\#$ form an order isomorphism between $S^{*\#}$ and $T$.*

*The composition of the inclusion and the isomorphism is the original reflection.*

The proposition is illustrated in Figure 3, which shows calculi $S$ and $T$, and the images $S^*$, $T^{\#}$, $S^{*\#}$, and $T^{\#*}$ of these calculi under maps $*$ and $\#$. The kernel $S^{*\#}$ of $S$ is isomorphic to $T$.

The summary illustrated in Figure 1 demonstrates repeated use of this proposition. Each reflection is factored into an inclusion (shown vertically) and an order isomorphism (shown horizontally).

## 4 Monads: The Problem

Let's review the traditional translation from the call-by-value calculus into the monad calculus, and see why it fails to be a reflection.

Plotkin's call-by-value calculus $\lambda_v$ is summarized in Figure 4. We let $x, y, z$ range over variables, $L, M, N$ range over terms, and $V, W$ range over values. A term is either a value or an application, and a value is either a variable or an abstraction. The call-by-value nature of the calculus is expressed by limiting the argument to a value in $(\beta.v)$, and by limiting the function to a value in $(\eta.v)$. An important aspect of each calculus we deal with is that it is confluent.

**Proposition 4.1** *The reductions of $\lambda_v$ are confluent.*

Moggi's monadic meta-language $\lambda_{ml}$ is summarized in Figure 5. This calculus distinguishes *values* from *computations*. Functions may accept and return either values or computations, and are defined by the call-by-name rules $(\beta)$ and $(\eta)$. Two terms relate values to computations: the term $[M]$ denotes the computation that does nothing save return the value $M$; and the term $\mathtt{let}\, x \Leftarrow L\, \mathtt{in}\, N$ denotes the computation that performs computation $L$, binds $x$ to the resulting value, and then performs computation $N$. The interaction of these terms is described by the three rules $(\beta.let)$, $(\eta.let)$, and $(assoc)$.

**Proposition 4.2** *The reductions of $\lambda_{ml}$ are confluent.*

The translation $*$ from $\lambda_v$ to $\lambda_{ml}$ is described in Figure 6. Translation $*$ on terms uses an auxiliary translation $\dagger$ on values. The two translations are related by a substitution lemma, $N^*[x := V^\dagger] \equiv (N[x := V])^*$, which is easily checked by induction over the structure of terms.

The translation $*$ is *sound* in that $M \longrightarrow\!\!\!\!\!\rightarrow_v N$ implies $M^* \longrightarrow\!\!\!\!\!\rightarrow_{ml} N^*$. This is easily checked by looking at the translation of $(\beta.v)$,

$$\begin{array}{ll} & ((\lambda x.N)V)^* \\ \equiv & \mathtt{let}\, y \Leftarrow [\lambda x.N^*] \\ & \mathtt{in}\, \mathtt{let}\, z \Leftarrow [V^\dagger]\, \mathtt{in}\, yz \\ \longrightarrow\!\!\!\!\!\rightarrow_{(\beta.let)} & (\lambda x.N^*)V^\dagger \\ \longrightarrow_{(\beta)} & N^*[x := V^\dagger] \\ \equiv & (N[x := V])^*, \end{array}$$

and similarly for $(\eta.v)$.

However, the translation is not *complete* even in the weak sense required by equational correspondence: $M^* =_{ml} N^*$ does not imply $M =_v N$. For example, it is easy to check that

$$((\lambda x.\, xM)L)^* \quad =_{ml} \quad (LM)^*$$

while if $L$ is not a value then the equivalence does not hold in the call-by-value calculus.

**Proposition 4.3** *The translation $* : \lambda_v \to \lambda_{ml}$ is sound, but does not generate an equational correspondence.*

## 5 Monads: The Solution

To refine the translation $* : \lambda_v \to \lambda_{ml}$ into a reflection requires three steps: first, grow $\lambda_v$ into $\lambda_c$; second, shrink $\lambda_{ml}$ into $\lambda_{ml*}$; third, fine tune the translation $*$.

Step one grows $\lambda_v$ into $\lambda_c$, which is summarized in Figure 7. The new calculus was carefully designed by Moggi to model directly the effect of translation into $\lambda_{ml}$. This is achieved by adding to $\lambda_c$ a term $\mathtt{let}\, x = M\, \mathtt{in}\, N$ which mimics the term $\mathtt{let}\, x \Leftarrow M\, \mathtt{in}\, N$ of $\lambda_{ml}$; by adding to $\lambda_c$ reductions corresponding to each reduction of $\lambda_{ml}$; and by adding to $\lambda_c$ two more reductions, $(let.1)$ and $(let.2)$, which mimic the effect of the translation from $\lambda_v$ into $\lambda_{ml}$. Let $P, Q$ range over non-values. Rules $(let.1)$ and $(let.2)$ are restricted to act on non-values, since values yield a reduction in the opposite direction via $(\beta.let)$.

You may wonder: Is the new form $(\mathtt{let}\, x = M\, \mathtt{in}\, N)$ necessary, or could it be represented by $(\lambda x.N)M$ instead? The latter is possible, but then the rules $(\beta.let)$, $(\eta.let)$, $(assoc)$, $(let.1)$, and $(let.2)$ all become more difficult to read. Further, we prefer for historical reasons to stick to Moggi's formulation.

You may also wonder: Do the rules $(let.1)$ and $(let.2)$ point in the right direction? They do: the direction is dictated by the desire that reduction be confluent. For example,

$$\begin{array}{ll} & \mathtt{let}\, y = (\mathtt{let}\, x = L\, \mathtt{in}\, M)\, \mathtt{in}\, yN \\ \longrightarrow_{(rev.let.1)} & (\mathtt{let}\, x = L\, \mathtt{in}\, M)N, \\ & \mathtt{let}\, y = (\mathtt{let}\, x = L\, \mathtt{in}\, M)\, \mathtt{in}\, yN \\ \longrightarrow_{(assoc)} & \mathtt{let}\, x = L\, \mathtt{in}\, (\mathtt{let}\, y = M\, \mathtt{in}\, yN) \\ \longrightarrow_{(rev.let.1)} & \mathtt{let}\, x = L\, \mathtt{in}\, MN \end{array}$$

shows that confluence fails if $(let.1)$ is reversed. But the system as given is confluent, as was shown by Moggi (1988).

**Proposition 5.1** *The reductions of $\lambda_c$ are confluent.*

Step two shrinks $\lambda_{ml}$ into $\lambda_{ml*}$, which is summarized in Figure 8. The grammar is the smallest one that contains all terms in the image of the translation $* : \lambda_v \to \lambda_{ml}$ of the previous section, and that is closed under the reductions of $\lambda_{ml}$. The new grammar differs from the old in two key ways: applications $MN$ in $\lambda_{ml}$ are restricted to the form $VW$ in $\lambda_{ml*}$, and computations $[M]$ in $\lambda_{ml}$ are restricted to the form $[V]$ in $\lambda_{ml*}$.

The reduction rules are restricted accordingly. Since all applications have the form $VW$, rules $(\beta)$ and $(\eta)$ and rules $(\beta.v)$ and $(\eta.v)$ have the same effect on the calculus. This provides an analogue of

$$
\begin{array}{lllll}
\text{terms} & L, M, N & ::= & V \mid LM \\
\text{values} & V, W & ::= & x \mid \lambda x.\, N
\end{array}
$$

$$
\begin{array}{lllll}
(\beta.v) & (\lambda x.\, N)V & \longrightarrow & N[x := V] & \\
(\eta.v) & \lambda x.\, (Vx) & \longrightarrow & V, & \text{if } x \notin fv(V)
\end{array}
$$

Figure 4: The call-by-value calculus, $\lambda_v$

$$
\text{terms} \quad L, M, N \quad ::= \quad x \mid \lambda x.\, M \mid MN \mid \texttt{[}M\texttt{]} \mid \texttt{let } x \Leftarrow M \texttt{ in } N
$$

$$
\begin{array}{lllll}
(\beta) & (\lambda x.\, N)M & \longrightarrow & N[x := M] & \\
(\eta) & \lambda x.\, (Mx) & \longrightarrow & M, & \text{if } x \notin fv(M) \\
(\beta.let) & \texttt{let } x \Leftarrow \texttt{[}M\texttt{] in } N & \longrightarrow & N[x := M] & \\
(\eta.let) & \texttt{let } x \Leftarrow M \texttt{ in [}x\texttt{]} & \longrightarrow & M, & \text{if } x \notin fv(M) \\
(assoc) & \texttt{let } y \Leftarrow (\texttt{let } x \Leftarrow L \texttt{ in } M) \texttt{ in } N & \longrightarrow & \texttt{let } x \Leftarrow L \texttt{ in } (\texttt{let } y \Leftarrow M \texttt{ in } N) &
\end{array}
$$

Figure 5: The monadic calculus, $\lambda_{ml}$

$$
\begin{array}{lll}
 & * : \lambda_v \to \lambda_{ml} & \\
V^* & \equiv & \texttt{[}V^\dagger\texttt{]} \\
(LM)^* & \equiv & \texttt{let } x \Leftarrow L^* \texttt{ in let } y \Leftarrow M^* \texttt{ in } xy \\
(x)^\dagger & \equiv & x \\
(\lambda x.\, N)^\dagger & \equiv & \lambda x.\, N^*
\end{array}
$$

Figure 6: Translation of $\lambda_v$ into $\lambda_{ml}$

Plotkin's *indifference* property, which states that call-by-value and call-by-name evaluation yield the same result for terms in CPS.

**Proposition 5.2** *The grammar of $\lambda_{ml*}$ is closed under the reductions of $\lambda_{ml}$. That is, if $M \longrightarrow N$ in $\lambda_{ml}$ and $M$ is in $\lambda_{ml*}$, then $N$ is also in $\lambda_{ml*}$, and $M \longrightarrow N$ in $\lambda_{ml*}$.*

The proof is an easy case analysis. It follows as a corollary from Proposition 4.2 that the reductions of $\lambda_{ml*}$ are confluent.

Step three adapts the translation $* : \lambda_v \to \lambda_{ml}$ to the new calculi. The straightforward choice is to leave the translation as is, adding a line for `let`.

$$
\begin{array}{lll}
V^* & \equiv & \texttt{[}V^\dagger\texttt{]} \\
(LM)^* & \equiv & \overline{\texttt{let}}\ x \Leftarrow L^* \\
 & & \texttt{in } \overline{\texttt{let}}\ y \Leftarrow M^* \\
 & & \texttt{in } xy \\
(\texttt{let } x = M \texttt{ in } N)^* & \equiv & \texttt{let } x \Leftarrow M^* \texttt{ in } N^*
\end{array}
$$

Here $\dagger$ is as in Figure 6. The meaning of the overbars will be explained shortly.

Alas, this translation is not even sound in our stronger sense: it preserves the equalities of $\lambda_c$ but not reductions. The key problem is with rule $(let.2)$, which requires $(\beta.let)$ reductions in both directions.

$$
\begin{array}{lll}
 & (VQ)^* & \\
\equiv & \texttt{let } x \Leftarrow \texttt{[}V^\dagger\texttt{] in let } y \Leftarrow Q^* \texttt{ in } xy & \\
\longrightarrow_{\beta.let} & \texttt{let } y \Leftarrow Q^* \texttt{ in } V^\dagger y & \\
\twoheadleftarrow_{\beta.let} & \texttt{let } y \Leftarrow Q^* & \\
 & \texttt{in let } x' \Leftarrow \texttt{[}V^\dagger\texttt{]} & \\
 & \texttt{in let } y' \Leftarrow \texttt{[}y\texttt{] in } x'y' & \\
\equiv & (\texttt{let } y = Q \texttt{ in } Vy)^* &
\end{array}
$$

The solution is to consider the $\beta.let$ reductions as part of the translation. The two overlined occurrences of `let` introduced in the translation of applications are regarded as *administrative* occurrences. A second stage is added to the translation where all administrative $(\beta.let)$ redexes (that is, ones where the relevant `let` is overlined) are reduced.

This somewhat awkward description as a two-stage translation with administrative redexes may be replaced by the equivalent translation given in the first half of Figure 9. This was derived by a simple case analysis on applications, with according simplification. The obvious homomorphism, shown in the second half of the figure, serves as the inverse translation. It is now straightforward to verify that these constitute a reflection.

**Proposition 5.3** *Translations $*$ and $\#$ (Figure 9) form a reflection in $\lambda_c$ of $\lambda_{ml*}$.*

| | | | |
|---|---|---|---|
| terms | $L, M, N$ | $::=$ | $V \mid P$ |
| values | $V, W$ | $::=$ | $x \mid \lambda x.\, N$ |
| non-values | $P, Q$ | $::=$ | $LM \mid \texttt{let}\ x = M\ \texttt{in}\ N$ |

| | | | | |
|---|---|---|---|---|
| $(\beta.v)$ | $(\lambda x.\, N)V$ | $\longrightarrow$ | $N[x := V]$ | |
| $(\eta.v)$ | $\lambda x.\, (Vx)$ | $\longrightarrow$ | $V,$ | if $x \notin fv(V)$ |
| $(\beta.let)$ | $\texttt{let}\ x = V\ \texttt{in}\ N$ | $\longrightarrow$ | $N[x := V]$ | |
| $(\eta.let)$ | $\texttt{let}\ x = M\ \texttt{in}\ x$ | $\longrightarrow$ | $M,$ | if $x \notin fv(M)$ |
| $(assoc)$ | $\texttt{let}\ y = (\texttt{let}\ x = L\ \texttt{in}\ M)\ \texttt{in}\ N$ | $\longrightarrow$ | $\texttt{let}\ x = L\ \texttt{in}\ (\texttt{let}\ y = M\ \texttt{in}\ N)$ | |
| $(let.1)$ | $PM$ | $\longrightarrow$ | $\texttt{let}\ x = P\ \texttt{in}\ xM$ | |
| $(let.2)$ | $VQ$ | $\longrightarrow$ | $\texttt{let}\ y = Q\ \texttt{in}\ Vy$ | |

Figure 7: The computational calculus, $\lambda_c$

| | | | |
|---|---|---|---|
| terms | $L, M, N$ | $::=$ | $[V] \mid P$ |
| values | $V, W$ | $::=$ | $x \mid \lambda x.\, N$ |
| non-values | $P, Q$ | $::=$ | $VW \mid \texttt{let}\ x \Leftarrow M\ \texttt{in}\ N$ |

| | | | | |
|---|---|---|---|---|
| $(\beta.v)$ | $(\lambda x.\, N)V$ | $\longrightarrow$ | $N[x := V]$ | |
| $(\eta.v)$ | $\lambda x.\, (Vx)$ | $\longrightarrow$ | $V,$ | if $x \notin fv(V)$ |
| $(\beta.let)$ | $\texttt{let}\ x \Leftarrow [V]\ \texttt{in}\ N$ | $\longrightarrow$ | $N[x := V]$ | |
| $(\eta.let)$ | $\texttt{let}\ x \Leftarrow M\ \texttt{in}\ [x]$ | $\longrightarrow$ | $M,$ | if $x \notin fv(M)$ |
| $(assoc)$ | $\texttt{let}\ y \Leftarrow (\texttt{let}\ x \Leftarrow L\ \texttt{in}\ M)\ \texttt{in}\ N$ | $\longrightarrow$ | $\texttt{let}\ x \Leftarrow L\ \texttt{in}\ (\texttt{let}\ y \Leftarrow M\ \texttt{in}\ N)$ | |

Figure 8: The simplified monadic calculus, $\lambda_{ml*}$

$* : \lambda_c \to \lambda_{ml*}$

| | | |
|---|---|---|
| $V^*$ | $\equiv$ | $[V^\dagger]$ |
| $(PM)^*$ | $\equiv$ | $\texttt{let}\ x \Leftarrow P^*\ \texttt{in}\ (xM)^*$ |
| $(VQ)^*$ | $\equiv$ | $\texttt{let}\ y \Leftarrow Q^*\ \texttt{in}\ (Vy)^*$ |
| $(VW)^*$ | $\equiv$ | $V^\dagger W^\dagger$ |
| $(\texttt{let}\ x = M\ \texttt{in}\ N)^*$ | $\equiv$ | $\texttt{let}\ x \Leftarrow M^*\ \texttt{in}\ N^*$ |
| $(x)^\dagger$ | $\equiv$ | $x$ |
| $(\lambda x.\, M)^\dagger$ | $\equiv$ | $\lambda x.\, M^*$ |

$\# : \lambda_{ml*} \to \lambda_c$

| | | |
|---|---|---|
| $x^\#$ | $\equiv$ | $x$ |
| $(\lambda x.\, M)^\#$ | $\equiv$ | $\lambda x.\, M^\#$ |
| $(VW)^\#$ | $\equiv$ | $V^\# W^\#$ |
| $([V])^\#$ | $\equiv$ | $V^\#$ |
| $(\texttt{let}\ x \Leftarrow M\ \texttt{in}\ N)^\#$ | $\equiv$ | $\texttt{let}\ x = M^\#\ \texttt{in}\ N^\#$ |

Figure 9: Reflection in $\lambda_c$ of $\lambda_{ml*}$

To prove this we verify separately each of the four parts of Proposition 3.1. Parts (i) and (ii) are verified by induction over the structure of terms, and parts (iii) and (iv) are verified by induction over the structure of reductions.

## 6 Monads: The Factorization

Proposition 3.3 guarantees that there must be an isomorphic image of $\lambda_{ml*}$ within $\lambda_c$. It consists of exactly those terms of the form $M^{*\#}$. We name this calculus $\lambda_{c*}$, and it is summarized in Figure 10. The grammar is identical to $\lambda_c$, except general applications $MN$ are replaced by value applications $VW$, much as in the move from $\lambda_{ml}$ to $\lambda_{ml*}$.

The terms of $\lambda_{c*}$ can be characterized as the terms of $\lambda_c$ in $(let.1)$ and $(let.2)$ normal form. Once a term has been normalized with respect to these two rules, they are never required again — that is, none of the other rules will introduce a $(let.1)$ or $(let.2)$ redex.

As guaranteed by Proposition 3.3, the reflection $* : \lambda_c \to \lambda_{ml*}$ factors into an inclusion $*_1 : \lambda_c \to \lambda_{c*}$ and an order isomorphism $*_2 : \lambda_{c*} \to \lambda_{ml*}$, given in Figures 11 and 12. The proposition even shows how to compute these: $*_1$ is $*\#$, $\#_1$ is the identity, $*_2$ is a restriction of $*$, and $\#_2$ is $\#$. What is a pleasant bonus, not guaranteed by the proposition, is that $*_1$ has a simple interpretation: it reduces a term of $\lambda_c$ to $(let.1)$ and $(let.2)$ normal form, hence yielding a term of $\lambda_{c*}$.

## 7 Continuations

The development for continuations is paralleled by the development for monads. Just as $\lambda_v$ maps into $\lambda_{ml}$ via the traditional call-by-value monad translation, so does $\lambda_v$ map into $\lambda_v$ via the traditional

| | | | |
|---|---|---|---|
| terms | $L, M, N$ | ::= | $V \mid P$ |
| values | $V, W$ | ::= | $x \mid \lambda x.\, N$ |
| non-values | $P, Q$ | ::= | $VW \mid \mathtt{let}\ x = M\ \mathtt{in}\ N$ |

Reductions $(\beta.v), (\eta.v), (\beta.let), (\eta.let), (assoc)$, as in Figure 7

Figure 10: The kernel computational calculus, $\lambda_{c*}$

| $*_1 : \lambda_c \to \lambda_{c*}$ | | | $\#_1 : \lambda_{c*} \to \lambda_c$ |
|---|---|---|---|
| $V^*$ | $\equiv$ | $V^\dagger$ | the trivial inclusion |
| $(PM)^*$ | $\equiv$ | $\mathtt{let}\ x = P^*\ \mathtt{in}\ (xM)^*$ | |
| $(VQ)^*$ | $\equiv$ | $\mathtt{let}\ y = Q^*\ \mathtt{in}\ (Vy)^*$ | |
| $(VW)^*$ | $\equiv$ | $V^\dagger W^\dagger$ | |
| $(\mathtt{let}\ x = M\ \mathtt{in}\ N)^*$ | $\equiv$ | $\mathtt{let}\ x = M^*\ \mathtt{in}\ N^*$ | |
| $x^\dagger$ | $\equiv$ | $x$ | |
| $(\lambda x.\, M)^\dagger$ | $\equiv$ | $\lambda x.\, M^*$ | |

Figure 11: Inclusion in $\lambda_c$ of $\lambda_{c*}$

| $*_2 : \lambda_{c*} \to \lambda_{ml*}$ | | | $\#_2 : \lambda_{ml*} \to \lambda_{c*}$ |
|---|---|---|---|
| $V^*$ | $\equiv$ | $[V^\dagger]$ | same as # from Figure 9 |
| $(VW)^*$ | $\equiv$ | $V^\dagger W^\dagger$ | |
| $(\mathtt{let}\ x = M\ \mathtt{in}\ N)^*$ | $\equiv$ | $\mathtt{let}\ x \Leftarrow M^*\ \mathtt{in}\ N^*$ | |
| $x^\dagger$ | $\equiv$ | $x$ | |
| $(\lambda x.\, M)^\dagger$ | $\equiv$ | $\lambda x.\, M^*$ | |

Figure 12: Isomorphism of $\lambda_{c*}$ and $\lambda_{ml*}$

call-by-value CPS translation:

$$
\begin{array}{lcl}
V^* & \equiv & \lambda k.\, kV^\dagger \\
(LM)^* & \equiv & \lambda k.\, L^*(\lambda x.\, M^*(\lambda y.\, xyk)) \\
\\
x^\dagger & \equiv & x \\
(\lambda x.\, N)^\dagger & \equiv & \lambda x.\, N^*
\end{array}
$$

A remarkable property of this translation is that one may always choose $k$ to be exactly the same name, without fear of name clash. Again, this translation is sound but not complete.

To refine the translation $* : \lambda_v \to \lambda_v$ into a reflection also requires three steps: first, grow the source $\lambda_v$ into $\lambda_c$; second, shrink the target $\lambda_v$ into $\lambda_{cps}$; third, fine tune the translation $*$. Step one was already accomplished as part of the monad development. Steps two and three are best considered in reverse order, as the calculus of step two should be the image of the modified translation of step three.

To refine the translation $* : \lambda_v \to \lambda_v$, it is suitably extended for `let`, and has some reductions labeled as administrative.

$$
\begin{array}{lcl}
V^* & \equiv & \overline{\lambda} k.\, kV^\dagger \\
(LM)^* & \equiv & \overline{\lambda} k.\, L^*(\overline{\lambda} x.\, M^*(\overline{\lambda} y.\, xyk)) \\
(\mathtt{let}\ x = M\ \mathtt{in}\ N)^* & \equiv & \overline{\lambda} k.\, M^*(\lambda x.\, N^* k) \\
x^\dagger & \equiv & x \\
(\lambda x.\, N)^\dagger & \equiv & \lambda x.\, N^*
\end{array}
$$

The translation now takes place in three stages. Stage one applies the translation proper. Stage two reduces any administrative $(\beta.v)$ redexes — that is, ones where the relevant $\lambda$ is overlined. Stage three strips the leading '$\lambda k$', which always appears and so is redundant.

This three-stage translation may be replaced by the equivalent translation given in the first half of Figure 14. The old translation relates to the new as follows: $M^{*\mathrm{old}} \equiv \lambda k.\, M^{*\mathrm{new}}$, where $k$ is the distinguished continuation variable. The auxiliary translation $M\!:\!K$ closely resembles a translation introduced by Plotkin (1975) to capture the effect of administrative reductions.

Step two shrinks the target $\lambda_v$ into $\lambda_{cps}$, which is summarized in Figure 13. The grammar is the smallest one that is in the image of the refined translation $*$. An additional class of non-terminals,

$$
\begin{array}{lllll}
\text{terms} & L, M, N & ::= & KV \mid VWK \\
\text{values} & V, W & ::= & x \mid \lambda x.\, \lambda k.\, M \\
\text{continuations} & K & ::= & k \mid \lambda x.\, M
\end{array}
$$

$$
\begin{array}{llll}
(\beta.v) & (\lambda x.\, \lambda k.\, M)VK & \longrightarrow & M[x := V][k := K] \\
(\eta.v) & \lambda x.\, \lambda k.\, Vxk & \longrightarrow & V, \quad \text{if } x \notin fv(V) \\
(\beta.let) & (\lambda x.\, M)V & \longrightarrow & M[x := V] \\
(\eta.let) & \lambda x.\, Kx & \longrightarrow & K, \quad \text{if } x \notin fv(K)
\end{array}
$$

Figure 13: Continuation-passing style calculus $\lambda_{cps}$

$* : \lambda_c \to \lambda_{cps}$

$$
\begin{array}{lcl}
M^* & \equiv & M : k \\
V : K & \equiv & KV^\dagger \\
(PM) : K & \equiv & P : (\lambda x.\, ((xM) : K)) \\
(VQ) : K & \equiv & Q : (\lambda y.\, ((Vy) : K)) \\
(VW) : K & \equiv & V^\dagger W^\dagger K \\
(\texttt{let}\ x = M\ \texttt{in}\ N) : K & \equiv & M : (\lambda x.\, (N : K)) \\
x^\dagger & \equiv & x \\
(\lambda x.\, M)^\dagger & \equiv & \lambda x.\, \lambda k.\, M^*
\end{array}
$$

$\# : \lambda_{cps} \to \lambda_c$

$$
\begin{array}{lcl}
(KV)^\# & \equiv & K^\flat[V^\natural] \\
(VWK)^\# & \equiv & K^\flat[V^\natural W^\natural] \\
x^\natural & \equiv & x \\
(\lambda x.\, \lambda k.\, M)^\natural & \equiv & \lambda x.\, M^\# \\
k^\flat & \equiv & [\,] \\
(\lambda x.\, N)^\flat & \equiv & \texttt{let}\ x = [\,]\ \texttt{in}\ N^\#
\end{array}
$$

Figure 14: Reflection in $\lambda_c$ of $\lambda_{cps}$

$K$, is added to the grammar, ranging over continuations. This class contains the distinguished free variable $k$, and contains those lambda expressions which may be substituted for $k$.

Examination reveals that each of the rules $(\beta.v)$ and $(\eta.v)$ arises in exactly two situations, yielding four rules in the target $\lambda_{cps}$ corresponding to four of the seven rules in the source $\lambda_c$. It is easily verified that the grammar is indeed closed under reduction. Every application has a value as an argument, so call-by-value and call-by-name reductions have the same effect on the language, which explains Plotkin's indifference property.

**Proposition 7.1** *The grammar of $\lambda_{cps}$ is closed under the reductions of $\lambda_v$. That is, if $M \longrightarrow N$ in $\lambda_v$ and $M$ is in $\lambda_{cps}$, then $N$ is also in $\lambda_{cps}$, and $M \longrightarrow N$ in $\lambda_{cps}$.*

The proof is an easy case analysis; it follows as a corollary that the reductions of $\lambda_{cps}$ are confluent. Further, the same result holds if the call-by-value calculus $\lambda_v$ is replaced by the call-by-name calculus $\lambda_n$.

The inverse translation $\#$ of Figure 14 is now easily derived. It has three parts, one for each component of the target grammar. A term $M$ in $\lambda_{cps}$ maps to a term $M^\#$ in $\lambda_c$; a value $V$ in $\lambda_{cps}$ maps to a value $V^\natural$ in $\lambda_c$; and a continuation $K$ in $\lambda_{cps}$ maps to an evaluation context $K^\flat$ in $\lambda_c$. An evaluation context $C$ is a term with a hole $[\,]$, and if $C$ is an evaluation context then $C[M]$ denotes the result of replacing the hole in $C$ by the term $M$. The filling operation is straightforward since our evaluation contexts do not bind any variables.

We can now verify that the maps of Figure 14 constitute a reflection.

**Proposition 7.2** *Translations $*$ and $\#$ (Figure 14) form a reflection in $\lambda_c$ of $\lambda_{cps}$.*

As before, we prove each of the four parts of Proposition 3.1 separately. To prove part (i),

$$M \longrightarrow\!\!\!\!\!\rightarrow M^{*\#}$$

requires we strengthen the inductive hypothesis to

$$K^\flat[M] \longrightarrow\!\!\!\!\!\rightarrow (M : K)^\#.$$

Each proof is now straightforward, given the following two lemmas, which are equally straightforward.

**Lemma 7.3** *Let $M, V$ be in $\lambda_c$, and $K$ be in $\lambda_{cps}$, then $(M : K)[x := V^\dagger] \equiv (M[x := V]) : K$.*

**Lemma 7.4** *Let $M, V, K$ be in $\lambda_{cps}$, then:*

$$K^\flat[M^\#[x := V^\natural]] \longrightarrow\!\!\!\!\!\rightarrow (M[x := V][k := K])^\#.$$

This completes the parallel development of the reflection; there is also a parallel development of the factorization. Proposition 3.3 guarantees that there must be an isomorphic image of $\lambda_{cps}$ within $\lambda_c$. We name this calculus $\lambda_{c**}$, and it is summarized in Figure 15. Just as the grammar of $\lambda_{cps}$ has

$$
\begin{array}{lll}
\text{terms} & L, M, N & ::= \quad K[V] \mid K[VW] \\
\text{values} & V, W & ::= \quad x \mid \lambda x.\, M \\
\text{contexts} & K & ::= \quad [\,] \mid \texttt{let}\ x = [\,]\ \texttt{in}\ M
\end{array}
$$

$$
\begin{array}{lll}
(\beta.v) & K[(\lambda x.\, M)V] & \longrightarrow \quad M[x := V] : K \\
(\eta.v) & \lambda x.\, (Vx) & \longrightarrow \quad V, \text{ if } x \notin fv(V) \\
(\beta.let) & \texttt{let}\ x = V\ \texttt{in}\ M & \longrightarrow \quad M[x := V] \\
(\eta.let) & \texttt{let}\ x = [\,]\ \texttt{in}\ K[x] & \longrightarrow \quad K, \text{if } x \notin fv(K)
\end{array}
$$

$$
\begin{array}{lll}
V : K & \equiv & K[V] \\
(VW) : K & \equiv & K[VW] \\
(\texttt{let}\ x = V\ \texttt{in}\ M) : K & \equiv & \texttt{let}\ x = V\ \texttt{in}\ (M : K) \\
(\texttt{let}\ x = VW\ \texttt{in}\ M) : K & \equiv & \texttt{let}\ x = VW\ \texttt{in}\ (M : K)
\end{array}
$$

Figure 15: The kernel computational calculus $\lambda_{c**}$

| $*_1 : \lambda_c \to \lambda_{c**}$ | | | $\#_1 : \lambda_{c**} \to \lambda_c$ |
|---|---|---|---|
| $M^*$ | $\equiv$ | $M : [\,]$ | the trivial inclusion |
| $V : K$ | $\equiv$ | $K[V]$ | |
| $(PM) : K$ | $\equiv$ | $P : (\texttt{let}\ x = [\,]\ \texttt{in}\ ((xM) : K))$ | |
| $(VQ) : K$ | $\equiv$ | $Q : (\texttt{let}\ y = [\,]\ \texttt{in}\ ((Vy) : K))$ | |
| $(VW) : K$ | $\equiv$ | $K[V^\dagger W^\dagger]$ | |
| $(\texttt{let}\ x = M\ \texttt{in}\ N) : K$ | $\equiv$ | $M : (\texttt{let}\ x = [\,]\ \texttt{in}\ (N : K))$ | |
| $x^\dagger$ | $\equiv$ | $x$ | |
| $(\lambda x.\, M)^\dagger$ | $\equiv$ | $\lambda x.\, M^*$ | |

Figure 16: Inclusion in $\lambda_c$ of $\lambda_{c**}$

| $*_2 : \lambda_{c**} \to \lambda_{cps}$ | | | $\#_2 : \lambda_{cps} \to \lambda_{c**}$ |
|---|---|---|---|
| $(K[V])^*$ | $\equiv$ | $K^\ddagger V^\dagger$ | same as # from Figure 14 |
| $(K[VW])^*$ | $\equiv$ | $V^\dagger W^\dagger K^\ddagger$ | |
| $x^\dagger$ | $\equiv$ | $x$ | |
| $(\lambda x.\, M)^\dagger$ | $\equiv$ | $\lambda x.\, \lambda k.\, M^\dagger$ | |
| $[\,]^\ddagger$ | $\equiv$ | $k$ | |
| $(\texttt{let}\ x = [\,]\ \texttt{in}\ N)^\ddagger$ | $\equiv$ | $\lambda x.\, N^*$ | |

Figure 17: Isomorphism of $\lambda_{c**}$ and $\lambda_{cps}$

three components — terms, values, and continuations — so the grammar of $\lambda_{c**}$ has three components — terms, values, and contexts. Observe that, despite the introduction of contexts, each term still possesses a unique decomposition in terms of the syntax. (For instance, the term $xy$ corresponds to $K[VW]$, where $K$ is $[\,]$, $V$ is $x$, and $W$ is $y$.)

The terms of $\lambda_{c**}$ can be characterized as the terms of $\lambda_c$ in $(let.1)$, $(let.2)$, and $(assoc)$ normal form. As in the previous development, once a term has been normalized, no further reductions ever introduce a $(let.1)$ or $(let.2)$ redex. Alas, the same cannot be said of $(assoc)$. The reduction $(\beta.v)$ may indeed introduce a further $(assoc)$ redex, as shown by the counterexample,

$$
\begin{array}{ll}
 & \texttt{let}\ x = ((\lambda y.\, \texttt{let}\ z = ab\ \texttt{in}\ c)\ \ d)\ \texttt{in}\ e \\
\longrightarrow & \texttt{let}\ x = (\texttt{let}\ z = ab\ \texttt{in}\ c)\ \texttt{in}\ e.
\end{array}
$$

To gain insight into the problem, consider the corresponding CPS reduction,

$$
\begin{array}{ll}
 & ((\lambda y.\, \lambda k.\, (a\ b\ (\lambda z.\, kc)))\ d\ (\lambda x.\, ke)) \\
\longrightarrow & (a\ b\ (\lambda z.\, (\lambda x.\, ke)c)).
\end{array}
$$

The image of this in $\lambda_{c**}$ is

$$
\begin{array}{ll}
 & \texttt{let}\ x = ((\lambda y.\, \texttt{let}\ z = ab\ \texttt{in}\ c)\ \ d)\ \texttt{in}\ e \\
\longrightarrow & \texttt{let}\ z = ab\ \texttt{in}\ \texttt{let}\ x = c\ \texttt{in}\ e.
\end{array}
$$

where the right hand side is in $(assoc)$-normal form. The CPS language achieves this normalization us-

ing the meta-operation of substitution which traverses the CPS term to locate $k$ and replace it by the continuation thus effectively "pushing" the continuation deep inside the term.

In order to properly match the behavior of CPS, we therefore add a corresponding meta-operation to $\lambda_{c**}$, $M : K$ shown in Figure 15. Using the new meta-operation we can extend the problematic source reduction $\beta.v$ with a built-in $(assoc)$-normalization action that mirrors the action of $\beta.v$ on CPS terms.

(An alternative to adding meta-notation for substitution may be to move to calculi that use explicit substitution, but we have not explored this possibility.)

There is one pitfall to avoid: since reductions apply to any subterm, one may be tempted to apply rule $(\beta.v)$ to the subterm $((\lambda y.\,\mathtt{let}\ z = ab\ \mathtt{in}\ c)\ d)$ in the counterexample by taking $K$ to be $[\,]$ rather than $\mathtt{let}\ x = [\,]\ \mathtt{in}\ e$. But it is natural to insist that the reduction rule $(\beta.v)$ is only applied when $K$ is determined according to the unique decomposition afforded by the grammar of $\lambda_{c**}$, and this steps neatly around the pitfall.

Again, as guaranteed by Proposition 3.3, the reflection $* : \lambda_c \to \lambda_{cps}$ factors into an inclusion $*_1 : \lambda_c \to \lambda_{c**}$ and an order isomorphism $*_2 : \lambda_{c**} \to \lambda_{cps}$, given in Figures 16 and 17. Again, these can be computed directly from the proposition, and again there is a bonus: $*_1$ has a simple interpretation as reducing a term of $\lambda_c$ by applying rules $(let.1)$, $(let.2)$, and $(assoc)$ judiciously.

In addition, the CPS translation factors through the monad translation. One may translate $\lambda_{ml*}$ into $\lambda_{cps}$ as follows.

$$
\begin{array}{lcl}
[V]^* & \equiv & \overline{\lambda}k.\,kV^\dagger \\
(VW)^* & \equiv & \overline{\lambda}k.\,V^\dagger W^\dagger k \\
(\mathtt{let}\ x \Leftarrow M\ \mathtt{in}\ N)^* & \equiv & \overline{\lambda}k.\,M^*(\lambda x.\,N^* k) \\
x^\dagger & \equiv & x \\
(\lambda x.\,N)^\dagger & \equiv & \lambda x.\,N^*
\end{array}
$$

Regarding this as a two-stage translation with administrative reductions yields a reflection. As before, the reflection factors through a calculus $\lambda_{ml**}$, corresponding to the subset of $\lambda_{ml*}$ consisting of terms in $(assoc)$ normal form. The three calculi $\lambda_{c**}$, $\lambda_{ml**}$, and $\lambda_{cps}$ are isomorphic. Figure 1 diagrams the situation.

## 8 Related Work

Plotkin (1975), among other contributions, formalizes the call-by-value CPS translation and shows that it preserves but does not reflect equalities: if $M = N$ in $\lambda_v$ then $M^* = N^*$ in $\lambda_v$, but not conversely. (Here and throughout we write $*$ for the variant of the CPS translation under consideration, and hope this will lead to no confusion.)

Sabry and Felleisen (1993) strengthen Plotkin's result by making the implication above reversible. They extend the call-by-value lambda calculus $\lambda_v$ with a set of reductions $X$ such that $M^* = N^*$ in $\lambda_v X$ if and only if $M^* = N^*$ in $\lambda_v$. As they note, $\lambda_v X$ and $\lambda_c$ prove the same equalities; but they do not prove the same reductions. The rules of $\lambda_v X$ were derived by mapping back from $\lambda_{cps}$, and hence are somewhat convoluted; in contrast, Moggi's rules for $\lambda_c$ are pleasingly natural.

Sabry and Felleisen introduce the notion of *equational correspondence* described in Section 3, and they prove that their translation constitutes such a correspondence. In fact, they prove something stronger, making their translation almost, but not quite, a Galois connection: their translations satisfies all four conditions of Proposition 3.1, except that condition (ii), $P^{\#*} \longrightarrow\!\!\!\!\!\rightarrow P$, is replaced by the weaker $P^{\#*} = P$. (Compare this to the stronger $P^{\#*} \equiv P$ required of a reflection.)

Sabry and Felleisen single out a subset $A$ of $X$, and observe that these $A$ reductions correspond directly to the administrative reductions on CPS terms. Their terms in $A$ normal form roughly correspond to our kernel calculus $\lambda_{c**}$, of terms in $(let.1)$, $(let.2)$, and $(assoc)$ normal form.

Flanagan, Sabry, Duba, and Felleisen (1993) apply the results of Sabry and Felleisen. They suggest that CPS translation may not be so beneficial after all: it may be better to work directly in the source calculus. They show that terms in $A$ normal form behave similarly to CPS terms, demonstrating this via a sequence of abstract machines. They also briefly sketch possible applications of the full set of reductions $X$. But they fail to observe our central point: that for optimization purposes one wants a result showing correspondence of reductions rather than correspondence of equations.

Lawall and Danvy (1993) give a factoring of CPS similar to the one described here, and also refer to Galois connections. They relate four languages: a source language, a kernel source language, a kernel target language, and a target language. The first two relate via a Galois connection, the middle two are isomorphic, and the last two again relate via a Galois connection. The first two parts of their factorization are similar to our inclusion in $\lambda_c$ of $\lambda_{c**}$, and our order isomorphism from $\lambda_{c**}$ to $\lambda_{cps}$. Their third step schedules evaluation, determining for each application whether the function or argument evaluates first. Here we use a language where the function always evaluates before the argument, and so we need no counterpart of their third step.

The Galois connections of Lawall and Danvy are based on an artificial ordering induced directly from the translations. One might argue that they are misusing the notion of Galois connection, and are instead dealing with the somewhat weaker notion of a pair of translations $*$ and $\#$ satisfying $M^{*\#*} \equiv_T M^*$ and $P^{\#*\#} \equiv_S P^\#$. As they note,

their artificial ordering is unsatisfactory, because their middle isomorphism between the source kernel and target kernel does not respect this ordering: it is not an order isomorphism, and hence not a Galois connection. Thus, their factorization cannot be viewed as a composition of Galois connections. In contrast, we use a natural ordering relation, and our Galois connections do compose. Whereas their isomorphism *violates* the ordering of their Galois connection, our isomorphism arises as a *consequence* of our Galois connection, as shown by Proposition 3.3.

Hatcliff and Danvy (1994) consider translations analogous to our translations from $\lambda_v$ to $\lambda_{ml}$, and from $\lambda_{ml}$ to $\lambda_{cps}$. They also look at translations from other source languages into $\lambda_{ml}$, an issue we ignore. They show the translation from $\lambda_v$ to $\lambda_{ml}$ is sound; we give the stronger result that the translation from $\lambda_c$ to $\lambda_{ml}$ is a reflection. They also show that the translation from $\lambda_{ml}$ to $\lambda_{cps}$ is an equational correspondence; again, we give the stronger result that it is a reflection.

## 9 Conclusion

Our results might extend in a number of ways.

Most lambda calculi possess a notion of *standard reduction*, characterized by two properties. First, at most one standard reduction applies to a term. Second, if any sequence of reductions reduces a term to an answer, then the sequence of standard reductions will also do so. Hence standard reductions capture the behavior of an evaluator. Plotkin (1975) specified standard reductions for $\lambda_v$, and his results for CPS demonstrate not only that reductions are preserved, but also that standard reductions are preserved. (He expresses this in a different but equivalent form by saying that evaluation is preserved.) Hatcliff and Danvy (1994) give similar results for translation from Moggi's $\lambda_{ml}$ into CPS. It appears straightforward to extend the work here by specifying a suitable notion of standard reduction for each of the calculi involved, and to show that the given translations are still reflections if one replaces reductions by standard reductions.

Sabry and Felleisen (1993) deal with a variant of the CPS translation, due to Fisher (1972), that reverses the order in which the argument and the continuation are passed to functions. The effect of this reversal is to enable additional administrative reductions to be performed. Just as the ordinary CPS translation corresponds to a kernel of $\lambda_c$ where terms are normalized with regard to $(let.1)$, $(let.2)$, and $(assoc)$, so does the Fisher CPS translation correspond to a kernel of $\lambda_c$ where terms are normalized with respect to all three of these rules and $(\beta.let)$ as well.

Maraist *et al.* (1995) consider translations into a linear lambda calculus, stressing the analogue with the CPS translation. That paper studies a call-by-let calculus that is closely related to $\lambda_c$ and that translates into linear logic. By extending the call-by-let calculus with just one law,

$$\texttt{let}\ x = M\ \texttt{in}\ N \quad \longrightarrow \quad N, \qquad \text{if } x \notin fv(N),$$

the authors derive a call-by-need calculus (Ariola *et al.* 1995) that translates into affine logic. We conjecture that when augmented with the above law, $\lambda_c$ also yields a model of call-by-need.

The call-by-value translation of Maraist *et al* (1995) carries over to a reflection between $\lambda_c$ and a suitable linear lambda calculus; the kernel of the reflection is given by the calculus $\lambda_{c*}$, normalized with respect to $(let.1)$ and $(let.2)$, but not $(assoc)$. Hence, an analogue with the monad translation is more apposite than an analogue with CPS.

*A reflection on call-by-value.* Plotkin's original paper on $\lambda_v$ layed out two key properties of this calculus: first, it is adequate to describe evaluation, and second, it is inadequate to prove some equalities that we might reasonably expect to hold between terms. The first was demonstrated by a correspondence between $\lambda_v$ and Landin's SECD machine (Landin 1964). The second was demonstrated by observing that there are terms that are not provably equal in $\lambda_v$, but whose translations into CPS are provably equal.

Moggi defined $\lambda_c$ as an extension of $\lambda_v$ that is sound and complete for all monad models, and hence proves a reasonably large set of equalities. He picked a confluent calculus to ease symbolic manipulation, but made no claims that $\lambda_c$ was itself a reasonable model of computation. Sabry and Felleisen showed that $\lambda_c$ proves two terms equal exactly when their CPS translations are equal. This reinforces the claim that $\lambda_c$ yields a good theory of equality, but because they dealt only with equational correspondence, again says nothing about $\lambda_v$ as a model of computation. Our results here relate $\lambda_c$ reductions to reductions in $\lambda_{ml}$ and $\lambda_{cps}$, both widely accepted as models of computation. We hereby put forward $\lambda_c$ as a model of call-by-value computation that improves on $\lambda_v$.

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