The First Monad Tutorial

Philip Wadler

University of Edinburgh

YOW! Melbourne, Brisbane, Sydney

December 2013

SCALA MONADS

DECLUTTER YOUR CODE WITH MONADIC DESIGN

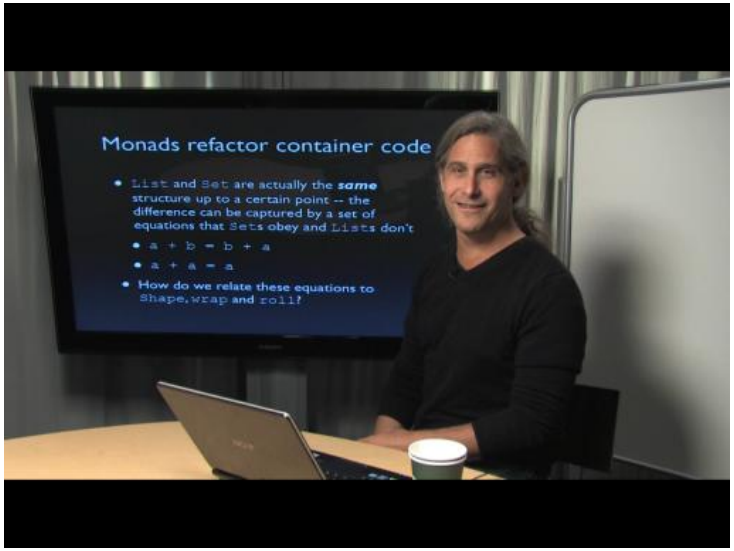
Dan Rosen, Marakana, Inc.

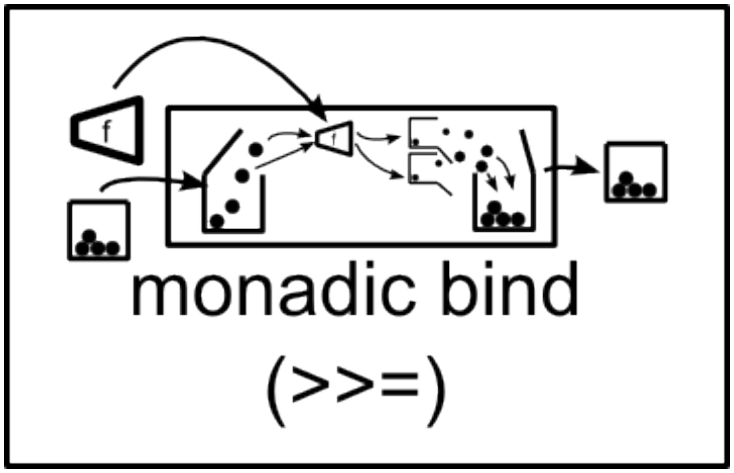
San Francicso, CA - February 09, 2012

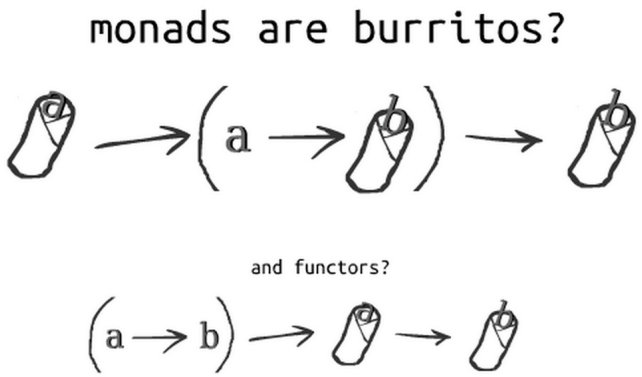
Monads in Perl

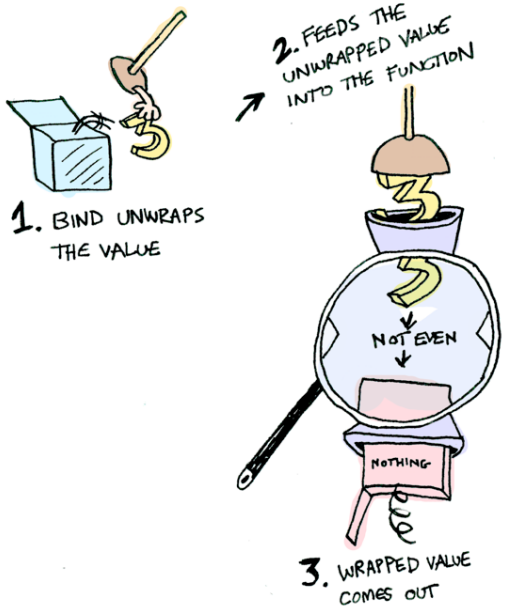
本間　雅洋

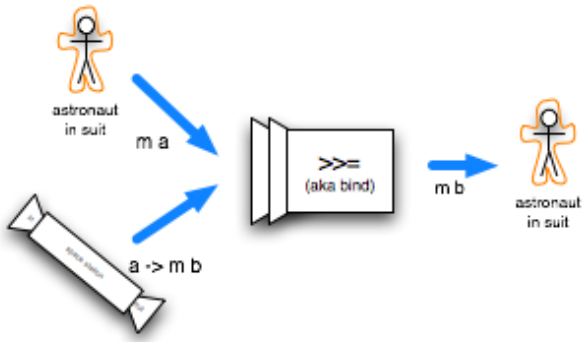
hiratara <hira.tara@gmail.com>











1. remove astronaut from suit
2. put naked astronaut in station
3. send out whatever the station sends out (well... almost)


Haskell


Clojure


Microsoft
Visual F#


Microsoft
C#.net


Scala

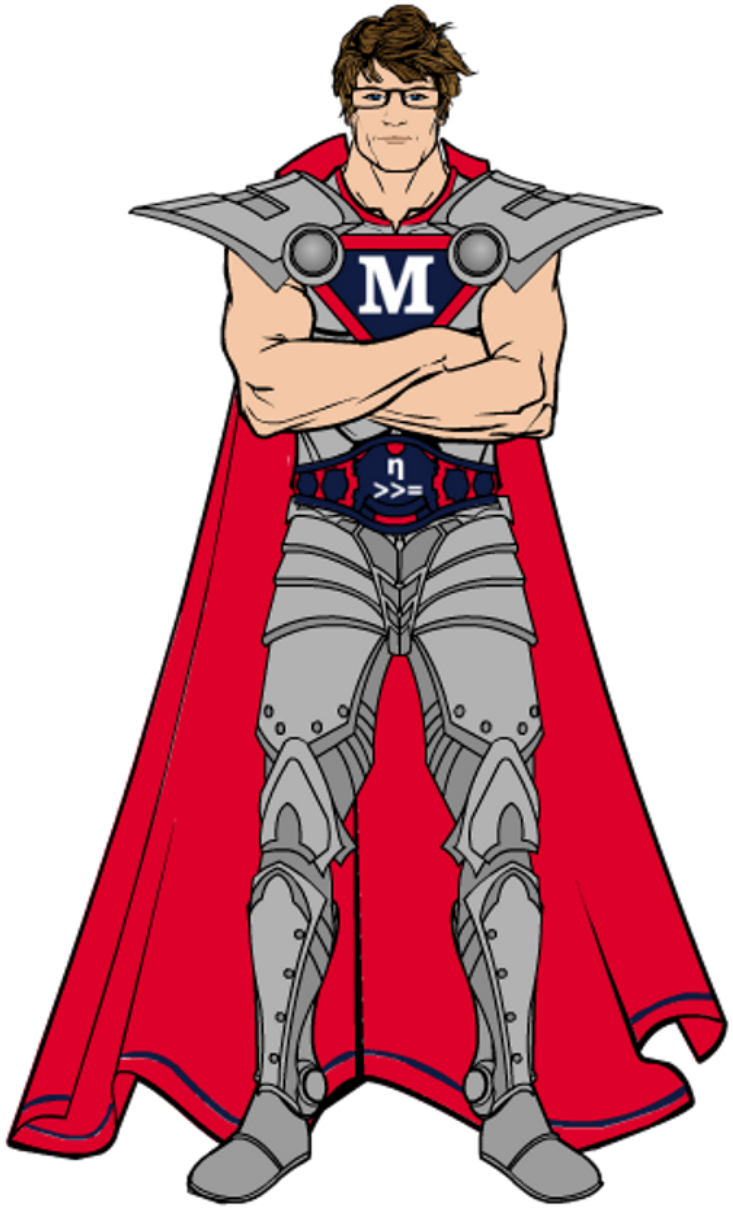

python

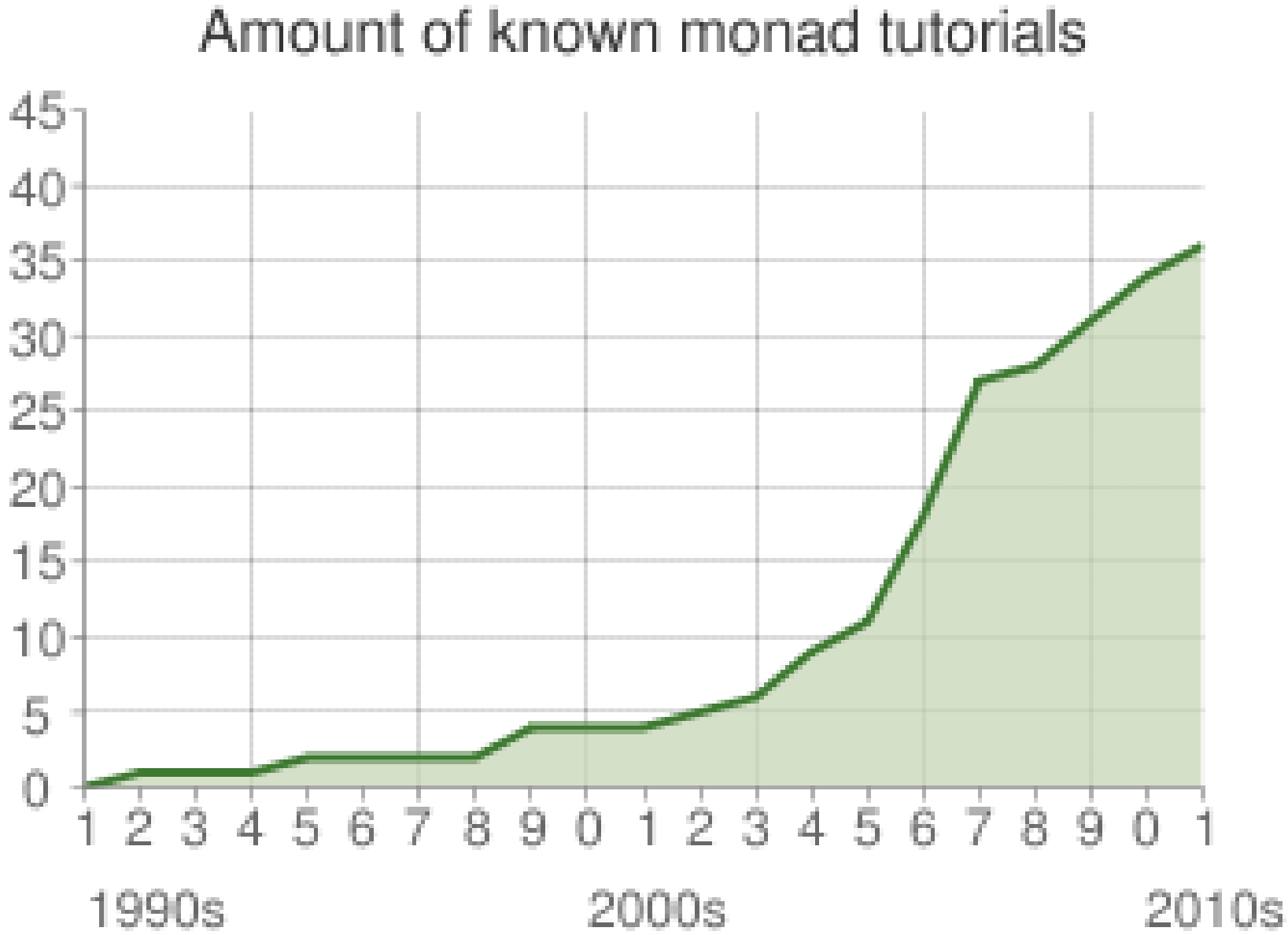

trapd in Monad tutorl
plz help


MONADS
MONADS EVERYWHERE
makeameme.org


trapd in IO monad
plz help


BRACE YOURSELVES
THE MONAD TUTORIALS ARE COMING
memegenerator.net


M
η
>>=


Amount of known monad tutorials
45
40
35
30
25
20
15
10
5
0
1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1
1990s
2000s
2010s

Church and State 1: Evaluating monads

Philip Wadler

~~University of Glasgow~~

Bell Labs, Lucent Technologies

Pure vs. impure

Modification	Impure language (Standard ML, Scheme)	Pure language (Miranda*, Haskell†)
Error messages	use exceptions	rewrite
Execution counts	use state	rewrite
Output	use output	rewrite
Backwards output	rewrite	modify output

*Miranda is a trademark of Research Software Limited.

†Haskell is not a trademark.

Variations on an evaluator

monad *n.* **1.** *Philosophy* **a.** any fundamental entity, esp. if autonomous.

– *Collins English Dictionary*

Variation zero: The basic evaluator

$$
\begin{array}{lcl}
\textbf{data}\ \mathit{Term} & = & \mathit{Con}\,\mathit{Int} \mid \mathit{Div}\ \mathit{Term}\ \mathit{Term} \\
\mathit{eval} & :: & \mathit{Term} \to \mathit{Int} \\
\mathit{eval}\,(\mathit{Con}\,a) & = & a \\
\mathit{eval}\,(\mathit{Div}\ t\ u) & = & \mathit{eval}\ t \div \mathit{eval}\ u
\end{array}
$$

Using the evaluator

Test data:

$$
\begin{array}{lcl}
\mathit{answer}, \mathit{error} & :: & \mathit{Term} \\
\mathit{answer} & = & (\mathit{Div}\,(\mathit{Div}\,(\mathit{Con}\,1972)\,(\mathit{Con}\,2))\,(\mathit{Con}\,23)) \\
\mathit{error} & = & (\mathit{Div}\,(\mathit{Con}\,1)\,(\mathit{Con}\,0))
\end{array}
$$

A sample run:

```
?eval answer
42

?eval error
< BOOM! >
```

Variation one: Exceptions

```
data  M a          =  Raise Exception | Return a
type  Exception    =  String

eval               ::  Term → M Int
eval (Con a)       =  Return a
eval (Div t u)     =  case  eval t  of
                        Raise e → Raise e
                        Return a →
                          case  eval u  of
                            Raise e → Raise e
                            Return b →
                              if  b == 0
                                then  Raise "divide by zero"
                                else  Return (a ÷ b)
```

Using the evaluator

A sample run:

?eval answer
Return 42

?eval error
Raise "divide by zero"

Variation two: State

$$
\begin{array}{lcl}
\textbf{type}\ \ M\,a & = & State \to (a, State) \\
\textbf{type}\ \ State & = & Int \\
 & & \\
eval & :: & Term \to M\,Int \\
eval\,(Con\,a)\,x & = & (a, x) \\
eval\,(Div\,t\,u)\,x & = & \textbf{let}\ \ (a, y) = eval\,t\,x\ \ \textbf{in} \\
 & & \textbf{let}\ \ (b, z) = eval\,u\,y\ \ \textbf{in} \\
 & & (a \div b, z + 1)
\end{array}
$$

Using the evaluator

Test data:

$$\begin{aligned} &\mathit{answer} :: \mathit{Term} \\ &\mathit{answer} = (\mathit{Div}\,(\mathit{Div}\,(\mathit{Con}\,1972)\,(\mathit{Con}\,2))\,(\mathit{Con}\,23)) \end{aligned}$$

A sample run:

$$\begin{aligned} &?\mathit{eval}\ \mathit{answer}\ 0 \\ &(42, 2) \end{aligned}$$

Variation three: Output

```
type  M a      =  (Output, a)
type  Output   =  String

eval           ::  Term → M Int
eval (Con a)   =  (line (Con a) a, a)
eval (Div t u) =  let  (x, a) = eval t  in
                  let  (y, b) = eval u  in
                  (x ++ y ++ line (Div t u) (a ÷ b), a ÷ b)

line           ::  Term → Int → Output
line t a       =  showterm t ++ "=" ++ showint a ++ "↩"
```

Using the evaluator

A sample run:

$$
\begin{aligned}
&?\mathit{eval}\ \mathit{answer}\\
&(\text{“}\mathit{Con}\,1972 = 1972\hookleftarrow\\
&\ \mathit{Con}\,2 = 2\hookleftarrow\\
&\ \mathit{Div}\,(\mathit{Con}\,1972)\,(\mathit{Con}\,2) = 986\hookleftarrow\\
&\ \mathit{Con}\,23 = 23\hookleftarrow\\
&\ \mathit{Div}\,(\mathit{Div}\,(\mathit{Con}\,1972)\,(\mathit{Con}\,2))\,(\mathit{Con}\,23)) = 42\hookleftarrow\\
&\text{''},42)
\end{aligned}
$$

Changing output order

Replace

$$x +\!\!+ y +\!\!+ line\,(Div\ t\ u)\,(a \div b)$$

with

$$line\,(Div\ t\ u)\,(a \div b) +\!\!+ y +\!\!+ x.$$

Modified sample run:

```
?eval answer
("Div (Div (Con 1972) (Con 2)) (Con 23)) = 42↩
Con 23 = 23↩
Div (Con 1972) (Con 2) = 986↩
Con 2 = 2↩
Con 1972 = 1972↩
", 42)
```

Monads

monad *n.* **1. b.** (in the metaphysics of Leibnitz) a simple indestructible nonspatial element regarded as the unit of which reality consists.

– *Collins English Dictionary*

What is a monad?

1. For each type a of *values*, a type $M\,a$ to represent *computations*.

   In general, $a \to b$ becomes $a \to M\,b$.
   In particular, $eval :: Term \to Int$ becomes $Term \to M\,Int$.

2. A way to turn values into computations.

$$unit \;::\; a \to M\,a$$

3. A way to combine computations.

$$(\star) \;::\; M\,a \to (a \to M\,b) \to M\,b$$

Compare $\star$ with **let**

$$\frac{\Gamma \vdash m :: M\,a \qquad \Gamma, x :: a \vdash n :: M\,b}{\Gamma \vdash (m \star \lambda x.\, n) :: M\,b}$$

$$\frac{\Gamma \vdash m :: a \qquad \Gamma, x :: a \vdash n :: b}{\Gamma \vdash (\textbf{let}\ \ x = m\ \ \textbf{in}\ \ n) :: b}$$

Monad laws

Left unit

$$unit\, x \star \lambda y.\, n = n[x/y]$$

Right unit

$$m \star \lambda x.\, unit\, x = m$$

Associative (when x is not free in o).

$$m \star (\lambda x.\, n \star \lambda y.\, o) = (m \star \lambda x.\, n) \star \lambda y.\, o$$

These resemble the laws for a *monoid*, except for λ binding.

Monad Laws

$$
\begin{aligned}
\mathit{return}\ v \gg\!= \lambda x.\, k\ x &= k\ v \\
m \gg\!= \lambda x.\, \mathit{return}\ x &= m \\
m \gg\!= (\lambda x.\, k\ x \gg\!= (\lambda y.\, h\ y)) &= (m \gg\!= (\lambda x.\, k\ x)) \gg\!= (\lambda y.\, h\ y)
\end{aligned}
$$

Do notation

$$
\begin{aligned}
\begin{pmatrix} \mathsf{do}\ x \leftarrow \mathit{return}\ v \\ \quad k\ x \end{pmatrix} &= k\ v \\
\begin{pmatrix} \mathsf{do}\ x \leftarrow m \\ \quad \mathit{return}\ x \end{pmatrix} &= m \\
\begin{pmatrix} \mathsf{do}\ y \leftarrow \mathsf{do}\ x \leftarrow m \\ \qquad\quad k\ x \\ h\ y \end{pmatrix} &= \begin{pmatrix} \mathsf{do}\ x \leftarrow m \\ \quad y \leftarrow k\ x \\ \quad h\ y \end{pmatrix}
\end{aligned}
$$

The evaluator revisited

monad *n.* **1. c.** (in the pantheistic philosophy of Giordano Bruno) a fundamental metaphysical unit that is spatially extended and psychically aware.

– *Collins English Dictionary*

Monadic evaluator

$$
\begin{array}{lcl}
\mathit{eval} & :: & \mathit{Term} \to \mathit{M}\ \mathit{Int} \\
\mathit{eval}\ (\mathit{Con}\ a) & = & \mathit{unit}\ a \\
\mathit{eval}\ (\mathit{Div}\ t\ u) & = & \mathit{eval}\ t \star \lambda a.\ \mathit{eval}\ u \star \lambda b.\ \mathit{unit}\ (a \div b)
\end{array}
$$

Variation zero, revisited: The basic evaluator

$$
\begin{array}{lcl}
\textbf{type}\ M\,a & = & a \\
unit & :: & a \to M\,a \\
unit\,a & = & a \\
(\star) & :: & M\,a \to (a \to M\,b) \to M\,b \\
a \star k & = & k\,a
\end{array}
$$

Variation one, revisited: Exceptions

```
data  M a          =  Raise Exception | Return a
type  Exception    =  String

unit               ::  a → M a
unit a             =   Return a

(⋆)                ::  M a → (a → M b) → M b
m ⋆ k              =   case m of
                         Raise e → Raise e
                         Return a → k a

raise              ::  Exception → M a
raise e            =   Raise e
```

Modifying the evaluator

$$
\begin{array}{lcl}
\mathit{eval} & :: & \mathit{Term} \to M\,\mathit{Int} \\
\mathit{eval}\,(\mathit{Con}\,a) & = & \mathit{unit}\,a \\
\mathit{eval}\,(\mathit{Div}\,t\,u) & = & \mathit{eval}\,t \star \lambda a. \\
 & & \quad \mathit{eval}\,u \star \lambda b. \\
 & & \quad \textbf{if}\ \ b == 0 \\
 & & \qquad \textbf{then}\ \ \mathit{raise}\ \text{“divide by zero”} \\
 & & \qquad \textbf{else}\ \ \mathit{unit}\,(a \div b)
\end{array}
$$

Variation two, revisited: State

$$
\begin{array}{lll}
\textbf{type}\ M\,a & = & State \to (a, State) \\
\textbf{type}\ State & = & Int \\
 & & \\
unit & :: & a \Rightarrow M\,a \\
unit\ a & = & \lambda x.\,(a, x) \\
 & & \\
(\star) & :: & M\,a \to (a \Rightarrow M\,b) \to M\,b \\
m \star k & = & \lambda x.\,\textbf{let}\ \ (a, y) = m\,x\ \ \textbf{in} \\
 & & \quad\ \textbf{let}\ \ (b, z) = k\,a\,y\ \ \textbf{in} \\
 & & \quad\ (b, z) \\
 & & \\
tick & :: & M\,() \\
tick & = & \lambda x.\,((), x + 1)
\end{array}
$$

Modifying the evaluator

$$
\begin{array}{lcl}
\mathit{eval} & :: & \mathit{Term} \to M\,\mathit{Int} \\
\mathit{eval}\,(\mathit{Con}\,a) & = & \mathit{unit}\,a \\
\mathit{eval}\,(\mathit{Div}\,t\,u) & = & \mathit{eval}\,t \star \lambda a. \\
 & & \mathit{eval}\,u \star \lambda b. \\
 & & \mathit{tick} \quad \star \lambda(). \\
 & & \mathit{unit}\,(a \div b)
\end{array}
$$

Variation three, revisited: Output

$$
\begin{array}{lcl}
\textbf{type}\ \ M\,a & = & (Output, a) \\
\textbf{type}\ \ Output & = & String \\
 & & \\
unit & :: & a \to M\,a \\
unit\,a & = & (\text{“”}, a) \\
 & & \\
(\star) & :: & M\,a \to (a \to M\,b) \to M\,b \\
m \star k & = & \textbf{let}\ \ (x, a) = m\ \ \textbf{in} \\
 & & \textbf{let}\ \ (y, b) = k\,a\ \ \textbf{in} \\
 & & (x +\!\!+ y, b) \\
 & & \\
out & :: & Output \to M\,() \\
out\,x & = & (x, ())
\end{array}
$$

Modifying the evaluator

$$
\begin{array}{lcll}
\mathit{eval} & :: & \mathit{Term} \to M\,\mathit{Int} & \\
\mathit{eval}\,(\mathit{Con}\,a) & = & \mathit{out}\,(\mathit{line}\,(\mathit{Con}\,a)\,a) & \star\lambda().\\
 & & \mathit{unit}\,a & \\
\mathit{eval}\,(\mathit{Div}\,t\,u) & = & \mathit{eval}\,t & \star\lambda a.\\
 & & \mathit{eval}\,u & \star\lambda b.\\
 & & \mathit{out}\,(\mathit{line}\,(\mathit{Div}\,t\,u)\,(a \div b)) & \star\lambda().\\
 & & \mathit{unit}\,(a \div b) &
\end{array}
$$

Changing output order

$$
\begin{array}{lcl}
(\star) & :: & M\,a \to (a \to M\,b) \to M\,b \\
m \star k & = & \textbf{let}\ \ (x, a) = m\ \ \textbf{in} \\
 & & \textbf{let}\ \ (y, b) = k\,a\ \ \textbf{in} \\
 & & (y +\!\!+ x, b)
\end{array}
$$

Conclusions

monad *n.* **2.** a single-celled organism, especially a flagellate protozoan

– Collins English Dictionary

The Glasgow Haskell compiler

Joint work with

Cordy Hall, Kevin Hammond,
Will Partain, Simon Peyton Jones.

Glasgow Haskell compiler is written in Haskell.

Each phase uses a monad.

Has proved easy to modify in practice.

Monads in the Glasgow Haskell compiler

Type inference phase.

- Exceptions for errors,
- state for current substitution,
- state for fresh variable names,
- read-only state for current location.

Simplification phase.

- State for fresh variable names.

Code generator phase.

- Output for code generated so far,
- state for table mapping variables to addressing modes,
- state for table to cache known state of stack.

Origins

Eugenio Moggi, *Computational λ-calculus and monads*, 1989.

values (int) vs. computations ($T\ int$)
call-by-value ($int \to T\ int$)
call-by-name ($T\ int \to T\ int$)

Michael Spivey, *A functional theory of exceptions*, 1990.

John Reynolds, *The essence of Algol*, 1981.

data types (int) vs. phrase types ($int\ exp$)
call-by-value ($int \to int\ exp$)
call-by-name ($int\ exp \to int\ exp$)

But Reynolds missed *unit* and $\star$.

monadism *or* **monadology** *n.* (esp. in writings of Leibnitz) the philosophical doctrine that monads are the ultimate units of reality.

– *Collins English Dictionary*

Monads

$$
\begin{array}{lrcl}
(1) & \mathit{return}\ v \gg\!= \lambda x.\, k\ x & = & k\ v \\
(2) & m \gg\!= \lambda x.\, \mathit{return}\ x & = & m \\
(3) & m \gg\!= (\lambda x.\, k\ x \gg\!= (\lambda y.\, h\ y)) & = & (m \gg\!= (\lambda x.\, k\ x)) \gg\!= (\lambda y.\, h\ y)
\end{array}
$$

- Eugenio Moggi, Computational Lambda Calculus and Monads, *Logic in Computer Science*, 1989.
- Philip Wadler, Comprehending Monads, *International Conference on Functional Programming*, 1990.
- Philip Wadler, The Essence of Functional Programming, *Principles of Programming Languages*, 1992.

Arrows

$$
\begin{array}{lrcl}
(1) & \mathit{arr}\ \mathit{id} \ggg f & = & f \\
(2) & f \ggg \mathit{arr}\ \mathit{id} & = & f \\
(3) & (f \ggg g) \ggg h & = & f \ggg (g \ggg h) \\
(4) & \mathit{arr}\ (g \cdot f) & = & \mathit{arr}\ f \ggg \mathit{arr}\ g \\
(5) & \mathit{first}\ (\mathit{arr}\ f) & = & \mathit{arr}\ (f \times \mathit{id}) \\
(6) & \mathit{first}\ (f \ggg g) & = & \mathit{first}\ f \ggg \mathit{first}\ g \\
(7) & \mathit{first}\ f \ggg \mathit{arr}\ (\mathit{id} \times g) & = & \mathit{arr}\ (\mathit{id} \times g) \ggg \mathit{first}\ f \\
(8) & \mathit{first}\ f \ggg \mathit{arr}\ \mathit{fst} & = & \mathit{arr}\ \mathit{fst} \ggg f \\
(9) & \mathit{first}\ (\mathit{first}\ f) \ggg \mathit{arr}\ \mathit{assoc} & = & \mathit{arr}\ \mathit{assoc} \ggg \mathit{first}\ f
\end{array}
$$

- John Hughes, Generalising Monads to Arrows, *Science of Computer Programming*, 2000.

Idioms (Applicative Functors)

$$
\begin{array}{lrcl}
(1) & u & = & \mathit{pure}\ \mathit{id} \otimes u \\
(2) & \mathit{pure}\ f \otimes \mathit{pure}\ p & = & \mathit{pure}\ (f\ p) \\
(3) & u \otimes (v \otimes w) & = & \mathit{pure}\ (\cdot) \otimes u \otimes v \otimes w \\
(4) & u \otimes \mathit{pure}\ x & = & \mathit{pure}\ (\lambda f.\ f\ x) \otimes u
\end{array}
$$

- Conor McBride and Ross Patterson, Applicative Programming with Effects, *Journal of Functional Programming*, 2008.